Review Article

Geometrical and P.D.E. Methods in the Treatment of the Theory of Shells: Comparing Euclidean and Affine Approaches

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The use of differential equations methods in the approach, treatment, and solution of problems in diverse areas of geometry, particularly in affine differential geometry is well known and prolific, where they have proven to be quite fruitful when it comes to the obtainment of definite results. It is perhaps lesser known that the same kind of those very same methods has been and is currently being used to treat developments in some specific areas of applied sciences, such as the theory of shells where, similarly, they can be proven to be quite effective as well. In this paper we precisely show that such is the case in two particular, related instances: the historic approach of the classical, Euclidean part of the theory pursued by Fritz John, in the past century, and the more recent expositions that we ourselves have dedicated to the affine counterpart of the theory.

1. Introduction

The use of differential equations methods in the approach, treatment, and solution of problems in diverse aspects of geometry, particularly in affine differential geometry, is quite well known and efficient. See, for instance, [1–21] and further references therein. On the other hand, it is not quite well known that the applications of geometrical combined with P.D.E. methods may be quite efficient as well in the consideration and treatment of a very specific topic of applied mathematics, namely, the theory of shells. In fact, most of the people working in the area of engineering are not aware of that particular approach initiated in the past century by John [22, 23], for the usual Euclidean case, that is, invariant under the action of the Euclidean group. On our hand, inspired by that very same source, we have developed in recent years an alternative approach to the theory which is, in this new situation and context, invariant under the action of the unimodular affine group. See [24, 25] and further references therein.

In fact the usual, general theory of thin shells has been historically developed in a great variety of ways and accessed by different authors based, from the geometrical point of view, on the classical theory of surfaces in three-dimensional space, particularly with respect to the invariants of the Euclidean group, $ASO(3, \mathbb{R})$, that is, the group of transformations generated by translations and rotations of the space [22, 23, 26–39]. Hence, what is called “normal” is the Euclidean one, and the “distance” is measured with respect to the norm induced by the usual, positive definite scalar product of vectors, which is the main, fundamental invariant in Euclidean geometry. Moreover, it should also be emphasized that its realm deals with a topic of mathematics with a rich history and many, diverse applications to the real world: engineering, industry, avionics, and so on.

On our part, we have been working more recently on an alternative foundation and development of the theory of shells which is invariant under the action of the unimodular affine group, $ASL(3, \mathbb{R})$. Thus, for the case in treatment, this gives rise to the so-called affine geometry of surfaces. For a

given surface in the three-dimensional space we use, within this context, concepts such as “affine normal” and “affine distance,” corresponding to the above mentioned ones in Euclidean geometry. See our previous articles [24, 25], for full details.

In this, mainly intended, survey paper we make an exposition of that alternative foundation and development of the theory. In brief summary, the contents presented here are the following. We start by introducing, in Section 2, an abbreviated version of the concepts of Euclidean and affine shells, previously treated in the latter cited articles. The exposition of compatibility conditions occupies Section 3, already showing from the beginning that the mainstream of the subject lies precisely in geometry and partial differential equations, since those very important conditions derive from the very well-known concept of integrability conditions. The basic inequalities of the theory, a topic inspired precisely in P.D.E. methods, are presented in Section 4. The further development of the theory requires the strain-stress relations in affine shells, with a combination and intercourse of geometrical and physical concepts, which are treated in Section 5. Once again, the use of P.D.E. methods appears with the estimates for the L_2 -norms of second order derivatives, which occupy Section 6, while the corresponding estimates of higher order derivatives take place in Section 7. Finally, Section 8 is dedicated to presenting the recovery of the deformed middle surface, which is obtained through the contribution of the fundamental theorem of unimodular affine surfaces.

2. Euclidean and Affine Shells

We consider the middle surface of a (solid) shell in its original (undeformed) state, denoted by M_0 , parametrized locally by a vector function $X_0 : U \rightarrow \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, which is assumed to be smooth enough. Coordinates in the domain are denoted by (u^1, u^2) . Thus, we can write locally $M_0 = X_0(U)$ and assume besides, as it is usually done, that X_0 is topological immersion (embedding).

Particles in the original state have curvilinear Lagrange coordinates (U^1, U^2, U^3) that for our present purposes shall be chosen in a special way: $(U^1, U^2, U^3) = (u^1, u^2, u)$ if we represent them by equation $X(u^1, u^2, u) = X_u(u^1, u^2) = X_0(u^1, u^2) + u\vec{n}$, where we have obviously extended the previous function to $X : U \times (-h, h) \rightarrow \mathbb{R}^3$, and $\vec{n}$ is the vector field normal to the middle surface. This normal can be the *Euclidean normal*, N_{eu} , of the classical, Euclidean theory of surfaces or the *unimodular affine normal*, N_{ua} , of our own, current development. In each case we shall clarify the situation when we deal with one or the other.

In the Euclidean case we shall use the following notations regarding the main geometrical objects, defined on the middle surface prior to deformation, which take part in the formulation of the theory [22–25]:

$$I_{eu} = \sum_{\alpha, \beta} a_{\alpha\beta} du^\alpha du^\beta \quad \text{with } a_{\alpha\beta} = \frac{\partial X_0}{\partial u^\alpha} \cdot \frac{\partial X_0}{\partial u^\beta} \quad (1)$$

denotes the Euclidean first fundamental form, while with the expression

$$II_{eu} = \sum_{\alpha, \beta} L_{\alpha\beta} du^\alpha du^\beta \quad \text{where } L_{\alpha\beta} = N_{eu} \cdot \frac{\partial^2 X_0}{\partial u^\alpha \partial u^\beta} \quad (2)$$

we represent the second fundamental form and with

$$III_{eu} = \sum_{\alpha, \beta} M_{\alpha\beta} du^\alpha du^\beta, \quad (3)$$

where $M_{\alpha\beta} = \sum_{\lambda} L_{\alpha\lambda} L_{\beta}^\lambda = \sum_{\gamma\lambda} a^{\gamma\lambda} L_{\alpha\lambda} L_{\beta\gamma}$,

the Euclidean third fundamental form.

In the state previous to deformation the border of the shell is made up of two “faces,” which are surfaces parallel to the middle surface M_0 at respective distance h , measured along the Euclidean normal N_{eu} , and of the “border” constituted by segments normal to the faces. Therefore, along the normal to M_0 coordinates U^1 and U^2 remain constant while $U^3 := u$ measures the signed distance from M_0 . Faces can be represented, then, by equations $U^3 = u = \pm h$ while the middle surface is given by $U^3 = u = 0$.

Now if $a_{\alpha\beta}$, $L_{\alpha\beta}$, and $M_{\alpha\beta}$ are, respectively, the coefficients of the first, second, and third Euclidean fundamental forms of the middle surface M_0 , the Euclidean structure of the ambient space induces a Riemannian structure on the shell and we can obtain, by means of straightforward computations, the following expressions in the above normal coordinates $(U^1, U^2, U^3) = (U^1, U^2, u)$ [22, 23]:

$$\begin{aligned} A_{\alpha\beta} &= \frac{\partial X}{\partial u^\alpha} \cdot \frac{\partial X}{\partial u^\beta} = a_{\alpha\beta} - 2uL_{\alpha\beta} + u^2M_{\alpha\beta}, \\ A_{\alpha 3} &= A_{3\alpha} = \frac{\partial X}{\partial u^\alpha} \cdot \frac{\partial X}{\partial u} = \frac{\partial X}{\partial u^\alpha} \cdot N_{eu} = 0, \\ A_{33} &= \frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial u} = N_{eu} \cdot N_{eu} = 1. \end{aligned} \quad (4)$$

Corresponding to the shell and its middle surface, in the state previous to deformation, we can consider the geometrical objects belonging to the shell in the deformed state that we shall denote with an upper right asterisk. Thus, for example, $X_0^* : U \rightarrow \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, represents the parametrization of the deformed middle surface $M_0^* = X_0^*(U)$, and we remark that the domain of definition of this immersion, $U \subset \mathbb{R}^2$, and the parameters (u^1, u^2) used in it are the same as those belonging to the middle surface of the shell in the original state, previous to deformation.

Consequently, the rest of geometrical objects change from one state to the other and the problem is to determine the nature and extension of such changes for every one of them reducing, under appropriate hypotheses, the obtainable information to both middle surfaces. One such hypothesis is the one concerning the comparison of the thickness parameter h , which is usually assumed to be small with respect to the other dimensions of the shell. This introduces in the theory the concept of “thin” shell which has important uses and applications.

In order to consider next the unimodular affine geometry of surfaces, we assume a defined, in the ambient space $\mathbb{R}^3$, exterior 3-form or nontrivial determinant function, denoted by the symbol $[\cdot, \cdot, \cdot] = \det$. Then, given the same previous mean surface, we can represent the objects of that geometry by the following expressions.

First, for the construction of the unimodular first fundamental form we define

$$h_{\alpha\beta} := \left[\frac{\partial X_0}{\partial u^1}, \frac{\partial X_0}{\partial u^2}, \frac{\partial^2 X_0}{\partial u^\alpha \partial u^\beta} \right]. \quad (5)$$

Then, by assuming that the surface is nondegenerate, that is, $H = \det(h_{\alpha\beta}) \neq 0$, we can write $g_{\alpha\beta} = |H|^{-1/4} h_{\alpha\beta}$, obtaining the unimodular affine first fundamental form expressed by equation

$$I_{ua} = \sum_{\alpha, \beta} g_{\alpha\beta} du^\alpha du^\beta. \quad (6)$$

It follows that the latter turns out to be a semi-Riemannian structure, which we shall also refer to as pseudometric. See [2–4, 7, 8, 11, 12, 24, 25].

