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Research Article

Algorithm for Solving a New System of Generalized Nonlinear Quasi-Variational-Like Inclusions in Hilbert Spaces

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We introduce and study a new system of generalized nonlinear quasi-variational-like inclusions with $H(\cdot,\cdot)$-cocoercive operator in Hilbert spaces. We suggest and analyze a class of iterative algorithms for solving the system of generalized nonlinear quasi-variational-like inclusions. An existence theorem of solutions for the system of generalized nonlinear quasi-variational-like inclusions is proved under suitable assumptions which show that the approximate solutions obtained by proposed algorithms converge to the exact solutions.

1. Introduction

Variational inclusion problems are important generalization of classical variational inequalities and have wide applications to many fields including mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences; see, for example, [1]. For these reasons, various variational inclusions have been intensively studied in recent years. Many efficient ways have been studied to find solutions for variational inclusions. Those methods include the projection method and its various forms, linear approximation, descent and Newton's method, and the method based on auxiliary principle technique. The method based on the resolvent operator technique is a generalization of the projection method and has been widely used to solve variational inclusions. For details, we refer to see [2–19].

Recently, Fang and Huang, Kazmi and Khan, and Lan et al. investigated several resolvent operators for generalized operators such as H-monotone [3, 17], H-accretive [4], A-maximal relaxed accretive [14], (H,η)-monotone [5], (A,η)-accretive [13], (P,η)-proximal point [8], and (P,η)-accretive [9] operators. Very recently, Zou and Huang [19] introduced and studied $H(\cdot,\cdot)$-accretive operators, Kazmi et al. [10–12] introduced and studied generalized $H(\cdot,\cdot)$-accretive operators and $H(\cdot,\cdot)$-η-proximal point mapping, and Xu and Wang [18] introduced and studied $(H(\cdot,\cdot),\eta)$-monotone operators. Ahmad et al. [2, 8] introduced and studied $H(\cdot,\cdot)$-cocoercive operators, showed some properties of the resolvent operator for the $H(\cdot,\cdot)$-cocoercive operators, and obtained an application for solving variational inclusions in Hilbert spaces. They also gave some examples to illustrate their results.

Inspired and motivated by the researches going on in this area, we introduce and discuss a new system of generalized nonlinear quasi-variational-like inclusions involving $H(\cdot,\cdot)$-cocoercive operators in Hilbert spaces. By using the resolvent operators associated with $H(\cdot,\cdot)$-cocoercive operators due to Ahmad et al. [2], we prove that the approximate solutions obtained by the iterative algorithms converge to the exact solutions of our system of generalized nonlinear quasi-variational-like inclusions. Our results can be viewed as an extension and generalization of some known results in the literature.

2. Preliminaries

Throughout this paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$, 2^X (resp., $CB(X)$) is

the family of all the nonempty (resp., closed and bounded) subsets of X and $\mathscr{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathscr{D}(P,Q) = \max\left\{\sup_{x\in P} d(x,Q), \sup_{y\in P} d(P,y)\right\}, \quad \forall P,Q \in CB(X), \tag{1}$$

where $d(x,Q) = \inf_{y\in Q}\|x-y\|$ and $d(P,y) = \inf_{x\in P}\|x-y\|$.

In the sequel, let us recall some concepts.

Definition 1 (see [20]). A mapping $g : X \to X$ is said to be

(i) monotone if

$$\langle g(x) - g(y), x-y\rangle \geq 0, \quad \forall x,y \in X; \tag{2}$$

(ii) ξ-strongly monotone if there exists a constant $\xi > 0$ such that

$$\langle g(x) - g(y), x-y\rangle \geq \xi\|x-y\|^2, \quad \forall x,y \in X; \tag{3}$$

(iii) μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle g(x) - g(y), x-y\rangle \geq \mu\|g(x)-g(y)\|^2, \quad \forall x,y \in X; \tag{4}$$

(iv) γ-relaxed cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), x-y\rangle \geq (-\gamma)\|g(x)-g(y)\|^2, \quad \forall x,y \in X; \tag{5}$$

(v) λ_g-Lipschitz continuous if there exists a constant $\lambda_g > 0$ such that

$$\|g(x)-g(y)\| \leq \lambda_g\|x-y\|, \quad \forall x,y \in X; \tag{6}$$

(vi) α-expansive if there exists a constant $\alpha > 0$ such that

$$\|g(x)-g(y)\| \geq \alpha\|x-y\|, \quad \forall x,y \in X; \tag{7}$$

if $\alpha = 1$, then it is expansive.

