Research Article

L_p -Approximation ($p \geq 1$) by q -Kantorovich Operators

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For a new q -Kantorovich operator we establish direct approximation theorems in the space $L_p[0, 1]$, $1 \leq p \leq \infty$, via Ditzian-Totik modulus of smoothness of second order.

1. Introduction

Let $q > 0$. Then for each non-negative integer n , the q -integer $[n] \equiv [n]_q$ and the q -factorial $[n]! \equiv [n]_q!$ are defined by $[n] = 1 + q + \dots + q^{n-1}$ ($n = 1, 2, \dots$), $[0] = 0$, and $[n]! = [1][2] \dots [n]$ ($n = 1, 2, \dots$), $[0]! = 1$. For integers $0 \leq k \leq n$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!}. \quad (1)$$

Further, we set

$$p_{n,k}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-qx) \dots (1-q^{n-k-1}x) \quad (2)$$

for $k = 0, 1, \dots, n-1$ and $p_{n,n}(q; x) = x^n$. Following Phillips [1], the operators $B_{n,q}: C[0, 1] \rightarrow C[0, 1]$ defined by

$$(B_{n,q}f)(x) \equiv B_{n,q}(f, x) = \sum_{k=0}^n p_{n,k}(q; x) f\left(\frac{[k]}{[n]}\right) \quad (3)$$

are called q -Bernstein operators. They quickly gained the popularity and were studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [2]. The q -Bernstein operators are not defined

for $f \in L_p[0, 1]$. For this reason we introduce the following operators based on q -integers:

$$\begin{aligned} (K_{n,q}f)(x) &\equiv K_{n,q}(f, x) \\ &= [n+1] \sum_{k=0}^n p_{n,k}(q; x) q^{-k} \int_{[k]/[n+1]}^{[k+1]/[n+1]} f(u) du. \end{aligned} \quad (4)$$

For $q = 1$, we recover the well-known Kantorovich operators [3]. In [4, 5] other Kantorovich type operators involving q -integers are considered, which are defined with the aid of q -Riemann integral (see [6]), and their rate of convergence are studied for only continuous functions on $[0, 1]$.

The aim of the paper is to establish direct approximation theorems for (4) in the space $L_p[0, 1]$, $1 \leq p \leq \infty$. To describe our results, we will give the definitions of Ditzian-Totik modulus of smoothness of second order and the corresponding K -functionals (cf. [7]). For $f \in L_p[0, 1]$, $1 \leq p \leq \infty$, and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$, we set

$$\begin{aligned} \omega_\varphi^2(f, \delta)_p &= \sup_{0 < h \leq \delta} \|f(x + h\varphi(x)) - 2f(x) \\ &\quad + f(x - h\varphi(x))\|_p, \end{aligned} \quad (5)$$

$$K_{2,\varphi}(f, \delta)_p = \inf_{g \in W_p^2} \left\{ \|f - g\|_p + \delta \|\varphi^2 g''\|_p \right\},$$

$$\bar{K}_{2,\varphi}(f, \delta)_p = \inf_{g \in W_p^2} \left\{ \|f - g\|_p + \delta \|\varphi^2 g''\|_p + \delta^2 \|g''\|_p \right\},$$

where $W_p^2 = \{g \in L_p[0, 1] : g' \in AC_{\text{loc}}[0, 1], \|g''\|_p < \infty\}$ and $g' \in AC_{\text{loc}}[0, 1]$ means that g is differentiable such that g' is absolutely continuous in every interval $[a, b] \subseteq [0, 1]$. It is known from [7] that $\omega_\varphi^2(f, \sqrt{\delta})_p$, $K_{2,\varphi}(f, \delta)_p$ and $\bar{K}_{2,\varphi}(f, \delta)_p$ are equivalent; that is, there exists $C > 0$ such that

$$C^{-1}\omega_\varphi^2(f, \sqrt{\delta})_p \leq K_{2,\varphi}(f, \delta)_p \leq C\omega_\varphi^2(f, \sqrt{\delta})_p, \quad (6)$$

$$C^{-1}\omega_\varphi^2(f, \sqrt{\delta})_p \leq \bar{K}_{2,\varphi}(f, \delta)_p \leq C\omega_\varphi^2(f, \sqrt{\delta})_p. \quad (7)$$

Here C denotes a positive constant independent of n and x , but it is not necessarily the same in different cases. Finally, we define the modulus of continuity of $f \in C[0, 1]$ by

$$\omega(f, \delta) = \sup \{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq \delta\}. \quad (8)$$

The paper is organized as follows. In Section 2 some auxiliary results are established and in Section 3 we give the main results, establishing direct approximation theorems for (4) in $L_p[0, 1]$, $1 \leq p \leq \infty$.