The unimodular affine normal is defined next by the expression

$$N_{ua} = \frac{1}{2} \Delta(X_0), \quad (7)$$

where Δ is the Laplacian operator with respect to the pseudometric I_{ua} ; that is,

$$\begin{aligned} \Delta X_0 &= \frac{1}{\sqrt{|g|}} \\ &\times \sum_{\alpha, \beta=1}^2 \frac{\partial}{\partial u^\alpha} \left(\sqrt{|g|} \sum_{\beta=1}^2 g^{\alpha\beta} \frac{\partial X_0}{\partial u^\beta} \right) \quad \text{with } g = \det(g_{\alpha\beta}). \end{aligned} \quad (8)$$

It is clear that, from the above developments, we obtain three connections defined on the immersed middle surface:

- (1) the Levi-Civita connection with respect to the Euclidean metric I_{eu} , which we shall label here as ∇_{EU} and which coincides with the projection over M_0 of the usual, flat connection D of $\mathbb{R}^3$ in the direction of the classical Euclidean normal N_{EU} ,
- (2) the Levi-Civita connection with respect to the pseudometric I_{ua} , $\tilde{\nabla}$,
- (3) the *affine normal induced* connection, ∇ , that is, the projection of D in the direction of the affine normal N_{ua} . Thus, for example, for vector fields X and Y tangent to M_0 we may write

$$\nabla_{X_p} Y = \text{proy}_{N_{ua}}(D_{X_p} Y). \quad (9)$$

We define next the unimodular affine second fundamental form [7, 8, 24, 25]

$$\nabla(I_{ua}) := II_{ua} \quad (10)$$

that we may also represent in local coordinates by

$$II_{ua} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma} du^\alpha du^\beta du^\gamma, \quad (11)$$

with the coefficients $g_{\alpha\beta\gamma}$ totally symmetric in their indices. Some authors prefer to refer to the latter as the *cubic form* [11, 12].

Finally, we consider the *affine third fundamental form*, which we can describe in the following way: similar to the Euclidean case regarding Weingarten's equation, it turns out too, in affine geometry of surfaces, that the local derivatives of the affine normal belong to the tangent plane of the surface at each point; that is, we can write

$$\frac{\partial N_{ua}}{\partial u^\alpha} = - \sum_{\beta} B_{\alpha}^{\beta} \frac{\partial X_0}{\partial u^\beta} = -B_{\alpha}^1 \frac{\partial X_0}{\partial u^1} - B_{\alpha}^2 \frac{\partial X_0}{\partial u^2} \quad (12)$$

and define the affine third fundamental form by the expression:

$$III_{ua} = \sum_{\alpha, \beta} B_{\alpha\beta} du^\alpha du^\beta, \quad \text{with } B_{\alpha\beta} = \sum_{\gamma} g_{\alpha\gamma} B_{\beta}^{\gamma}. \quad (13)$$

Similarly, *Weingarten's affine shape operator* S is defined by [7, 12]

$$S = S_{ua} = - \sum_{\alpha, \beta} B_{\alpha}^{\beta} du^\alpha \otimes \frac{\partial}{\partial u^\beta}. \quad (14)$$

As we have previously seen, the definition of shell as a three-dimensional body and, in particular, the Riemannian structure induced on that object by the ambient space metric is generated in a natural fashion. In the present case of unimodular affine geometry that extension is not at all that immediate. However, as we shall see, it can also be realized in a canonical way.

Proposition 1. *With the above conditions and notations, one can assert that there exists a positive $h > 0$ such that the given pseudometric, originally defined on the middle surface M_0 , may be extended to all points at affine distance less than or equal to h from M_0 .*

Proof. By starting from the unimodular affine invariant pseudometric I_{ua} , defined on the middle surface M_0 ,

$$g_{\alpha\beta} = I_{ua} \left(\frac{\partial X_0}{\partial u^\alpha}, \frac{\partial X_0}{\partial u^\beta} \right), \quad (15)$$

we seek, in the present context, to define on the shell a pseudometric, which is a unimodular affine invariant, to be denoted by

$$G = \sum G_{ij} du^i du^j. \quad (16)$$

That is,

$$G_{ij} := G \left(\frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \right). \quad (17)$$

Observe that, since bilinearity must be preserved, we have to write in *affine normal coordinates* of the shell

$$\begin{aligned} G_{\alpha\beta} &= G\left(\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta}\right) \\ &= G\left(\frac{\partial X_0}{\partial u^\alpha} + u \frac{\partial N_{ua}}{\partial u^\alpha}, \frac{\partial X_0}{\partial u^\beta} + u \frac{\partial N_{ua}}{\partial u^\beta}\right), \end{aligned} \quad (18)$$

whence, by definition, we should have

$$G_{\alpha\beta} := g_{\alpha\beta} - 2uB_{\alpha\beta} + u^2 \sum_{\lambda} B_{\alpha}^{\lambda} B_{\beta\lambda} \quad (19)$$

and where, as stated previously, Greek indices run from 1 to 2. Thus, in order to extend that definition to the third index, we also write

$$G_{3\alpha} = G_{\alpha 3} = G\left(\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^3}\right) = G(X_\alpha, N_{ua}) := 0 \quad (20)$$

and, finally,

$$G_{33} = G\left(\frac{\partial X}{\partial u^3}, \frac{\partial X}{\partial u^3}\right) = G(N_{ua}, N_{ua}) := 1. \quad (21)$$

Then, it is easy to see that there exists $h > 0$ such that, for $|u| = |u^3| < h$, it holds that

$$\det(G_{ij}) \neq 0 \quad (22)$$

and, consequently, the latter is a pseudo-Riemannian, unimodular affine invariant metric defined on the shell with the suitable data, as it was our purpose to construct. $\square$

3. Compatibility Conditions [24]

One of the main aspects in the theory of shells is the determination of *compatibility conditions*. These are restraints on the behavior of the various difference tensors that can be defined by comparing the two states of the shell. The natural tool here is represented by the integrability conditions that must be satisfied, in all cases, by both middle surfaces. These conditions are very well known in the case of Euclidean shells (see, e.g., [22, 23, 31, 32, 36]) and can be described, in our notation, as follows.

For the strain tensor with components defined by $\varepsilon_{\alpha\beta} = (1/2)(a_{\alpha\beta}^* - a_{\alpha\beta})$, it is proven that

$$\begin{aligned} \varepsilon_{\beta,\delta}^{\beta,\delta} - \varepsilon_{\beta,\delta}^{\delta,\beta} &= L_{\beta}^{*\delta} L_{\delta}^{*\beta} - L_{\delta}^{*\delta} L_{\beta}^{*\beta} \\ &+ g_{\mu}^{*\beta} \left(L_{\delta}^{\delta} L_{\beta}^{\mu} - L_{\beta}^{\delta} L_{\delta}^{\mu} \right) \\ &- g_{\mu\nu}^* g^{\beta\alpha} g^{\delta\gamma} \left(C_{\alpha\beta}^{\mu} C_{\gamma\delta}^{\nu} - C_{\alpha\delta}^{\mu} C_{\beta\gamma}^{\nu} \right) \end{aligned} \quad (23)$$

while for the difference tensor $w_{\alpha\beta} := L_{\alpha\beta}^* - L_{\alpha\beta}$ it holds that

$$\begin{aligned} w_{\beta,\gamma}^{\alpha} - w_{\gamma,\beta}^{\alpha} &= g^{\alpha\rho} (w_{\rho\beta,\gamma} - w_{\rho\gamma,\beta}) \\ &= g^{\alpha\rho} \left(C_{\rho\gamma}^{\mu} L_{\mu\beta}^* - C_{\rho\beta}^{\mu} L_{\mu\gamma}^* \right). \end{aligned} \quad (24)$$

In both equations the symbol $C_{\rho\beta}^{\mu}$ represents the components of the difference tensor between the Levi-Civita connection of M_0^* and that of M_0 .

Now, for the case of affine shells the corresponding compatibility conditions were obtained in our previous article [24], where the basis for our analysis and development was mainly given by the following two lemmas, with somewhat more general validity and context.

Lemma 2 (relation between the Levi-Civita connections). *Let (M, g) and (M, g^*) be (pseudo-) Riemannian structures defined on the differentiable manifold M , denote with ∇ and ∇^* the Levi-Civita connections associated with them, and let Γ_{BC}^A and Γ_{BC}^{*A} be the corresponding Christoffel symbols. Then, both connections are related by the equation*

$$\Gamma_{BC}^{*A} = \Gamma_{BC}^A + T_{BC}^A, \quad (25)$$

where T_{BC}^A represent the components of the tensor defined by

$$T_{BC}^A = \frac{1}{2} \bar{g}^{*AR} (g_{RB,C}^* + g_{RC,B}^* - g_{BC,R}^*), \quad (26)$$

and $\bar{g}^{*UV}$ are the components of the tensor which represents the inverse matrix of g_{UV}^* .

Lemma 3 (relation between the curvature tensors). *In the conditions of the previous lemma, the components of the respective associated tensors of the Riemannian curvature, R_{MNP}^L and R_{ABCD}^* , are related by the following equation:*

$$\begin{aligned} R_{ACDB}^* &= \varepsilon_{AB;CD} + \varepsilon_{CD;AB} - \varepsilon_{AD;CB} - \varepsilon_{BC;AD} \\ &- \frac{1}{2} (g_{AM}^* R_{CBD}^M + g_{CM}^* R_{ADB}^M) \\ &+ g_{LS}^* (T_{AB}^L T_{CD}^S - T_{AD}^L T_{BC}^S), \end{aligned} \quad (27)$$

where, by definition, $\varepsilon_{AB} := (1/2)(g_{AB}^* - g_{AB})$, and the double semicolon denotes the “second covariant derivative” with respect to the Levi-Civita connection associated with the metric g .

Then, with the help of those two previous lemmas, the main result of this section may be stated as follows.