Definition 2. Let $M : X \to 2^X$ be set-valued mapping. Then, M is said to be μ'-cocoercive if there exists a constant $\mu' > 0$ such that

$$\langle u-v, \eta(x,y)\rangle \geq \mu'\|u-v\|^2, \quad \forall x,y \in X,\ u \in M(x),\ v \in M(y). \tag{8}$$

Definition 3 (see [2]). Let $H : X\times X \to X$ and $A, B : X \to X$ be the single-valued mappings. Then,

(i) $H(A,\cdot)$ is said to be μ-cocoercive with respect to A if there exists a constant $\mu > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), x-y\rangle \geq \mu\|Ax-Ay\|^2, \quad \forall x,y \in X; \tag{9}$$

(ii) $H(\cdot,B)$ is said to be γ-relaxed cocoercive with respect to B if there exists a constant $\gamma > 0$ such that

$$\langle H(u,Bx) - H(u,By), x-y\rangle \geq (-\gamma)\|Bx-By\|^2, \quad \forall x,y \in X; \tag{10}$$

(iii) $H(A,\cdot)$ is said to be r_1-Lipschitz continuous with respect to A if there exists a constant $r_1 > 0$ such that

$$\|H(Ax,\cdot) - H(Ay,\cdot)\| \leq r_1\|x-y\|, \quad \forall x,y \in X; \tag{11}$$

(iv) $H(\cdot,B)$ is said to be r_2-Lipschitz continuous with respect to B if there exists a constant $r_2 > 0$ such that

$$\|H(\cdot,Bx) - H(\cdot,By)\| \leq r_2\|x-y\|, \quad \forall x,y \in X. \tag{12}$$

Definition 4 (see [2]). Let $H : X\times X \to X$ and $A, B : X \to X$ be the single-valued mappings. Then, the set-valued mapping $M : X \to 2^X$ is said to be $H(\cdot,\cdot)$-cocoercive with respect to A and B (or simply $H(\cdot,\cdot)$-cocoercive in the sequel), if

(i) M is cocoercive;

(ii) $(H(A,B) + \lambda M)(X) = X$, for every $\lambda > 0$.

Example 5. Let $X = \mathbb{R}^2$ with the usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = (2x_1 - 2x_2, -2x_1 + 4x_2), \quad By = (-y_1 + y_2, -y_2), \quad \forall x,y \in \mathbb{R}^2. \tag{13}$$

Suppose that $H(A,B) : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$H(Ax,By) = Ax + By, \quad \forall x,y \in \mathbb{R}^2. \tag{14}$$

Then, it is easy to check that $H(A,B)$ is 1/6-cocoercive with respect to A and 1/2-relaxed cocoercive with respect to B.

Let $M = I$, where I is the identity mapping. Then, M is $H(\cdot,\cdot)$-cocoercive mapping with respect to A and B.

Example 6. Let $X = \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the space of all 2×2 real symmetric matrices. Let $H(Ax,By) = x^2 - y$, for all $x,y \in \mathbb{S}^2$ and $M = I$. Then, for $\lambda = 1$, we have

$$(H(A,B) + M)(x) = x^2 - x + x = x^2, \tag{15}$$

but

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \notin (H(A,B) + M)(\mathbb{S}^2), \tag{16}$$

because $\left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is not the square of any 2×2 real symmetric matrix. Hence, M is not $H(\cdot,\cdot)$-cocoercive with respect to A and B.

Proposition 7 (see [2]). *Let $H(A,B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B; A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be an $H(\cdot,\cdot)$-cocoercive operator. If the inequality*

$$\langle u-v, x-y\rangle \geq 0 \tag{17}$$

holds for all $(y, v) \in Graph(M)$, *then* $u \in Mx$, *where*

$$Graph(M) = \{(u, x) \in X \times X : x \in M(u)\}. \quad (18)$$

Theorem 8 (see [2]). *Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B; A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then, the operator $(H(A, B) + \lambda M)^{-1}$ is single-valued.*

Definition 9 (see [2]). Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B; A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. The resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)} : X \to X$ is defined by

$$R_{\lambda,M}^{H(\cdot,\cdot)}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X. \quad (19)$$

Theorem 10 (see [2]). *Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^2 - \gamma\beta^2$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then, the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)} : X \to X$ is $1/(\mu\alpha^2 - \gamma\beta^2)$-Lipschitz continuous; that is,*

$$\left\| R_{\lambda,M}^{H(\cdot,\cdot)}(u) - R_{\lambda,M}^{H(\cdot,\cdot)}(v) \right\| \le \frac{1}{\mu\alpha^2 - \gamma\beta^2} \|u - v\|, \quad \forall u, v \in X. \quad (20)$$