2. Auxiliary Results

In the sequel we need some lemmas.

Lemma 1. Let $K_{n,q}$, $0 < q < 1$, be defined by (4) and $\delta_{q,n}(x) = \varphi(x) + [n+1]^{-1/2}$. Then $K_{n,q}(1, x) = 1$, $|K_{n,q}(t-x, x)| \leq 2/[n+1]$ and $K_{n,q}((t-x)^2, x) \leq (8/[n+1])\delta_{q,n}^2(x)$.

Proof. By (4), we obtain

$$K_{n,q}(f, x) = \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 f\left(\frac{[k] + q^k u}{[n+1]}\right) du. \quad (9)$$

Recall some properties of the q -Bernstein operators (see [1]):

$$\begin{aligned} B_{n,q}(1, x) &= 1, & B_{n,q}(t-x, x) &= 0, \\ B_{n,q}((t-x)^2, x) &= \frac{\varphi^2(x)}{[n]}. \end{aligned} \quad (10)$$

Then, by (9) and (10), we find $K_{n,q}(1, x) = B_{n,q}(1, x) = 1$ and

$$\begin{aligned} K_{n,q}(t-x, x) &= \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 \left(\frac{[k] + q^k u}{[n+1]} - x\right) du \\ &= \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 \left(\frac{[k] + q^k u}{[n+1]} - \frac{[k]}{[n]}\right) du \\ &\quad + B_{n,q}(t-x, x) \\ &= \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 \left(\frac{[k] + q^k u}{[n+1]} - \frac{[k]}{[n]}\right) du. \end{aligned} \quad (11)$$

But $[n+1] = [n] + q^n$ implies that

$$\left|\frac{[k] + q^k u}{[n+1]} - \frac{[k]}{[n]}\right| = \frac{|q^k [n] u - q^n [k]|}{[n][n+1]} \leq \frac{2}{[n+1]} \quad (12)$$

for $k = 0, 1, \dots, n$. Hence, by (10),

$$|K_{n,q}(t-x, x)| \leq B_{n,q}(1, x) \frac{2}{[n+1]} \leq \frac{2}{[n+1]}. \quad (13)$$

Further, by (9), (10) and (12), we have

$$\begin{aligned} K_{n,q}((t-x)^2, x) &= \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 \left(\frac{[k] + q^k u}{[n+1]} - x\right)^2 du \\ &\leq 2 \sum_{k=0}^n p_{n,k}(q; x) \\ &\quad \times \int_0^1 \left(\frac{[k] + q^k u}{[n+1]} - \frac{[k]}{[n]}\right)^2 du \\ &\quad + 2B_{n,q}((t-x)^2, x) \\ &\leq \frac{8}{[n+1]^2} + \frac{2[n+1]}{[n]} \frac{\varphi^2(x)}{[n+1]} \\ &\leq \frac{8}{[n+1]} \delta_{n,q}^2(x), \end{aligned} \quad (14)$$

because

$$[n+1] = 1 + q[n] \leq 2[n]. \quad (15)$$

□

Lemma 2. Let $f \in L_p[0, 1]$ and let $K_{n,q}f$ be given by (4). Then

- (i) $\|K_{n,q}f\|_p \leq C\|f\|_p$, where $1 \leq p < \infty$, $q = q_n \in (0, 1)$ and there exists $C_0 > 0$ such that $n^3(1-q^n) \leq C_0 < 1$ for all $n = 1, 2, \dots$;
- (ii) $\|K_{n,q}f\|_\infty \leq C\|f\|_\infty$, where $0 < q < 1$.

Proof. By Riesz-Thorin theorem (see, e.g., [8]), we separate the proof for $p = 1$ and $p = \infty$.