Theorem 4 (compatibility conditions for affine shells). *For the difference tensors defined by the various expressions that establish comparisons between the respective components of the first, second, and third fundamental forms, that is,*

$$\varepsilon_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta}^* - g_{\alpha\beta}), \quad \sigma_{\alpha\beta\gamma} := g_{\alpha\beta\gamma}^* - g_{\alpha\beta\gamma}, \quad (28)$$

$$w_{\alpha\beta} := B_{\alpha\beta}^* - B_{\alpha\beta},$$

and the tensor defined by comparison between the corresponding Levi-Civita connections, represented by equation

$$\tilde{\Gamma}_{\alpha\beta}^{*\mu} = C_{\alpha\beta}^{\mu} + \tilde{\Gamma}_{\alpha\beta}^{\mu}, \quad (29)$$

there hold the following conditions:

(1) *affine Gauss condition*

$$\begin{aligned} \varepsilon_{\beta,\delta}^{\beta,\delta} - \varepsilon_{\beta,\delta}^{\delta,\beta} &= \frac{1}{2} \left(B_{\beta}^{*\beta} g_{\delta}^{*\delta} - B_{\delta}^{*\beta} g_{\beta}^{*\delta} + B_{\delta}^{*\delta} g_{\beta}^{*\beta} - B_{\beta}^{*\delta} g_{\delta}^{*\beta} \right) \\ &\quad - \frac{1}{2} g^{\beta\alpha} g^{\delta\gamma} \left(A_{\gamma\beta}^{*\eta} \cdot g_{\alpha\eta}^{*} - A_{\gamma\delta}^{*\eta} \cdot g_{\alpha\eta}^{*} \right) \\ &\quad + \frac{1}{2} g_{\mu}^{*\beta} \left(-B_{\delta}^{\delta} \delta_{\beta}^{\mu} + A_{\beta}^{\delta\eta} \cdot A_{\eta\delta}^{\mu} \right) \\ &\quad - g_{\lambda\mu}^{*} g^{\beta\alpha} g^{\delta\gamma} \left(C_{\alpha\beta}^{\lambda} C_{\gamma\delta}^{\mu} - C_{\alpha\delta}^{\lambda} C_{\beta\gamma}^{\mu} \right), \end{aligned} \quad (30)$$

(2) *affine Mainardi-Codazzi condition*

$$\begin{aligned} \sigma_{\alpha\beta\gamma,\delta} - \sigma_{\alpha\beta\delta,\gamma} &= g_{\mu\beta\gamma}^{*} C_{\alpha\delta}^{\mu} + g_{\mu\alpha\gamma}^{*} C_{\beta\delta}^{\mu} - g_{\mu\beta\delta}^{*} C_{\alpha\gamma}^{\mu} \\ &\quad - g_{\mu\alpha\delta}^{*} C_{\beta\gamma}^{\mu} + B_{\alpha\delta}^{*} g_{\beta\gamma}^{*} + B_{\beta\delta}^{*} g_{\alpha\gamma}^{*} \\ &\quad - B_{\alpha\gamma}^{*} g_{\beta\delta}^{*} - B_{\beta\gamma}^{*} g_{\alpha\delta}^{*} - B_{\alpha\delta} g_{\beta\gamma} \\ &\quad - B_{\beta\delta} g_{\alpha\gamma} + B_{\alpha\gamma} g_{\beta\delta} + B_{\beta\gamma} g_{\alpha\delta}, \end{aligned} \quad (31)$$

(3) *Codazzi condition for the affine shape operators*

$$\begin{aligned} w_{\beta,\alpha}^{\alpha} - w_{\alpha,\beta}^{\alpha} &= g^{\alpha\rho} \left[B_{\beta\mu}^{*} \left(C_{\rho\alpha}^{\mu} + A_{\rho\alpha}^{*\mu} \right) - B_{\alpha\mu}^{*} \left(C_{\rho\beta}^{\mu} + A_{\rho\beta}^{*\mu} \right) + B_{\alpha\mu} A_{\rho\beta}^{\mu} \right]. \end{aligned} \quad (32)$$

4. Basic Inequalities for Affine Shells

The following basic inequalities, involving the geometrical objects treated above, were previously obtained and may be seen in [25].

When represented in the form of Monge's, that is, as a graph, the middle surface of the shell M_0 has all of its geometrical properties related to a given function f assumed to be differentiable enough and, in the present context of affine geometry, satisfying a partial differential equation of Monge-Ampère type:

$$\det(\partial_{\alpha\beta} f) = \pm F, \quad (33)$$

and for such a kind of equations, with boundary conditions as in the present case, there hold bounds for the function f and its derivatives. Also, since the function F is strictly positive in the complete domain where f is defined, there exist lower and upper bounds for F as well.

As a consequence, we can also assume that the second derivatives of f are bounded, that is, the components of the Hessian matrix $(\partial_{\alpha\beta} f)$, as well as the components of the inverse matrix of the latter, denoted by $(f^{\alpha\beta})$, and also the components of the pseudometric tensor, covariant as well as contravariant, that is, $g_{\alpha\beta}$ and $g^{\alpha\beta}$. These facts are expressed in the following inequalities:

$$|\partial_{\alpha\beta} f| < K, \quad |f^{\alpha\beta}| < K, \quad |g_{\alpha\beta}| < K, \quad |g^{\alpha\beta}| < K. \quad (34)$$

Besides, since the higher order derivatives are also bounded and in order to unify notation, we shall assume that there exists a *generalized affine upper bound of curvature*, intimately related to the upper bound for the affine principal curvatures of the middle surface M_0 , which we shall also denote by R and which for the present, affine case remains specified by the conditions that

$$|\partial_{\alpha\beta\gamma} f| < \frac{1}{R^{1/2}} \quad (35)$$

and for the successive derivatives,

$$\begin{aligned} |\partial_{\alpha\beta\gamma\eta} f| &\leq R^{-1}, \\ |\partial_{\alpha\beta\gamma\eta\lambda} f| &\leq (R^{-1/2})^3, \dots, \end{aligned} \quad (36)$$

are satisfied for as high order of derivatives as needed in the development of the theory.

By using these hypotheses one obtains the corresponding bounds for the components of the tensor representing the third fundamental form

$$B_{\alpha\beta} = -\frac{1}{4} \left(\partial_{\alpha\beta} (\log F) + \frac{1}{4} \partial_{\alpha} (\log F) \partial_{\beta} (\log F) \right. \quad (37)$$

$$\left. - \sum_{\sigma,\lambda} f^{\sigma\lambda} \partial_{\alpha\beta\sigma} f \partial_{\lambda} (\log F) \right),$$

if we have in mind, besides, the two following, well-known identities:

$$\partial_{\alpha} \log F = \sum_{\rho,\sigma} f^{\rho\sigma} \partial_{\alpha\rho\sigma} f, \quad (38)$$

$$\partial_{\alpha\beta} \log F = \sum_{\rho,\sigma} f^{\rho\sigma} \partial_{\alpha\beta\rho\sigma} f - \sum_{\rho,\sigma} f^{\rho\theta} f^{\tau\sigma} \partial_{\theta\tau\alpha} f \partial_{\rho\sigma\beta} f,$$

with which it turns out that

$$|B_{\alpha\beta}| \leq \frac{1}{4} \frac{1}{R} (4K + 32K^2) = (K + 8K^2) \frac{1}{R}. \quad (39)$$

We compute next the partial derivatives of these components:

$$\begin{aligned}
& \partial_\gamma B_{\alpha\beta} \\
&= -\frac{1}{4} \begin{pmatrix} \partial_\gamma f^{\rho\tau} \partial_{\alpha\beta\rho\tau} f + f^{\rho\tau} \partial_{\alpha\beta\rho\tau\gamma} f \\ -(\partial_\gamma f^{\rho\eta} \partial_{\eta\mu\alpha} f) (f^{\mu\tau} \partial_{\beta\rho\tau} f) \\ - (f^{\rho\eta} \partial_{\eta\mu\alpha\gamma} f) (f^{\mu\tau} \partial_{\beta\rho\tau} f) \\ - (f^{\rho\eta} \partial_{\eta\mu\alpha} f) (\partial_\gamma f^{\mu\tau} \partial_{\beta\rho\tau} f) \\ - (f^{\rho\eta} \partial_{\eta\mu\alpha} f) (f^{\mu\tau} \partial_{\beta\rho\tau\gamma} f) \end{pmatrix} \\
&+ \frac{1}{16} \begin{pmatrix} (\partial_\gamma f^{\rho\tau} \partial_{\tau\rho\alpha} f) (f^{\sigma\lambda} \partial_{\beta\sigma\lambda} f) \\ + (f^{\rho\tau} \partial_{\tau\rho\alpha\gamma} f) (f^{\sigma\lambda} \partial_{\beta\sigma\lambda} f) \\ + (f^{\rho\tau} \partial_{\tau\rho\alpha} f) (\partial_\gamma f^{\sigma\lambda} \partial_{\beta\sigma\lambda} f) \\ + (f^{\rho\tau} \partial_{\tau\rho\alpha} f) (f^{\sigma\lambda} \partial_{\beta\sigma\lambda\gamma} f) \end{pmatrix} \quad (40) \\
&- \frac{1}{4} \begin{pmatrix} (\partial_\gamma f^{\sigma\lambda} \partial_{\alpha\beta\sigma} f) (f^{\rho\tau} \partial_{\lambda\rho\tau} f) \\ + (f^{\sigma\lambda} \partial_{\alpha\beta\sigma\gamma} f) (f^{\rho\tau} \partial_{\lambda\rho\tau} f) \\ + (f^{\sigma\lambda} \partial_{\alpha\beta\sigma} f) (\partial_\gamma f^{\rho\tau} \partial_{\lambda\rho\tau} f) \\ + (f^{\sigma\lambda} \partial_{\alpha\beta\sigma} f) (f^{\rho\tau} \partial_{\lambda\rho\tau\gamma} f) \end{pmatrix}.
\end{aligned}$$

Then, by using the identity

$$\sum_\lambda f^{\lambda\sigma} \partial_{\lambda\mu} f = \delta_\mu^\sigma, \quad (41)$$

from which it follows that

$$\partial_\alpha f^{\sigma\gamma} = -\sum_\lambda f^{\lambda\gamma} f^{\mu\sigma} \partial_{\alpha\lambda\mu} f, \quad (42)$$

we find by direct computation the following estimate:

$$|\partial_\gamma B_{\alpha\beta}| \leq \frac{1}{R^{3/2}} (K + 19K^2 + 24K^3). \quad (43)$$

With the development done so far, we can also obtain estimates for the components of the pseudometric, that is, the components of the pseudo-Riemannian tensor $G = \sum_{i,j} G_{ij} du^i du^j$ of the shell in the undeformed state and its successive derivatives, partial as well as covariant. For example, we observe first from the very definition that

$$G_{\alpha\beta} = g_{\alpha\beta} - 2u B_{\alpha\beta} + u^2 \sum_{\lambda,\mu} g^{\lambda\mu} B_{\alpha\mu} B_{\beta\lambda} \quad (44)$$

and, consequently, obtain

$$|G_{\alpha\beta}| \leq K + 2K(1 + 8K) \frac{h}{R} + 4K^2(1 + 8K) \left(\frac{h}{R}\right)^2. \quad (45)$$

In a similar fashion, one can obtain other estimates for the higher order derivatives. All of these estimates shall be used in the attainment of one of our important objectives: to find estimates for the higher order covariant derivatives for the strain and the stress tensors.