3. The System of Generalized Nonlinear Quasi-Variational-Like Inclusions and Iterative Algorithm

Let X_1 and X_2 be real Hilbert spaces. Let $P : X_1 \times X_2 \to X_1$, $Q : X_1 \times X_2 \to X_2$, $g, A, B : X_1 \to X_1$, $h, C, D : X_2 \to X_2$, $G : X_1 \times X_1 \to X_1$, and $H : X_2 \times X_2 \to X_2$ be the single-valued mappings. Let $S : X_1 \to CB(X_1)$ and $T : X_2 \to CB(X_2)$ be the set-valued mappings, let $M : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each $a \in X_1$, $M(\cdot, a)$ is a $G(\cdot,\cdot)$-cocoercive operator with respect to A and B, and let $N : X_2 \times X_2 \to 2^{X_2}$ be a set-valued mapping such that, for each $b \in X_2$, $N(\cdot, b)$ is an $H(\cdot,\cdot)$-cocoercive operator with respect to C and D. Assume that $g(X_1) \cap \mathrm{dom}(M(\cdot, a)) \neq \emptyset$ for each $a \in X_1$, and $h(X_2) \cap \mathrm{dom}(N(\cdot, b)) \neq \emptyset$ for each $b \in X_2$. Then, we consider the following system of generalized nonlinear quasi-variational-like inclusions.

Find $(a, b) \in X_1 \times X_2$ with $p \in S(a)$, $q \in T(b)$ such that

$$\begin{aligned} 0 &\in P(p, q) + M(g(a), a), \\ 0 &\in Q(p, q) + N(h(b), b). \end{aligned} \quad (21)$$

Next are some special cases of problem (21).

(1) Let S, T be identity mappings, for each $(a, b) \in X_1 \times X_2$, $M(g(a), a) = M(x)$ and $N(h(b), b) = N(x)$; then problem (21) reduces to the following problem considered in [15]:

$$\begin{aligned} 0 &\in P(p, q) + M(a), \\ 0 &\in Q(p, q) + N(b). \end{aligned} \quad (22)$$

(2) $X_1 = X_2$, $a = b$, $S = T$ is an identity mapping, $g = h$, $P(p, q) = Q(p, q) = P(\cdot)$, for each $(a, b) \in X_1 \times X_2$, $M(g(a), a) = N(h(b), b) = M(g(x))$; then problem (21) reduces to the following problem considered in [16]:

$$0 \in P(x) + M(g(a)). \quad (23)$$

For a suitable choice of the mappings g, h, P, Q, G, H, A, B, C, D, S, T and the space $X_1 = X_2$, a number of known systems of quasi-variational inequalities, systems of variational inequalities, systems of quasi-variational inclusions, and variational inclusions can be obtained as special cases of the generalized nonlinear quasi-variational inclusion problem (21). We would like to mention that the problem of finding zero of the sum of two maximal monotone operators is also a special case of problem (21). Furthermore, these types of variational inclusion enable us to study many important problems arising in mathematical, physical, and engineering science in a general and unified framework.

Lemma 11. *Let X_1 and X_2 be real Hilbert spaces. Let $P : X_1 \times X_2 \to X_1$, $Q : X_1 \times X_2 \to X_2$, $g, A, B : X_1 \to X_1$, $h, C, D : X_2 \to X_2$ be the single-valued mappings. Let $G : X_1 \times X_1 \to X_1$ be a single-valued mapping such that $G(A, B)$ is μ_1-cocoercive with respect to A and γ_1-relaxed cocoercive with respect to B; A is α_1-expansive, B is β_1-Lipschitz continuous, and $\mu_1 > \gamma_1$, $\alpha_1 > \beta_1$. Let $H : X_2 \times X_2 \to X_2$ be a single-valued mapping such that $H(C, D)$ is μ_2-cocoercive with respect to C and γ_2-relaxed cocoercive with respect to D; C is α_2-expansive, D is β_2-Lipschitz continuous, and $\mu_2 > \gamma_2$, $\alpha_2 > \beta_2$. Let $S : X_1 \to CB(X_1)$ and $T : X_2 \to CB(X_2)$ be the set-valued mappings, let $M : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each $a \in X_1$, $M(\cdot, a$ is a $G(\cdot,\cdot)$-cocoercive operator, and let $N : X_2 \times X_2 \to 2^{X_2}$ be a set-valued mapping such that, for each $b \in X_2$, $N(\cdot, b)$ is an $H(\cdot,\cdot)$-cocoercive operator. Assume that $g(X_1) \cap \mathrm{dom}(M(\cdot, a)) \neq \emptyset$ for each $a \in X_1$, and $h(X_2) \cap \mathrm{dom}(N(\cdot, b)) \neq \emptyset$ for each $b \in X_2$. Then, for any $(a, b) \in X_1 \times X_2$ with $p \in S(a)$, $q \in T(b)$ is a solution of the problem (21), if and only if*