Case 1 ($p = 1$). Because the q -binomial coefficients are increasing functions of q , we get for the polynomials $p_{n,k}(q; x)$ and $p_{n,k}(1; x)$ of degree n that

$$\begin{aligned} \frac{p_{n,k}(q; x)}{p_{n,k}(1; x)} &= \frac{\begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-qx) \cdots (1-q^{n-k-1}x)}{\begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)^{n-k}} \\ &\leq \frac{1-qx}{1-x} \cdots \frac{1-q^{n-k-1}x}{1-x} \leq \left(\frac{1-q^n x}{1-x}\right)^{n-k-1} \\ &\leq \left(1 + (1-q^n) \frac{x}{1-x}\right)^n. \end{aligned} \quad (16)$$

Now let $x \in [1/(2n^2), 1 - 1/(2n^2)]$ and $k \in \{0, 1, \dots, n-2\}$. Then, in view of $n^3(1 - q^n) \leq C_0$, $n = 1, 2, \dots$, we obtain

$$\begin{aligned} \frac{p_{n,k}(q; x)}{p_{n,k}(1; x)} &\leq (1 + 2n^2(1 - q^n))^n \\ &= \left\{ (1 + 2n^2(1 - q^n))^{1/(2n^2(1 - q^n))} \right\}^{2n^3(1 - q^n)} \quad (17) \\ &\leq C. \end{aligned}$$

Using [7, Theorem 8.4.8, page 108] translated from $[-1, 1]$ to $[0, 1]$, we find that

$$\begin{aligned} \|p_{n,k}(q; x)\|_1 &\leq C \|p_{n,k}(q; x)\|_{L_1[1/(2n^2), 1 - 1/(2n^2)]} \\ &\leq C \|p_{n,k}(1; x)\|_{L_1[1/(2n^2), 1 - 1/(2n^2)]} \quad (18) \\ &\leq C \|p_{n,k}(1; x)\|_1 = \frac{C}{n+1}. \end{aligned}$$

On the other hand $n^3(1 - q^n) \leq C_0 < 1$, $n = 1, 2, \dots$, implies that $q^n \geq 1 - C_0 > 0$, $n = 1, 2, \dots$. Hence, by (4) and (18), we get

$$\begin{aligned} \|K_{n,q}f\|_1 &\leq [n+1] \sum_{k=0}^n q^{-k} \|p_{n,k}(q; x)\|_1 \\ &\quad \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} |f(u)| du \quad (19) \\ &\leq C \frac{[n+1]}{q^n(n+1)} \|f\|_1 \leq C \|f\|_1. \end{aligned}$$

Case 2 ($p = \infty$). Due to (9) and (10), we have

$$\begin{aligned} |K_{n,q}(f, x)| &\leq \sum_{k=0}^n p_{n,k}(q; x) \int_0^1 \left| f\left(\frac{[k] + q^k u}{[n+1]}\right) \right| du \quad (20) \\ &\leq B_{n,q}(1, x) \|f\|_\infty = \|f\|_\infty. \end{aligned}$$

□

Lemma 3. If $K_{n,q}f$ is defined by (4), $E_{[n+1]} = [1/[n+1], 1 - 1/[n+1]]$, $g \in W_p^2$ and $1 \leq p < \infty$, then

$$\left\| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right\|_{L_p(E_{[n+1]})} \leq \frac{C}{[n+1]} \|\varphi^2 g''\|_p, \quad (21)$$

where $q = q_n \in (0, 1)$ satisfies that there exists $C_0 > 0$ such that $n^3(1 - q^n) \leq C_0 < 1$ for all $n = 1, 2, \dots$

Proof. Consider the following cases.