5. Strain-Stress Relations in Euclidean and Affine Shells

For the case of Affine shells, the contravariant components of the stress tensor, t^{mk} , are connected with the components of the strain tensor, ε_{mk} , by means of the stress-strain relations

$$t^{mk} := \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_{mk}}, \quad (46)$$

which are defined in a similar fashion as to the Euclidean case, introduced by John [22, 23]. In the latter equation W is the strain energy density of the given material that relates both the stress and the strain tensors, defined by

$$t := \sum_{i,j} t_{ij} du^i du^j = \sum_{i,j} t^{ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \quad (47)$$

$$\varepsilon := \sum_{i,j} \varepsilon_{ij} du^i du^j,$$

with $\varepsilon_{ij} = (1/2)(G_{ij}^* - G_{ij})$.

The same expression in terms of the corresponding (1,1)-tensors is

$$t_i^m = \sum_k G_{ik} t^{mk} = \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_m^i}. \quad (48)$$

Next, we introduce the components of the “pseudo-stress tensor” defined by

$$T_i^m := \sqrt{\frac{G^*}{G}} t_i^m - \delta_i^m W, \quad (49)$$

and we may also write

$$\begin{aligned}
T_i^m &= (W_{s_1} - W) \delta_i^m + (2W_{s_1} + 2W_{s_2}) \varepsilon_i^m \\
&+ (4W_{s_2} + 3W_{s_3}) \sum_k \varepsilon_i^k \varepsilon_k^m + 6W_{s_3} \sum_{s,k} \varepsilon_i^s \varepsilon_s^k \varepsilon_k^m, \quad (50)
\end{aligned}$$

where

$$\begin{aligned}
s_1 &= \sum_i \varepsilon_i^i, \\
s_2 &= \sum_{i,j} \varepsilon_j^i \varepsilon_i^j, \\
s_3 &= \sum_{i,j,k} \varepsilon_j^i \varepsilon_k^j \varepsilon_i^k. \quad (51)
\end{aligned}$$

Then, the equations of equilibrium [25] can be written:

$$t_{,,j}^{ij} + c_{hj}^i t^{hj} + c_{hj}^h t^{ij} = 0, \quad (52)$$

where the double comma „ indicates covariant differentiation with respect to the Levi-Civita connection determined by the pseudo-Riemannian structure G and where

$$c_{jk}^i = \frac{1}{2} G^{*ir} (G_{rj,k}^* + G_{rk,j}^* - G_{jk,r}^*). \quad (53)$$

Therefore, we also have as a consequence that

$$\begin{aligned} \sum_m T_{i;m}^m &= \left(\sqrt{\frac{G^*}{G}} \right) \\ &\times \sum_{m,r,s} \left(c_{mr}^r t^{ms} G_{si}^* - t^{rs} c_{rm}^m G_{si}^* \right. \\ &\quad \left. - t^{mr} c_{rm}^s G_{si}^* + t^{ms} G_{si;m}^* - \frac{1}{2} t^{ms} G_{sm;i}^* \right) = 0. \end{aligned} \quad (54)$$

Additional notations are needed in order to compare the components of the stress and strain tensors, even those belonging to different spaces of definition, for example, and very particularly, in order to compare the components of type (0,2) tensors t^{ij} with those with components of type (1,1) t_i^j . Thus, we follow in this respect the kind of notation previously introduced by John in [22, 23], in particular the so-called “general form of an expression” like

$$F(p, q)(u + v + w), \quad (55)$$

representing a vector in a suitable space, where u, v, w, p , and q are vectors themselves. The notation indicates that each of the components of $F(p, q)(u + v + w)$ is a sum of a linear form in the components of u , a linear form in the components of v , and a linear form in the components of w . The coefficients of these linear forms are functions of the components of the vectors p and q defined and differentiable as often as needed for all sufficiently small “lengths” $|p|$ and $|q|$. The letter F stands for a different expression in every equation to be considered. Thus, for example, we can write for the components of the stress and strain tensors of type (1,1)

$$t = (t_k^i), \quad \varepsilon = (\varepsilon_k^i). \quad (56)$$

Thus, in terms of the Lamé coefficients λ , and μ , the following equation is valid:

$$t_i^m = \lambda \sum_j \varepsilon_j^j \delta_i^m + 2\mu \varepsilon_i^m + F(\varepsilon) \varepsilon^2, \quad (57)$$

since such coefficients are defined by the relation

$$W := \frac{\lambda}{2} (s_1)^2 + \mu s_2 + F(\varepsilon) \varepsilon^3, \quad (58)$$

where, as before,

$$s_1 = \sum_i \varepsilon_i^i, \quad s_2 = \sum_{i,j} \varepsilon_i^j \varepsilon_j^i, \quad s_3 = \sum_{i,j,k} \varepsilon_i^j \varepsilon_j^k \varepsilon_k^i \quad (59)$$

and where we observe that the first two terms, on the right-hand side, are quadratic with regards to the strain tensor (operator) $\varepsilon = (\varepsilon_k^i)$, while the third term involves all of those components of order higher than two, representing otherwise the “remainder,” of paramount importance when coming to the corresponding numerical estimates.

From now on we establish that in the same sense all of the expressions to follow have to be interpreted.

Hence, by taking partial derivatives, we can write

$$W_{s_1} = \frac{\partial W}{\partial s_1} = \partial_{s_1} W = \lambda s_1, \quad (60)$$

$$W_{s_2} = \frac{\partial W}{\partial s_2} = \partial_{s_2} W = \mu.$$

From the latter we obtain, successively, first that

$$\begin{aligned} 2W_{s_1} + 2W_{s_2} &= 2\mu + 2\lambda s_1, \\ 4W_{s_2} + 3W_{s_3} &= 2\mu + F(t) t^3, \end{aligned} \quad (61)$$

$$W_{s_1} - W = \lambda s_1 - \frac{1}{2} \lambda (s_1)^2 - \frac{1}{2} \mu s_2 + F(\varepsilon) \varepsilon^3.$$

Then, by using Taylor’s series development

$$(1+x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)x + \frac{3/4}{2}x^2 + \dots, \quad (62)$$

we can express

$$\sqrt{\frac{G}{G^*}} = \left(\frac{G}{G^*}\right)^{1/2} = \left(\frac{G^*}{G}\right)^{-1/2} = 1 - \frac{1}{2}s_1 + \frac{3}{8}(s_1)^2 + \dots, \quad (63)$$

and also

$$t_i^m = \sqrt{\frac{G}{G^*}} \left(W_{s_1} \delta_i^m + 2W_{s_2} \varepsilon_i^m + 3W_{s_3} \sum_j \varepsilon_j^m \varepsilon_j^j \right) \quad (64)$$

becomes, first,

$$\begin{aligned} t_i^m &= \left(1 - \frac{1}{2}s_1 + \frac{3}{8}(s_1)^2 + \dots \right) \\ &\times \left(W_{s_1} \delta_i^m + 2W_{s_2} \varepsilon_i^m + 3W_{s_3} \sum_j \varepsilon_j^m \varepsilon_j^j \right) \end{aligned} \quad (65)$$

and, afterwards,

$$\begin{aligned} t_i^m &= W_{s_1} \delta_i^m + 2W_{s_2} \varepsilon_i^m + 3W_{s_3} \sum_j \varepsilon_j^m \varepsilon_j^j \\ &= \frac{\lambda}{2} s_1 \delta_i^m + 2\mu \varepsilon_i^m + 3W_{s_3} \sum_j \varepsilon_j^m \varepsilon_j^j \\ &= \frac{\lambda}{2} \sum_j \varepsilon_j^j \delta_i^m + 2\mu \varepsilon_i^m + F(\varepsilon) \varepsilon^2. \end{aligned} \quad (66)$$

From the latter, the trace of the stress tensor (operator) can be written:

$$\sum_j t_j^j = \left(\frac{3}{2} \lambda + 2\mu \right) \sum_j \varepsilon_j^j + F(\varepsilon) \varepsilon^2, \quad (67)$$

where $\sum_j \varepsilon_j^j$ itself represents the trace of the strain tensor (operator).

Hence, from the above we can also write

$$\epsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{\lambda}{2\mu} s_1 \delta_i^m + F(t) t^2, \quad (68)$$

or, also,

$$\epsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{1-2\mu}{2\mu} \sum_j t_j^j \delta_i^m + F(t) t^2. \quad (69)$$

Then, the expression for the components of the pseudo-stress tensor is

$$\begin{aligned} T_i^m &= t_i^m + \frac{2}{\mu} \sum_i t_i^m t_i^s + \left(\frac{5\mu-2}{\mu} \right) \sum_j t_j^j t_i^s \\ &\quad - \frac{1}{2} \left(\frac{1}{2\mu} \sum_{r,s} t_r^r t_s^s - \left(\frac{1-2\mu}{2\mu} \right) \sum_r t_r^r \sum_s t_s^s \right) \delta_i^m + F(t) t^3. \end{aligned} \quad (70)$$

We introduce next the “vector” $\eta = (\eta_k^i)$ by means of the following relation:

$$G_{ik} = \delta_k^i + \eta_k^i, \quad (71)$$

which clearly measures the difference between the metric matrix and that corresponding to the identity.