$$\begin{aligned} g(a) &= R_{\lambda_1,M(\cdot,a)}^{G(\cdot,\cdot)} \left[G(A(g(a)), B(g(a))) - \lambda_1 P(p, q) \right], \\ h(b) &= R_{\lambda_2,N(\cdot,b)}^{H(\cdot,\cdot)} \left[H(C(h(b)), D(h(b))) - \lambda_2 Q(p, q) \right], \end{aligned} \quad (24)$$

where $R_{\lambda_1,M(\cdot,a)}^{G(\cdot,\cdot)} = (G(\cdot,\cdot) + \lambda_1 M(\cdot, a))^{-1}$, $R_{\lambda_2,N(\cdot,b)}^{H(\cdot,\cdot)} = (H(\cdot,\cdot) + \lambda_2 N(\cdot, b))^{-1}$, and $\lambda_1, \lambda_2 > 0$ are constants.

Proof. By using the definitions of the resolvent operators $R_{\lambda_1,M(\cdot,a)}^{G(\cdot,\cdot)}$ and $R_{\lambda_2,N(\cdot,b)}^{H(\cdot,\cdot)}$, the conclusion follows directly. □

The preceding lemma allows us to suggest the following iterative algorithm for problem (21).

Algorithm 12. For $(a_0, b_0) \in X_1 \times X_2$ with $p \in S(a), q \in T(b)$, compute the sequences $\{a_n\}$, $\{b_n\}$, $\{p_n\}$, $\{q_n\}$ as follows:

$$\begin{aligned} g(a_{n+1}) &= R^{G(\cdot,\cdot)}_{\lambda_1, M(\cdot,a)} [G(A(g(a_n)), B(g(a_n))) \\ &\qquad - \lambda_1 P(p_n, q_n)], \quad \lambda_1 > 0, \\ h(b_{n+1}) &= R^{H(\cdot,\cdot)}_{\lambda_2, N(\cdot,b)} [H(C(h(b_n)), D(h(b_n))) \\ &\qquad - \lambda_2 Q(p_n, q_n)], \quad \lambda_2 > 0, \\ p_n &\in S(a_n), \quad \|p_n - p_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) \\ &\qquad \times \mathfrak{D}(S(a_n), S(a_{n+1})), \\ q_n &\in T(b_n), \quad \|q_n - q_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) \\ &\qquad \times \mathfrak{D}(T(b_n), T(b_{n+1})), \end{aligned} \tag{25}$$

for all $n = 0, 1, 2, \ldots$, and λ_1 and λ_2 are constants.

Definition 13. Let $S : X_1 \to CB(X_1)$ and $T : X_2 \to CB(X_2)$ be two set-valued mappings. A single-valued mapping $P : X_1 \times X_2 \to X_1$ is said to be

(i) ϵ_1-Lipschitz continuous in the first argument with respect to S, if there exists a constant $\epsilon_1 > 0$ such that

$$\begin{aligned} &\|P(p_1, \cdot) - P(p_2, \cdot)\| \le \epsilon_1 \|p_1 - p_2\|, \\ &\forall p_1 \in S(a_1),\ p_2 \in S(a_2),\ a_1, a_2 \in X_1; \end{aligned} \tag{26}$$

(ii) ϵ_2-Lipschitz continuous in the second argument with respect to T, if there exists a constant $\epsilon_2 > 0$ such that

$$\begin{aligned} &\|P(\cdot, q_1) - P(\cdot, q_2)\| \le \epsilon_2 \|q_1 - q_2\|, \\ &\forall q_1 \in T(b_1),\ q_2 \in T(b_2),\ b_1, b_2 \in X_2. \end{aligned} \tag{27}$$

Definition 14. Let $S : X_1 \to CB(X_1)$ and $T : X_2 \to CB(X_2)$ be two set-valued mappings. A single-valued mapping $Q : X_1 \times X_2 \to X_2$ is said to be

(i) δ_1-Lipschitz continuous in the first argument with respect to S, if there exists a constant $\delta_1 > 0$ such that

$$\begin{aligned} &\|Q(p_1, \cdot) - Q(p_2, \cdot)\| \le \delta_1 \|p_1 - p_2\|, \\ &\forall p_1 \in S(a_1),\ p_2 \in S(a_2),\ a_1, a_2 \in X_1; \end{aligned} \tag{28}$$