Then, we obtain the following estimate for the components of the corresponding inverse matrix $(G^{ik}) := (G_{ik})^{-1}$:

$$G^{ik} = \delta_k^i + \eta_k^i + F(\eta) (\eta^2). \quad (72)$$

Next, a straightforward computation shows that the Christoffel symbols associated with the Levi-Civita connection determined by the pseudometric G satisfy the estimate

$$\Gamma_{kr}^i = F(\eta) (\eta^r). \quad (73)$$

Then, it also holds the following estimate:

$$t_{ik} = t_k^i + F(\eta, t) (\eta t). \quad (74)$$

For the metric tensor in the deformed “strained” state we have, by definition,

$$G_{ik}^* = G_{ik} + 2\epsilon_{ik}. \quad (75)$$

Hence, we can also estimate that

$$\begin{aligned} G_{ik}^* &= G_{ik} + \frac{2}{\mu} t_{ik} \\ &\quad - 2 \left(\frac{1-2\mu}{2\mu} \right) \left(\sum_j t_j^j \right) \delta_{ik} + F(t, \eta) (t^2 + \eta t). \end{aligned} \quad (76)$$

For the tensor with components c_{kr}^i measuring the change in the Levi-Civita connections, from the “unstrained” natural state to the deformed “strained” state, we estimate

$$c_{kr}^i = F(\eta, t) (t^r + \eta^r t). \quad (77)$$

Then, we also obtain the two following estimates:

$$\begin{aligned} \sum_m t_{im;m} &= F(\eta, t) (tt' + \eta' t + \eta t'), \\ \sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} &= F(\eta, t) (\eta t'' + (t')^2 + t\eta' t' + \eta' t' + \eta'' t \\ &\quad + (\eta')^2 t + (\eta')^2 t^2 + tt''), \end{aligned} \quad (78)$$

since, in the present case, the three-dimensional compatibility equations are given in terms of the comparison between the Riemannian curvature tensors of the affine shell, when passing from the natural (undeformed) to the deformed state.

Let us remark here on an important comparison that holds between the expression obtained by John [22], in Euclidean geometry, by using the corresponding compatibility conditions, which in that context furnish for both states of the shell that the curvature tensor is vanishing, as expressed in (7) of the cited article; that is,

$$\begin{aligned} 0 &= R_{acdb}^* = \epsilon_{ab;cd} + \epsilon_{cd;ab} - \epsilon_{ad;cb} \\ &\quad - \epsilon_{bc;ad} + \sum_{l,s} G_{ls}^* (c_{ab}^l c_{cd}^s - c_{ad}^l c_{bc}^s), \end{aligned} \quad (79)$$

while for us, in the present context of affine geometry, the same kind of geometrical object is in general not vanishing, so that we have to include both by writing

$$\begin{aligned} R_{acdb}^* &= \epsilon_{ab;cd} + \epsilon_{cd;ab} - \epsilon_{ad;cb} - \epsilon_{bc;ad} \\ &\quad - \frac{1}{2} \left(\sum_m G_{am}^* R_{cbd}^m + \sum_m G_{cm}^* R_{adb}^m \right) \\ &\quad + \sum_{l,s} G_{ls}^* (c_{ab}^l c_{cd}^s - c_{ad}^l c_{bc}^s) \end{aligned} \quad (80)$$

obtained by direct application of Lemma 3 above. Because of this significant fact, both of them have to be estimated.

Thus, if we denote by $\epsilon_{ab;cd}$ the second covariant derivatives with respect to the Levi-Civita connection associated with the pseudometric G , we further obtain from the latter equation

$$\begin{aligned} &\epsilon_{ab;cd} + \epsilon_{cd;ab} - \epsilon_{ad;cb} - \epsilon_{bc;ad} \\ &= - \sum_{l,s} G_{ls}^* (c_{ab}^l c_{cd}^s - c_{ad}^l c_{bc}^s) + R_{acdb}^* \\ &\quad - \frac{1}{2} \left(\sum_m G_{am}^* R_{cbd}^m + \sum_m G_{cm}^* R_{adb}^m \right), \end{aligned} \quad (81)$$

and it is not difficult to get the following estimates for those tensors:

$$\begin{aligned} R &= F(\eta) (\eta'' + (\eta')^2 + (\eta')^2), \\ R^* &= F(\eta, t) ((t')^2 + (\eta')^2 + t'' + \eta'' t + \eta' t' + \eta t'). \end{aligned} \quad (82)$$

In what follows we shall also denote by ε an upper bound for the absolute values of the principal strains at all points of the shell. Let P_0 be a point on the undeformed middle surface M_0 and D the closest affine distance from P_0 to the lateral surface of the shell. Also, $2h$ represents the thickness of the shell and R is the typical length associated with the middle surface, all these quantities having been previously introduced above and in our article [25].

Then, we introduce the quantity

$$\theta = \max \left(\frac{h}{D}, \sqrt{\frac{h}{R}}, \sqrt{\varepsilon} \right) \quad (83)$$

and assume, besides, that the circumstances are such that

$$\theta < \theta_0, \quad (84)$$

where θ_0 is a constant which depends only on the choice of the strain energy density W .

We assume that all of the calculations shall be done for a system of affine normal coordinates, as indicated previously and also fully described in [24].

The middle surface of the shell is represented then in the form of Monge's, that is, as a graph function, where the origin of coordinates is located precisely at $P_0 \in M_0$, with the first two axes of coordinates chosen to lie at the tangent plane to M_0 at that point and with the third axis in the affine normal direction at the same point. The estimates to be computed for the partial derivatives of the function representing M_0 , in the system chosen, shall be used immediately to make the corresponding estimates for the successive covariant derivatives $t_{ik;r,s,\dots}$, so that the latter estimates shall be independent of the system originally used.

Thus, we define

$$\lambda = \frac{\theta_0 h}{\theta} = \theta_0 \min \left(D, \sqrt{Rh}, \frac{h}{\sqrt{\varepsilon}} \right) \quad (85)$$

and obtain the inequalities

$$h < \lambda < \sqrt{Rh}, \quad h < \frac{R}{4}, \quad \lambda < \frac{D}{2}, \quad (86)$$

which are easily seen to be satisfied if we assume, for example, that

$$\theta_0 < \frac{1}{2}. \quad (87)$$

From those we obtain that

$$\frac{1}{R} \leq (\theta_0)^2 \frac{h}{\lambda^2} \leq \theta_0 \theta \frac{1}{\lambda} \leq \frac{1}{2} \frac{\theta}{\lambda}, \quad \varepsilon < (\theta_0)^2 \frac{h^2}{\lambda^2} < \frac{h^2}{4\lambda^2}. \quad (88)$$

It is to be further assumed next that θ_0 is chosen so small that for the given strain energy function W all of the above formulae are valid in the region defined by

$$M = \{(u^1, u^2, u^3) : \sum (u^\alpha)^2 < \lambda^2, |u^3| < h\}. \quad (89)$$

Also, from now on we shall use the same symbols of approximation as described by John [22], represented by “ O ” and “ o ”; that is, the first symbol is used in the conventional, classical way except that dependence on W is allowed. Thus, the relation

$$A = O(B), \quad (90)$$

where $B \geq 0$, means that for a given strain energy function W there exists a positive number K such that

$$|A| \leq KB. \quad (91)$$

The second one shall be used in an unconventional sense and only in combination with the first. The relation

$$A = O(B) + o(C), \quad (92)$$

where $B \geq 0$ and $C \geq 0$, shall mean that for a given strain energy function W there exists a function $K(k)$, defined for all positive k such that

$$|A| \leq K(k)B + kC, \quad (93)$$

for all $k > 0$.

Besides, we shall assume that the strain energy function $W(s_1, s_2, s_3)$ is defined for all values of $|s_i|$ small enough and is as differentiable as needed. Here s_i are the traces of the successive powers of the strain operator. By definition, the “length” of such “strain operator,” (1,1)-tensor with components ε_i^m , is $|\varepsilon| := \sqrt{\sum_{i,m} \varepsilon_i^m \varepsilon_i^m}$. For the pseudometric tensor G sufficiently close to the unit matrix, that is, for $|\eta|$ sufficiently small, we can estimate $|\varepsilon|$ in terms of the eigenvalues of the matrix (ε_i^m) , that is, in terms of the so-called principal strains.

Then, there exists a positive ε_0 only depending on the choice of the strain energy function W such that the strain-stress relations hold for $|\varepsilon| < \varepsilon_0$, and it also follows that, for such values, $t_i^m = O(|\varepsilon|)$.

Hence, for a given function W we can also find bounds t_0 and η_0 such that for $|t| < t_0$ and $|\eta| < \eta_0$ all of the previously stated estimates are valid and, besides, $|\varepsilon| < \varepsilon_0$.

6. Estimates for the L_2 -Norms of Second Order Derivatives

In what follows, we shall use the following expression of the norm $\|w\|$ for any vector $w = w(u^1, u^2, u^3)$ defined in the working region M specified above:

$$\|w\| = \sqrt{\iiint_M |w| du^1 du^2 du^3}. \quad (94)$$

The symbol w' shall denote the gradient of w , that is, the vector whose components are the first derivatives of the components of w with respect to u^1, u^2 , and u^3 . We shall denote, besides, with w^* the “surface” coordinates gradient of w , that is, the vector of first derivatives with respect to u^1 and u^2 only.