(ii) δ_2-Lipschitz continuous in the second argument with respect to T, if there exists a constant $\delta_2 > 0$ such that

$$\begin{aligned} &\|Q(\cdot, q_1) - Q(\cdot, q_2)\| \le \delta_2 \|q_1 - q_2\|, \\ &\forall q_1 \in T(b_1), q_2 \in T(b_2),\ b_1, b_2 \in X_2. \end{aligned} \tag{29}$$

Definition 15. A set-valued mapping $A : X \to CB(X)$ is said to be $\mathfrak{D}$-Lipschitz continuous, if there exists a constant $\rho > 0$ such that

$$\mathfrak{D}(A(x), A(y)) \le \rho \|x - y\|, \quad \forall x, y \in X. \tag{30}$$

Theorem 16. *Let X_1 and X_2 be real Hilbert spaces. Let $P : X_1 \times X_2 \to X_1$, $Q : X_1 \times X_2 \to X_2$, g, A, $B : X_1 \to X_1$, $h, C, D : X_2 \to X_2$, $G : X_1 \times X_1 \to X_1$, and $H : X_2 \times X_2 \to X_2$ be the single-valued mappings. Let $S : X_1 \to CB(X_1)$ and $T : X_2 \to CB(X_2)$ be the set-valued mappings, let $M : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each $a \in X_1$, $M(\cdot, a)$ is a $G(\cdot,\cdot)$-cocoercive operator with respect to A and B, and let $N : X_2 \times X_2 \to 2^{X_2}$ be a set-valued mapping such that for each $b \in X_2$, $N(\cdot, b)$ is an $H(\cdot,\cdot)$-cocoercive operator with respect to C and D. Assume that $g(X_1) \cap \mathrm{dom}(M(\cdot, a)) \neq \emptyset$ for each $a \in X_1$, and $h(X_2) \cap \mathrm{dom}(N(\cdot, b)) \neq \emptyset$ for each $b \in X_2$. Assume that*

(i) *S, T are ρ, τ-Lipschitz continuous in the Hausdorff metric $\mathfrak{D}(\cdot,\cdot)$, respectively;*

(ii) *$G(A, B)$ is μ_1-cocoercive with respect to A and γ_1-relaxed cocoercive with respect to B;*

(iii) *$H(C, D)$ is μ_2-cocoercive with respect to C and γ_2-relaxed cocoercive with respect to D;*

(iv) *A, C is α_1, α_2-expansive, respectively;*

(v) *B, D is β_1, β_2-Lipschitz continuous, respectively;*

(vi) *g is λ_g-Lipschitz continuous and ξ_1-strongly monotone;*

(vii) *h is λ_h-Lipschitz continuous and ξ_2-strongly monotone;*

(viii) *$G(A, B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B;*

(ix) *$H(C, D)$ is s_1-Lipschitz continuous with respect to C and s_2-Lipschitz continuous with respect to D;*

(x) *$P(\cdot,\cdot)$ is ϵ_1-Lipschitz continuous in the first argument and ϵ_2-Lipschitz continuous in the second argument;*

(xi) *$Q(\cdot,\cdot)$ is δ_1-Lipschitz continuous in the first argument and δ_2-Lipschitz continuous in the second argument;*

(xii)

$$\begin{aligned} 0 &< \frac{(r_1 + r_2)\lambda_g + \lambda_1 \epsilon_1 \rho}{\xi_1(\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2)} + \frac{\lambda_2 \delta_1 (1 + 1/n)\rho}{\xi_2(\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2)} < 1, \\ 0 &< \frac{(s_1 + s_2)\lambda_h + \lambda_2 \delta_2 \tau}{\xi_2(\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2)} + \frac{\lambda_1 \epsilon_2 (1 + 1/n)\tau}{\xi_1(\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2)} < 1. \end{aligned} \tag{31}$$

Then, the iterative sequences $\{a_n\}$, $\{b_n\}$, $\{p_n\}$, and $\{q_n\}$, generated by Algorithm 12, converge strongly to a, b, p, and q, respectively, and (a, b, p, q) is a solution of problem (21).

Proof. Since S, T are Lipschitz continuous with constants ρ, τ, respectively, it follows from Algorithm 12 that

$$\begin{aligned}\|p_n - p_{n+1}\| &\le \left(1+\frac{1}{n}\right)\mathfrak{D}\left(S\left(a_n\right), S\left(a_{n+1}\right)\right)\\ &\le \left(1+\frac{1}{n}\right)\rho\left\|a_n - a_{n+1}\right\|,\\ \|q_n - q_{n+1}\| &\le \left(1+\frac{1}{n}\right)\mathfrak{D}\left(T\left(b_n\right), T\left(b_{n+1}\right)\right)\\ &\le \left(1+\frac{1}{n}\right)\tau\left\|b_n - b_{n+1}\right\|,\end{aligned} \tag{32}$$

for all $n = 0, 1, 2, \ldots$.

Using the ξ-strong monotonicity of g, we have

$$\begin{aligned}&\|g\left(a_{n+1}\right) - g\left(a_n\right)\|\left\|a_{n+1} - a_n\right\|\\ &\quad\ge \left\langle g\left(a_{n+1}\right) - g\left(a_n\right), a_{n+1} - a_n\right\rangle\\ &\quad\ge \xi_1\left\|a_{n+1} - a_n\right\|^2,\end{aligned} \tag{33}$$

which implies that

$$\left\|a_{n+1} - a_n\right\| \le \frac{1}{\xi_1}\left\|g\left(a_{n+1}\right) - g\left(a_n\right)\right\|. \tag{34}$$

Now we estimate $\|g(a_{n+1})-g(a_n)\|$ by using Algorithm 12 and the Lipschitz continuity of $R^{G(\cdot,\cdot)}_{\lambda_1,M(\cdot,a)}$ as follows:

$$\begin{aligned}&\|g\left(a_{n+1}\right) - g\left(a_n\right)\|\\ &\quad= \left\|R^{G(\cdot,\cdot)}_{\lambda_1,M(\cdot,a)}\left[G\left(A\left(g\left(a_n\right)\right), B\left(g\left(a_n\right)\right)\right) - \lambda_1 P\left(p_n, q_n\right)\right]\right.\\ &\qquad\left.- \left[R^{G(\cdot,\cdot)}_{\lambda_1,M(\cdot,a)}\left\{G\left(A\left(g\left(a_{n-1}\right)\right), B\left(g\left(a_{n-1}\right)\right)\right)\right.\right.\right.\\ &\qquad\qquad\left.\left.\left.- \lambda_1 P\left(p_{n-1}, q_{n-1}\right)\right\}\right]\right\|\\ &\quad\le \frac{1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|G\left(A\left(g\left(a_n\right)\right), B\left(g\left(a_n\right)\right)\right)\right.\\ &\qquad\qquad\left.- G\left(A\left(g\left(a_{n-1}\right)\right), B\left(g\left(a_{n-1}\right)\right)\right)\right\|\\ &\qquad+ \frac{\lambda_1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|P\left(p_n, q_n\right) - P\left(p_{n-1}, q_{n-1}\right)\right\|\\ &\quad\le \frac{1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|G\left(A\left(g\left(a_n\right)\right), B\left(g\left(a_n\right)\right)\right)\right.\\ &\qquad\qquad\left.- G\left(A\left(g\left(a_{n-1}\right)\right), B\left(g\left(a_n\right)\right)\right)\right\|\\ &\qquad+ \frac{1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|G\left(A\left(g\left(a_{n-1}\right)\right), B\left(g\left(a_n\right)\right)\right)\right.\\ &\qquad\qquad\left.- G\left(A\left(g\left(a_{n-1}\right)\right), B\left(g\left(a_{n-1}\right)\right)\right)\right\|\\ &\qquad+ \frac{\lambda_1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|P\left(p_n, q_n\right) - P\left(p_{n-1}, q_n\right)\right\|\\ &\qquad+ \frac{\lambda_1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|P\left(p_{n-1}, q_n\right) - P\left(p_{n-1}, q_{n-1}\right)\right\|.\end{aligned} \tag{35}$$

Since $G(A, B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, P is ϵ_1-Lipschitz continuous in the first argument and ϵ_2-Lipschitz continuous in the second argument. g is λ_g-Lipschitz continuous and using (32), (34), and (35) it becomes