It is well known that the components of the stress tensor t_{ik} satisfy the symmetry condition

$$t_{ik} = t_{ki}. \quad (95)$$

We can represent the estimates obtained from the *equations of equilibrium for the Euclidean case* (see [22, 23] for full details) by

$$\sum_m t_{im;m} = P_i = F(\eta, t) (tt' + \eta' t + \eta t') \quad (96)$$

and the estimates resulting from the *compatibility conditions* [24] by

$$\begin{aligned} \sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} \\ = Q_{hk} = F(\eta, t) \left(\eta t'' + (t')^2 + t\eta' t' + \eta' t' \right. \\ \left. + \eta'' t + (\eta')^2 t + (\eta')^2 t^2 + tt'' \right). \end{aligned} \quad (97)$$

We obtain correspondingly for the *affine theory of shells*:

$$\begin{aligned} \sum_r t_{hk;rr} + 2\mu \sum_r t_{rr;hk} \\ = F(\eta, t) \left(\begin{aligned} &(t')^2 + \eta' tt' + (\eta' t)^2 + (\eta')^2 + t'' + \eta'' t \\ &+ \eta' t' + \eta t + \eta'' + (\eta)^2 + \eta'' t^2 + (\eta t)^2 \\ &+ (\eta' t)^2 + \eta \eta'' t + (\eta)^3 t + \eta t (\eta')^2 \end{aligned} \right). \end{aligned} \quad (98)$$

In fact, by using the previous estimates one may write, in a general form,

$$\varepsilon_{ab;cd} + \varepsilon_{cd;ab} - \varepsilon_{ad;cb} - \varepsilon_{bc;ad} = \dots \text{ estimated terms,} \quad (99)$$

and, consequently,

$$\begin{aligned} t_{hk;rr} + \frac{1}{1+\nu} t_{rr;hk} \\ = \frac{\nu}{1+\nu} (t_{1,rr} \delta_h^k - t_{1,rk} \delta_r^h - t_{1,kr} \delta_k^r) \\ + t_{hr;rk} + t_{rk;hr} + \text{higher order terms,} \end{aligned} \quad (100)$$

since

$$\begin{aligned} t_{ij;kl} = t_{ij,kl} - (\Gamma_{ik}^h)_{,l} t_{hj} - \Gamma_{ik}^h t_{hj,l} - (\Gamma_{jk}^m)_{,l} t_{im} \\ - \Gamma_{jk}^m t_{im,l} - \Gamma_{il}^h (t_{hk,j} - \Gamma_{hj}^r t_{rk} - \Gamma_{kj}^s t_{sh}) \\ - \Gamma_{jl}^m (t_{mi,k} - \Gamma_{mk}^r t_{ri} - \Gamma_{ik}^s t_{sm}) \\ - \Gamma_{kl}^q (t_{qi,j} - \Gamma_{qi}^r t_{rj} - \Gamma_{ij}^s t_{sq}). \end{aligned} \quad (101)$$

Thus, by using all of the above expressions one may write, for the case of affine shells, estimates which resemble the ones obtained for the Euclidean case.

In fact, we develop estimates for the norms of the second derivatives of the (symmetric) tensor components t_{ik} , in terms of the above quantities P_i and Q_{hk} , introducing the auxiliary function $\phi : M \rightarrow \mathbb{R}$, defined by

$$\phi = (1 - \lambda^{-2} u^\alpha u_\alpha)^2. \quad (102)$$

One verifies easily the following properties:

$$\begin{aligned} 0 \leq \phi \leq 1, \quad \partial_3 \phi = 0, \\ |\phi^*| = \sqrt{\partial_\alpha \phi \partial_\alpha \phi} \leq \frac{4}{\lambda} \phi^{1/2}, \end{aligned} \quad (103)$$

$$|\phi^{**}| = \sqrt{\partial_{\alpha\beta} \phi \partial_{\alpha\beta} \phi} \leq \frac{4\sqrt{10}}{\lambda^2}.$$

Next, we consider the integral

$$A =: \sqrt{\iiint_M \phi^2 \partial_{i\alpha} t_{3k} \partial_{i\alpha} t_{3k} du^1 du^2 du^3}, \quad (104)$$

whereas, in a similar fashion to the Euclidean case, one obtains the estimates

$$\|\phi t^{**}\| = O(A + \|\phi P'\| + \|\phi Q\| + \lambda^{-2} \|t\|), \quad (105)$$

with

$$\begin{aligned} t^{**} = (\partial_{\alpha\beta} t_{ik}), \quad P' = (\partial_k P_i), \quad Q = (Q_{ik}), \\ \|\phi t''\| = O(\lambda h^{-1} (A + \|\phi P'\| + \|\phi Q\| + \lambda^{-2} \|t\|)). \end{aligned} \quad (106)$$

Similarly, one may estimate the above expression A :

$$\begin{aligned} A = O(\lambda^{-2} \|t\| + \|\phi Q\| + \lambda h^{-1} \|\phi P^*\|) \\ + o(\|\phi t^{**}\| + \lambda^{-1} h \|\phi t''\|). \end{aligned} \quad (107)$$

Then, it follows that

$$\begin{aligned} \|\phi t^{**}\| = O(\lambda^{-2} \|t\| + \|\phi Q\| + \|\phi^* t^*\|^2 + \lambda h^{-1} \|\phi P^*\|), \\ \|\phi t''\| \\ = O(\lambda h^{-1} (\lambda^{-2} \|t\| + \|\phi Q\| + \|\phi^* t^*\|^2 + \lambda h^{-1} \|\phi P^*\|)). \end{aligned} \quad (108)$$

Next, we make use of the particular form of the second and third members in the equations above, by writing

$$\begin{aligned} P_i = F(\eta, t) (tt' + \eta' t + \eta t'), \\ Q_{hk} = F(\eta, t) \left(\begin{aligned} &(t')^2 + \eta' tt' + (\eta' t)^2 + (\eta')^2 + t'' + \eta'' t \\ &+ \eta' t' + \eta t + \eta'' + (\eta)^2 + \eta'' t^2 + (\eta t)^2 \\ &+ (\eta' t)^2 + \eta \eta'' t + (\eta)^3 t + \eta t (\eta')^2 \end{aligned} \right), \end{aligned} \quad (109)$$

as well as on the respective (gradient) derivatives P' and P^* .

Thus, by taking into account these differences with the Euclidean case we find, by taking at the same time a sufficiently small value for the constant θ_0 , that

$$\begin{aligned}\|\phi t^{\bullet\bullet}\| &= O(\lambda^{-2}\|t\|^2), \\ \|\phi t''\| &= O(\lambda^{-1}h^{-1}\|t\|^2).\end{aligned}\quad (110)$$

7. Estimates for the L_2 -Norms of Higher Order Derivatives

From the previous development we observe that a similar approach can be made by considering again the differential equations involving the quantities P_i and Q_{hk} above, and by taking partial derivatives, that is, for a fixed value of γ , the $\partial_\gamma t_{ik}$ quantities satisfy the same equations if we substitute P_i and Q_{hk} , respectively, by $\partial_\gamma P_i$ and $\partial_\gamma Q_{hk}$. Thus, we obtain estimates for $\|\phi \partial_{\gamma\alpha\beta} t_{ik}\|$ and $\|\phi \partial_{\gamma rs} t_{ik}\|$. On the other hand, the remaining component $\|\phi \partial_{333} t_{ik}\|$ may also be estimated if one observes that all those of the form $\partial_{333} t_{ik}$ are expressible in terms of the quantities $\partial_{\gamma rs} t_{mn}$ and suitable derivatives of the expressions P_i and Q_{hk} . In such a fashion we are led to obtain estimates like the following ones:

$$\begin{aligned}\|\phi t^{\bullet\bullet\bullet}\| &= O(\lambda^{-2}\|t^{\bullet\bullet}\| + \|\phi Q'\| \\ &\quad + \|\phi P''\| + \lambda h^{-1}\|\phi P^{\bullet\bullet}\|),\end{aligned}\quad (111)$$

$$\begin{aligned}\|\phi t'''\| &= O(\lambda h^{-1}(\lambda^{-2}\|t^{\bullet\bullet}\| + \|\phi Q'\| + \|\phi P''\| \\ &\quad + \lambda h^{-1}\|\phi P^{\bullet\bullet}\|)).\end{aligned}\quad (112)$$

So far, in all of the previous development we kept the value of λ fixed regarding the defining region M , the function ϕ , and the norm $\|\omega\|$, all of those depending on λ . However, for estimating higher order terms we shall need to reduce further and further the region of work. Thus, we are led to rename all of those objects, for example, by calling M_λ instead of M , ϕ_λ in place of ϕ , and $\|\omega\|_\lambda$ for $\|\omega\|$. Moreover, we shall also proceed in our work by replacing λ with $\lambda/2$. Consequently, we replace too θ_0 by $\theta_0/2$; that is, in doing so we are restricted now to values θ such that $\theta < \theta_0/2$. Next, we observe that $1 = O(\phi_\lambda)$ in the region $M_{\lambda/2}$, so that

$$\|\omega\|_{\lambda/2} = O(\|\phi_\lambda \omega\|_\lambda). \quad (113)$$

It follows that

$$\begin{aligned}\|\phi_{\lambda/2} t^{\bullet\bullet\bullet}\|_{\lambda/2} &= O(\lambda^{-2}\|\phi_\lambda t^{\bullet\bullet}\|_\lambda + \|\phi_{\lambda/2} Q'\|_{\lambda/2} \\ &\quad + \|\phi_{\lambda/2} P''\|_{\lambda/2} + \lambda h^{-1}\|\phi_{\lambda/2} P^{\bullet\bullet}\|_{\lambda/2}),\end{aligned}\quad (114)$$

with the corresponding expression for $\|\phi_{\lambda/2} t'''\|_{\lambda/2}$ and also the corresponding changes for the (gradient) derivatives P'' and $P^{\bullet\bullet}$.

Then, a similar argument to the one in the Euclidean case allows us to conclude first that

$$\begin{aligned}\|\phi_{\lambda/2} t'''\|_{\lambda/2} &= O(\lambda^{-2}h^{-1}\|t\|_\lambda^2), \\ \|\phi_{\lambda/2} t^{\bullet\bullet\bullet}\|_{\lambda/2} &= O(\lambda^{-3}\|t\|_\lambda^2).\end{aligned}\quad (115)$$

This kind of process can be repeated over and over again, with the corresponding changes. For example, by applying the above relations to $t^{\bullet\bullet}$, in place of t , and, at the same time, replacing λ by $\lambda/4$, P by $P^{\bullet\bullet}$, and Q by $Q^{\bullet\bullet}$, we find that

$$\begin{aligned}\|\phi_{\lambda/4} t^{\bullet\bullet\bullet\bullet}\|_{\lambda/4} &= O(\lambda^{-2}\|\phi_{\lambda/2} t^{\bullet\bullet}\|_{\lambda/2} + \|\phi_{\lambda/4} Q''\|_{\lambda/4} \\ &\quad + \|\phi_{\lambda/4} P'''\|_{\lambda/4} + \lambda h^{-1}\|\phi_{\lambda/4} P^{\bullet\bullet\bullet}\|_{\lambda/4}),\end{aligned}\quad (116)$$

with a similar expression for the case of $\|\phi_{\lambda/4} t'''\|_{\lambda/4}$.