$$\begin{aligned}\|g\left(a_{n+1}\right) - g\left(a_n\right)\| &\le \frac{\left(r_1 + r_2\right)\lambda_g}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|a_n - a_{n-1}\right\|\\ &\quad+ \frac{\lambda_1\epsilon_1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|p_n - p_{n-1}\right\|\\ &\quad+ \frac{\lambda_1\epsilon_2}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|q_n - q_{n-1}\right\|\\ &\le \frac{\left(r_1 + r_2\right)\lambda_g}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left\|a_n - a_{n-1}\right\|\\ &\quad+ \frac{\lambda_1\epsilon_1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left(1+\frac{1}{n}\right)\rho\left\|a_n - a_{n-1}\right\|\\ &\quad+ \frac{\lambda_1\epsilon_2}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\left(1+\frac{1}{n}\right)\tau\left\|b_n - b_{n-1}\right\|,\\ \left\|a_{n+1} - a_n\right\| &\le \frac{\left(r_1 + r_2\right)\lambda_g + \lambda_1\epsilon_1\left(1+1/n\right)\rho}{\xi_1\left(\mu_1\alpha_1^2 - \gamma_1\beta_1^2\right)}\left\|a_n - a_{n-1}\right\|\\ &\quad+ \frac{\lambda_1\epsilon_2\left(1+1/n\right)\tau}{\xi_1\left(\mu_1\alpha_1^2 - \gamma_1\beta_1^2\right)}\left\|b_n - b_{n-1}\right\|.\end{aligned} \tag{36}$$

Let

$$\left\|a_{n+1} - a_n\right\| \le \theta_{1n}\left\|a_n - a_{n-1}\right\| + \theta_{2n}\left\|b_n - b_{n-1}\right\|, \tag{37}$$

where

$$\begin{aligned}\theta_{1n} &= \frac{\left(r_1 + r_2\right)\lambda_g + \lambda_1\epsilon_1\left(1+1/n\right)\rho}{\xi_1\left(\mu_1\alpha_1^2 - \gamma_1\beta_1^2\right)},\\ \theta_{2n} &= \frac{\lambda_1\epsilon_2\left(1+1/n\right)\tau}{\xi_1\left(\mu_1\alpha_1^2 - \gamma_1\beta_1^2\right)}.\end{aligned} \tag{38}$$

Now we estimate $\|h(b_n) - h(b_{n-1})\|$ by using Algorithm 12 and the Lipschitz continuity of $R^{H(\cdot,\cdot)}_{\lambda_2,N(\cdot,b)}$ as follows:

$$\begin{aligned}&\|h\left(b_{n+1}\right) - h\left(b_n\right)\|\\ &\quad= \left\|R^{H(\cdot,\cdot)}_{\lambda_2,N(\cdot,b)}\left[H\left(C\left(h\left(b_n\right)\right), D\left(h\left(b_n\right)\right)\right) - \lambda_2 Q\left(p_n, q_n\right)\right]\right.\end{aligned}$$

$$
\begin{aligned}
&- R^{H(\cdot,\cdot)}_{\lambda_2,N(\cdot,b)}\left[H\left(C\left(h\left(b_{n-1}\right)\right),D\left(h\left(b_{n-1}\right)\right)\right)\right.\\
&\qquad\left.-\lambda_2 Q\left(p_{n-1},q_{n-1}\right)\right]\Big\|\\
&\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\Big\|H\left(C\left(h\left(b_n\right)\right),D\left(h\left(b_n\right)\right)\right)\\
&\qquad - H\left(C\left(h\left(b_{n-1}\right)\right),D\left(h\left(b_{n-1}\right)\right)\right)\Big\|\\
&\quad+\frac{\lambda_2}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left\|Q\left(p_n,q_n\right)-Q\left(p_{n-1},q_{n-1}\right)\right\|\\
&\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\Big\|H\left(C\left(h\left(b_n\right)\right),D\left(h\left(b_n\right)\right)\right)\\
&\qquad - H\left(C\left(h\left(b_{n-1}\right)\right),D\left(h\left(b_n\right)\right)\right)\Big\|\\
&\quad+\frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\Big\|H\left(C\left(h\left(b_{n-1}\right)\right),D\left(h\left(b_n\right)\right)\right)\\
&\qquad - H\left(C\left(h\left(b_{n-1}\right)\right),D\left(h\left(b_{n-1}\right)\right)\right)\Big\|\\
&\quad+\frac{\lambda_2}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left\|Q\left(p_n,q_n\right)-Q\left(p_{n-1},q_n\right)\right\|\\
&\quad+\frac{\lambda_2}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left\|Q\left(p_{n-1},q_n\right)-Q\left(p_{n-1},q_{n-1}\right)\right\|.
\end{aligned}
\tag{39}
$$

Since $H(C,D)$ is s_1-Lipschitz continuous with respect to C and s_2-Lipschitz continuous with respect to D, Q is δ_1-Lipschitz continuous in the first argument and δ_2-Lipschitz continuous in the second argument. h is λ_h-Lipschitz continuous and using (32), (34), and (39) it becomes

$$
\begin{aligned}
\left\|h\left(b_{n+1}\right)-h\left(b_n\right)\right\| \le{}& \frac{\left(s_1+s_2\right)\lambda_h}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left\|b_n-b_{n-1}\right\|\\
&+\frac{\lambda_2\delta_1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left(1+\frac{1}{n}\right)\rho\left\|a_n-a_{n-1}\right\|\\
&+\frac{\lambda_2\delta_2}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\left(1+\frac{1}{n}\right)\tau\left\|b_n-b_{n-1}\right\|.
\end{aligned}
\tag{40}
$$

In the light of (34), we have

$$
\begin{aligned}
\left\|b_{n+1}-b_n\right\| \le{}& \frac{\left(s_1+s_2\right)\lambda_h+\lambda_2\delta_2\left(1+1/n\right)\tau}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)}\left\|b_n-b_{n-1}\right\|\\
&+\frac{\lambda_2\delta_1}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)}\left(1+\frac{1}{n}\right)\rho\left\|a_n-a_{n-1}\right\|.
\end{aligned}
\tag{41}
$$

Let

$$
\left\|b_{n+1}-b_n\right\| \le \theta_{3n}\left\|a_n-a_{n-1}\right\|+\theta_{4n}\left\|b_n-b_{n-1}\right\|, \tag{42}
$$

where

$$
\begin{aligned}
\theta_{3n} &= \frac{\lambda_2\delta_1\left(1+1/n\right)\rho}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)},\\
\theta_{4n} &= \frac{\left(s_1+s_2\right)\lambda_h+\lambda_2\delta_2\left(1+1/n\right)\tau}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)}.
\end{aligned}
\tag{43}
$$

From adding (37) and (42), we get

$$
\begin{aligned}
\left\|a_{n+1}-a_n\right\|+\left\|b_{n+1}-b_n\right\| &\le \left(\theta_{1n}+\theta_{3n}\right)\left\|a_n-a_{n-1}\right\|\\
&\quad+\left(\theta_{2n}+\theta_{4n}\right)\left\|b_n-b_{n-1}\right\|\\
&\le \Theta_n\left\{\left\|a_n-a_{n-1}\right\|+\left\|b_n-b_{n-1}\right\|\right\},
\end{aligned}
\tag{44}
$$

where $\Theta_n=\max\{(\theta_{1n}+\theta_{3n}),(\theta_{2n}+\theta_{4n})\}$.

Letting $n\to\infty$, we obtain $\Theta_n\to\theta$, where

$$
\begin{aligned}
\theta &= \max\left\{\left(\theta_1+\theta_3\right),\left(\theta_2+\theta_4\right)\right\},\\
\theta_1 &= \frac{\left(r_1+r_2\right)\lambda_g+\lambda_1\epsilon_1\rho}{\xi_1\left(\mu_1\alpha_1^2-\gamma_1\beta_1^2\right)},\\
\theta_2 &= \frac{\lambda_1\epsilon_2\tau}{\xi_1\left(\mu_1\alpha_1^2-\gamma_1\beta_1^2\right)},\\
\theta_3 &= \frac{\lambda_2\delta_1\rho}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)},\\
\theta_4 &= \frac{\left(s_1+s_2\right)\lambda_h+\lambda_2\delta_2\tau}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)}.
\end{aligned}
\tag{45}
$$

By (31), $\theta\in(0,1)$, and (44) $\{a_n\}$ and $\{b_n\}$ both are Cauchy sequences. Thus, there exists $(a,b)\in X_1\times X_2$ such that $a_n\to a$, $b_n\to b$, as $n\to\infty$. From the Lipschitz continuity of S and T and (32), $\{p_n\}$,$\{q_n\}$ are also Cauchy sequences, and thus there exists $(p,q)\in X_1\times X_2$ such that $p_n\to p$, $q_n\to q$, as $n\to\infty$.

Now, we prove that $p\in S(a)$ and $q\in T(b)$. In fact, since $p_n\in S(a_n)$ and $q_n\in T(b_n)$, we have

$$
\begin{aligned}
d\left(p,S\left(a\right)\right) &\le \left\|p-p_n\right\|+d\left(p_n,S\left(a\right)\right),\\
d\left(p,S\left(a\right)\right) &\le \left\|p-p_n\right\|+\mathfrak{D}\left(S\left(a_n\right),S\left(a\right)\right),\\
d\left(p,S\left(a\right)\right) &\le \left\|p-p_n\right\|+\rho\left\|a_n-a\right\|\longrightarrow 0,\qquad \text{as } n\longrightarrow\infty,
\end{aligned}
\tag{46}
$$

which implies that $d(p,S(a))=0$. Since $S(a)\in CB(X_1)$, it follows that $p\in S(a)$. Similarly, we have $q\in T(b)$. By Lemma 11, it follows that a,b,p,q is a solution of problem (21), and this completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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