Therefore, the above development allows stating the following preliminary result, in the form of estimates for the corresponding partial derivatives of higher order.

Lemma 5. *With the above notations and conditions, there hold the estimates*

$$\begin{aligned}\partial_{k_1, k_2, i_1, i_2, \dots, i_n} t &= O(\varepsilon^2 \lambda^{1-n} h^{-1}), \\ \partial_{k_1, k_2, \alpha_1, \alpha_2, \dots, \alpha_n} t &= O(\varepsilon^2 \lambda^{-n}).\end{aligned}\quad (117)$$

Next and on the other hand, by using the estimate expressed in one of the above previous equations, namely, $\Gamma_{kr}^i = F(\eta)(\eta')$, and its successive derivatives, together with the other ones previously determined, that is,

$$\eta = O\left(\frac{h^2}{R^2}\right), \quad \eta' = O\left(\frac{1}{R^{5/2}}\right), \quad \eta'' = O\left(\frac{1}{R^{7/2}}\right), \dots, \quad (118)$$

we come to the conclusion that the same kind of estimates is valid for the corresponding covariant derivatives with respect to the Levi-Civita connection, the importance here being due to the fact that such kind of estimates is independent of the system of coordinates chosen. Thus, we state the main result of this section.

Theorem 6. *With the above notations and conditions, there holds the following estimate:*

$$t_{k_1, k_2, i_1, i_2, \dots, i_n} = O(\varepsilon^2 \lambda^{1-n} h^{-1}). \quad (119)$$

That is, in terms of a coordinate free expression,

$$\|\tilde{\nabla}^n t\| = O(\varepsilon^2 \lambda^{1-n} h^{-1}). \quad (120)$$

8. The Recovery of the Deformed Middle Surface

The main objective in the theory of shells, whether Euclidean or affine, is to determine the extent of deformation of the shell

once a solicitation is applied to it. And the best way of analyzing this is to see what happens to the corresponding middle surfaces. Thus, when the solicitation is produced on the shell in its original, undeformed state, it produces a three-dimensional stress tensor [22–39]:

$$t := \sum_{i,j} t_{ij} du^i du^j = \sum_{i,j} t^{ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \quad (121)$$

which is well known to be symmetric; that is,

$$t^{ij} = t^{ji}. \quad (122)$$

That first action produces, next, the deformation whose extent is contained as information in the corresponding strain tensor

$$\varepsilon_{ij} := \frac{1}{2} (G_{ij}^* - G_{ij}). \quad (123)$$

Let us recall that the relation between both of them is established through the stress-strain relation as described in [22, 23]:

$$t^{mk} := \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_{mk}^*}. \quad (124)$$

This, in terms of tensors with mixed components, may also be written as

$$t_i^m = \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_m^i}, \quad (125)$$

as already analyzed, for the affine case, in our previous articles [24, 25]. See also Section 5 above.

The earlier considerations, mainly the estimates exposed in the previous sections, together with the geometrical information available so far, shall be used in the recovery of the strained middle surface. We see, for example, from Section 2, that on the middle surface, characterized by the equation $u^3 := u = 0$, we have

$$G_{\alpha\beta}(u^1, u^2, 0) = g_{\alpha\beta}(u^1, u^2). \quad (126)$$

Hence and correspondingly, we may also write that

$$G_{\alpha\beta}^*(u^1, u^2, 0) := g_{\alpha\beta}^*(u^1, u^2). \quad (127)$$

The main geometrical information is being contained in the middle surfaces M_0 and M_0^* through the corresponding difference tensors $\varepsilon_{\alpha\beta} := (1/2)(g_{\alpha\beta}^* - g_{\alpha\beta})$, $\sigma_{\alpha\beta\gamma} := g_{\alpha\beta\gamma}^* - g_{\alpha\beta\gamma}$, and $w_{\alpha\lambda} := B_{\alpha\lambda}^* - B_{\alpha\lambda}$, defined on the parameter space and relating the coefficients of the first, second, and third fundamental forms, treated in [24] and also exposed above.

Thus, the known data are those corresponding to the undeformed middle surface M_0 , that is, $g_{\alpha\beta}$, $g_{\alpha\beta\gamma}$, and $B_{\alpha\lambda}$, and the problem is to determine the deformed ones, $g_{\alpha\beta}^*$, $g_{\alpha\beta\gamma}^*$, and $B_{\alpha\lambda}^*$, because these furnish a description of the deformed middle surface M_0^* , up to rigid motions in unimodular affine geometry. In case we want to obtain, that is, recover, the actual

deformed middle surface we can do so by applying one, or more, of the several existence fundamental theorems available, as treated, for example, in [7] and, with different notations, in [12]. In these two books the cases of general dimensions are considered, while the version for surfaces is originally due to Radon [14]. The corresponding method for the recovery of the deformed middle surface had been previously treated, in the Euclidean case, by John in papers [22, 23], based on the well-known fundamental theorem for (Euclidean) surfaces as developed, for example, in [40–42].

The affine pseudo-Riemannian structure induced on the deformed shell may be represented by writing

$$X^*(u_1, u_2, v) = X_0^*(u_1, u_2) + v N_{ua}^*. \quad (128)$$

So, we may express the affine objects components, $g_{\alpha\beta}^*$, $g_{\alpha\beta\gamma}^*$, and $B_{\alpha\beta}^*$, which are going to play a role comparable to the ones in the Euclidean case, $g_{\alpha\beta}^*$, $L_{\alpha\beta}^*$, and $M_{\alpha\beta}^*$, developed previously by John in [22, 23]. Here the treatment is made by assuming suitable integrability conditions exposed, for example, in [42] by Millman and Parker, where the fundamental theorem of existence and uniqueness (congruence) is stated, while complete proofs of the results can be found in [40] and also in [41]. On the other hand, the corresponding material for the affine case may be seen fully developed by Gigena, in [7], and, with a different kind of notation by Nomizu and Sasaki, in [12]. The expositions made in the latter two references are valid for dimension of the immersed hypersurface greater or equal than two, $n \geq 2$, while the original case of surfaces, that is, $n = 2$, was treated by Radon in [14].

If we consider that the shell deformation is the same, whatever the geometry is, but treated in a different way for each case, we have the following relations.

Euclidean Gauss equation

$$(X_0)_{\alpha\beta}^* = L_{\alpha\beta}^* N_{eu}^* + \Gamma_{\alpha\beta}^{*1} (X_0)_1^* + \Gamma_{\alpha\beta}^{*2} (X_0)_2^*, \quad (129)$$

where, in the middle surface ($u = 0$), we have

$$a_{\alpha\beta}^* = (X_0)_1^* \cdot (X_0)_2^* = X_1^*(u_1, u_2, 0) \cdot X_2^*(u_1, u_2, 0), \quad (130)$$

affine Gauss equation

$$(X_0)_{\alpha\beta}^* = g_{\alpha\beta}^* N_{ua}^* + \Gamma_{\alpha\beta}^{*1} (X_0)_1^* + \Gamma_{\alpha\beta}^{*2} (X_0)_2^*. \quad (131)$$

Notice that the meaning for the coefficients $\Gamma_{\alpha\beta}^{*\lambda}$ is different in one or the other of the above equations. However, we have preferred to use the same kind of symbol in order to avoid unnecessary complications in notation.

Thus, by the very definition of the corresponding second fundamental (cubic) form

$$\nabla(I_{ua}^*) = II_{ua}^* \quad (132)$$

we may write that

$$g_{\alpha\beta\gamma}^* = g_{\alpha\beta;\gamma}^* = \partial_\gamma g_{\alpha\beta}^* - \Gamma_{\alpha\gamma}^{*1} g_{1\beta}^* - \Gamma_{\beta\gamma}^{*2} g_{2\alpha}^*. \quad (133)$$

It follows that we may successively write, by always reducing the situation to the middle surface ($u = 0$) at the end of the process,

$$\begin{aligned}\partial_\beta G_{\alpha s}^* &:= \frac{\partial}{\partial u^\beta} G^*(X_\alpha^*, X_s^*) \\ &= G^*(X_{\alpha\beta}^*, X_s^*) + G^*(X_\alpha^*, X_{s\beta}^*)\end{aligned}\quad (134)$$

and, by rotating indexes, we also have

$$\begin{aligned}\partial_\alpha G_{\beta s}^* &= \partial_\alpha G^*(X_\beta^*, X_s^*) \\ &= G^*(X_{\beta\alpha}^*, X_s^*) + G^*(X_\beta^*, X_{s\alpha}^*), \\ \partial_s G_{\alpha\beta}^* &= \partial_s G^*(X_\alpha^*, X_\beta^*) \\ &= G^*(X_{\alpha s}^*, X_\beta^*) + G^*(X_\alpha^*, X_{\beta s}^*).\end{aligned}\quad (135)$$

Then, we readily get that

$$\frac{1}{2}(\partial_\beta G_{\alpha s}^* + \partial_\alpha G_{\beta s}^* - \partial_s G_{\alpha\beta}^*) = G^*(X_{\alpha\beta}^*, X_s^*). \quad (136)$$

Now, from the latter we get successively that

$$\begin{aligned}\frac{1}{2}G^{*3s}(\partial_\beta G_{\alpha s}^* + \partial_\alpha G_{\beta s}^* - \partial_s G_{\alpha\beta}^*) &= G^*(X_{\alpha\beta}^*, G^{*3s}X_s^*), \\ G^*(X_{\alpha\beta}^*, G^{*3s}X_s^*) &= G^{*33}g_{\alpha\beta}^*G^*(N_{ua}^*, X_3^*) \\ &\quad + \Gamma_{\alpha\beta}^{*1}(G^{*31}G_{11}^* + G^{*32}G_{21}^* + G^{*33}G_{31}^*) \\ &\quad + \Gamma_{\alpha\beta}^{*2}(G^{*21}G_{12}^* + G^{*22}G_{22}^* + G^{*23}G_{32}^*),\end{aligned}\quad (137)$$

where we further observe that the last two terms vanish since

$$\begin{aligned}G^{*3s}G_{s1}^* &= \delta_1^3 = 0, \\ G^{*3s}G_{s2}^* &= \delta_2^3 = 0.\end{aligned}\quad (138)$$

On the other hand we may also write similarly

$$\begin{aligned}G^*(X_{\alpha\beta}^*, G^{*2s}X_s^*) &= G^{*23}g_{\alpha\beta}^*G^*(N_{ua}^*, X_3^*) \\ &\quad + \Gamma_{\alpha\beta}^{*1}\underbrace{(G^{*21}G_{11}^* + G^{*22}G_{21}^* + G^{*23}G_{31}^*)}_{\delta_1^2=0} \\ &\quad + \Gamma_{\alpha\beta}^{*2}\underbrace{(G^{*21}G_{12}^* + G^{*22}G_{22}^* + G^{*23}G_{32}^*)}_{\delta_2^2=1}, \\ G^*(X_{\alpha\beta}^*, G^{*1s}X_s^*) &= G^{*13}g_{\alpha\beta}^*G^*(N_{ua}^*, X_3^*) \\ &\quad + \Gamma_{\alpha\beta}^{*1}\underbrace{(G^{*11}G_{11}^* + G^{*12}G_{21}^* + G^{*13}G_{31}^*)}_{\delta_1^1=1} \\ &\quad + \Gamma_{\alpha\beta}^{*2}\underbrace{(G^{*11}G_{12}^* + G^{*12}G_{22}^* + G^{*13}G_{32}^*)}_{\delta_2^1=0}.\end{aligned}\quad (139)$$

Now, in order to properly express the term $G^*(N_{ua}^*, X_3^*)$, appearing in all of these equations, we may proceed to use the very well-known Gram determinant identities:

$$\begin{aligned}[X_1^*, X_2^*, N_{ua}^*][X_1^*, X_2^*, X_3^*] \\ = \pm \det \begin{bmatrix} G_{11}^* & G_{12}^* & G_{13}^* \\ G_{12}^* & G_{22}^* & G_{23}^* \\ 0 & 0 & G^*(N_{ua}^*, X_3^*) \end{bmatrix},\end{aligned}\quad (140)$$

so that

$$[X_1^*, X_2^*, N_{ua}^*][X_1^*, X_2^*, X_3^*] = \pm G^*(N_{ua}^*, X_3^*) \det[G_{\alpha\beta}^*]. \quad (141)$$

Following the same reasoning, we also have

$$[X_1^*, X_2^*, X_3^*]^2 = \pm \det[G_{ij}^*], \quad (142)$$

$$[X_1^*, X_2^*, N_{ua}^*]^2 = \pm \det \begin{bmatrix} G_{11}^* & G_{12}^* & 0 \\ G_{12}^* & G_{22}^* & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm \det \begin{bmatrix} G_{11}^* & G_{12}^* \\ G_{12}^* & G_{22}^* \end{bmatrix}. \quad (143)$$

Thus, we obtain

$$G^*(N_{ua}^*, X_3^*) = \frac{[X_1^*, X_2^*, X_3^*]}{[X_1^*, X_2^*, N_{ua}^*]} = \sqrt{\frac{\det[G_{ij}^*]}{\det[G_{\alpha\beta}^*]}} = (G^{*33})^{-1/2}. \quad (144)$$

So far, from the above it follows that the coefficients of the affine Gauss equation may be expressed in terms of the components of the deformed pseudometric of the shell. In fact, see the expressions above containing the components $g_{\alpha\beta}^*$, $\Gamma_{\alpha\beta}^{*1}$, and $\Gamma_{\alpha\beta}^{*2}$.

It then follows that the rest of components of the first, second, and third affine fundamental forms may be expressed, too, in terms of the components of the deformed pseudometric shell G_{ij}^* , in every case reducing those terms to the middle surface ($u = 0$).

From the information recorded in this way, we can properly approximate the first, second, and third fundamental forms of affine geometry and, by the further use of the Fundamental Existence Theorem, recover the deformed middle surface. See [7, 12, 14], where the first two expositions are valid for the cases of the dimension of the immersed manifold $n \geq 2$, while the third one is valid for $n = 2$.

Theorem 7. *One may represent the deformed middle surface M_0^* as a real, 2-dimensional, oriented differentiable manifold, together with the following geometrical objects: a torsion-free affine connection ∇^* , a nondegenerate, pseudo-Riemannian structure, g^* , and a (1, 1)-tensor field, S^* , all of them defined in terms of the known, estimated coefficients of the deformed pseudometric exposed above, in Section 5:*

$$\begin{aligned}G_{ik}^* &= G_{ik} + \frac{2}{\mu}t_{ik} - 2\left(\frac{1-2\mu}{2\mu}\right)\left(\sum_j t_j^j\right)\delta_{ik} \\ &\quad + F(t, \eta)(t^2 + \eta t).\end{aligned}\quad (145)$$

In fact, those geometrical objects satisfy the following integrability conditions.

- (1) g^* is a Codazzi tensor field with respect to ∇^* , that is, in regard to the connection ∇^* , the covariant derivative of g^* is a symmetric $(0, 3)$ -tensor field, so that we can write

$$\psi^* := \nabla^* g^* = g^*_{\alpha\beta\gamma} du^\alpha du^\beta du^\gamma, \quad (146)$$

with the local components $g^*_{\alpha\beta\gamma}$ symmetric in all of their indices.

- (2) $\psi^* = \nabla^* g^*$ is apolar with respect to g^* ; namely, the Tschebyscheff covector field vanishes identically, or in local coordinates $\tau^* := \sum g^{*\alpha\beta} g^*_{\alpha\beta\gamma} du^\gamma \equiv 0$.
- (3) S^* is a $(1, 1)$ -Codazzi tensor field with respect to ∇^* ; that is, the equation following holds:

$$B^{*\alpha}_{\beta;\gamma} = B^{*\alpha}_{\gamma;\beta}. \quad (147)$$

- (4) The equation of Gauss holds, relating the given data, that is,

$$\begin{aligned} \bar{R}^{*\alpha}_{\beta\gamma\eta} = & \frac{1}{2} (B^{*\alpha}_{\eta} g^*_{\beta\gamma} - B^{*\alpha}_{\gamma} g^*_{\beta\eta} + B^{*\alpha}_{\beta\gamma} \delta^{\alpha}_{\eta} - B^{*\alpha}_{\beta\eta} \delta^{\alpha}_{\gamma}) \\ & - \sum (A^{*\sigma}_{\beta\eta} A^{*\alpha}_{\sigma\gamma} - A^{*\sigma}_{\beta\gamma} A^{*\alpha}_{\sigma\eta}), \end{aligned} \quad (148)$$

where $\bar{R}^{*\alpha}_{\beta\gamma\eta}$ represent the components of the curvature tensor associated with the Levi-Civita connection and $A^{*\alpha}_{\beta\gamma}$ are the components of the difference tensor, also equal to $(1/2)g^{*\alpha\rho} g^*_{\rho\beta\gamma}$.

Finally and consequently, it also follows that there exists a local unimodular affine immersion $X^* : (M^*, \nabla^*, g^*, S^*) \rightarrow \mathbb{R}^3$ for which ∇^* is the affine normal connection, g^* is the first fundamental form I^*_{ua} , and S^* is Weingarten's affine shape operator.

Proof. The integrability conditions stated are a direct consequence of the corresponding compatibility conditions, as exposed in Theorem 4 above, and the rest of the proof, in a far more general setting and for general dimension $n \geq 2$, may be found in [7] and also in [12], the latter with a different kind of notation. The original case of dimension $n = 2$ may be found in [14].

Moreover, for dimension $n = 2$, we may further reduce the proof to a two-step procedure, whereas we consider first the expression involving the affine Gauss equation, together with the expression of the local derivatives of the affine normal vector field:

$$\begin{aligned} (X_0)^*_{\alpha\beta} &= g^*_{\alpha\beta} N^*_{ua} + \Gamma^{*1}_{\alpha\beta} (X_0)^*_1 + \Gamma^{*2}_{\alpha\beta} (X_0)^*_2, \\ (N^*_{ua})_\alpha &= - \sum_{\beta} B^{*\beta}_{\alpha} (X_0)^*_\beta. \end{aligned} \quad (149)$$

This is a system of five partial differential equations, when considering that it involves the vector fields $(X_0)^*_{11}$,

$(X_0)^*_{12}$, $(X_0)^*_{22}$, $(N^*_{ua})_1$, and $(N^*_{ua})_2$. Correspondingly, we have a system of fifteen scalar partial differential equations, when expressing each one of the latter in terms of the ambient space coordinates. The fact that the system is integrable, and allows recovering firstly the vector fields $(X_0)^*_1$, $(X_0)^*_2$, and N^*_{ua} follows precisely from the integrability conditions exposed, as stated before and, for example, in [7] and also in [12], the latter with different notation.

Thus, with this first step of the procedure we obtain those three vector fields in terms of the components of the deformed pseudometric shell G^*_{ij} , having always in mind their approximate values and also by considering the reduction of those same values to the deformed middle surface ($u = 0$).

In the next, second step of the procedure, we use those expressions for $(X_0)^*_1$ and $(X_0)^*_2$ in order to recover the vector field representing the deformed middle surface $(X_0)^*$. Here the integrability conditions are obviously satisfied since $(X_0)^*_{12} = (X_0)^*_{21}$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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