Case 1 ($p = 1$). Because of [7, Lemma 9.6.1, page 140], we have $|u - v|/\varphi^2(v) \leq |u - x|/\varphi^2(x)$. Then, by (4),

$$\begin{aligned} &\int_{E_{[n+1]}} \left| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right| dx \\ &\leq \int_{E_{[n+1]}} \sum_{k=0}^n p_{n,k}(q; x) [n+1] q^{-k} \\ &\quad \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} \frac{|u - x|}{\varphi^2(x)} \\ &\quad \times \left| \int_x^u \varphi^2(v) |g''(v)| dv \right| du dx. \end{aligned} \quad (22)$$

But

$$\begin{aligned} \left| \frac{[k]}{[n+1]} - \frac{[k]}{[n]} \right| &= \frac{q^n [k]}{[n][n+1]} \leq \frac{1}{[n+1]}, \\ \left| \frac{[k+1]}{[n+1]} - \frac{[k]}{[n]} \right| &= \left| \frac{1 + q[k]}{1 + q[n]} - \frac{[k]}{[n]} \right| \quad (23) \\ &= \frac{[n] - [k]}{[n][n+1]} = \frac{q^k [n - k]}{[n][n+1]} \leq \frac{1}{[n+1]} \end{aligned}$$

imply

$$\begin{aligned} &[n+1] q^{-k} \int_{[k]/[n+1]}^{[k+1]/[n+1]} |u - x| du \\ &\leq [n+1] q^{-k} \int_{[k]/[n+1]}^{[k+1]/[n+1]} \left| u - \frac{[k]}{[n]} \right| du + \left| \frac{[k]}{[n]} - x \right| \quad (24) \\ &\leq \frac{1}{[n+1]} + \left| \frac{[k]}{[n]} - x \right|. \end{aligned}$$

Hence, by (22),

$$\begin{aligned} &\int_{E_{[n+1]}} \left| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right| dx \\ &\leq \int_{E_{[n+1]}} \sum_{k=0}^n p_{n,k}(q; x) \varphi^{-2}(x) \\ &\quad \times \left(\left| \frac{[k]}{[n]} - x \right| + \frac{1}{[n+1]} \right) \\ &\quad \times \left| \int_x^{[k^*]/[n+1]} \varphi^2(v) |g''(v)| dv \right| dx = I_{q,n}, \end{aligned} \quad (25)$$

where k^* is either k or $k+1$ such that

$$\begin{aligned} &\left| \int_x^{[k^*]/[n+1]} \varphi^2(v) |g''(v)| dv \right| \\ &= \max_{j=k, k+1} \left| \int_x^{[j]/[n+1]} \varphi^2(v) |g''(v)| dv \right|. \end{aligned} \quad (26)$$

Using the method in [7, pages 146-147], let

$$\begin{aligned} D(l, q, n, x) &= \left\{ k : l\varphi(x) [n+1]^{-1/2} \right. \\ &\quad \left. \leq \left| \frac{k}{n} - x \right| < (l+1)\varphi(x) [n+1]^{-1/2} \right\}. \end{aligned} \quad (27)$$

Then

$$\begin{aligned} I_{q,n} &= \int_{E_{[n+1]}} \varphi^{-2}(x) \sum_{l=0}^{\infty} \sum_{k \in D(l, q, n, x)} p_{n,k}(q; x) \\ &\quad \times \left(\left| \frac{[k]}{[n]} - x \right| + \frac{1}{[n+1]} \right) \\ &\quad \times \left| \int_x^{[k^*]/[n+1]} \varphi^2(v) |g''(v)| dv \right| dx. \end{aligned} \quad (28)$$

For $x \in E_{[n+1]}$ we prove that

$$\begin{aligned} \sum_{k \in D(l, q, n, x)} p_{n,k}(q; x) \left(\left| \frac{[k]}{[n]} - x \right| + \frac{1}{[n+1]} \right) \\ \leq \frac{C}{(l+1)^4} \varphi(x) [n+1]^{-1/2}. \end{aligned} \quad (29)$$

To obtain (29), we need some useful estimations:

$$\left| \frac{k}{n} - \frac{[k]}{[n]} \right| \leq \frac{4}{[n+1]}, \quad k = 0, 1, \dots, n, \quad (30)$$

$$[n+1]^{-1/2} \leq \sqrt{\frac{2}{1-C_0}} \varphi(x), \quad x \in E_{[n+1]}. \quad (31)$$

Indeed, for $k \in \{1, 2, \dots, n\}$ we have $0 \leq k - [k] = (1-q) + (1-q^2) + \dots + (1-q^{k-1}) \leq (k-1)(1-q^{k-1}) \leq n^3(1-q^n) \leq C_0 < 1$. Hence, by (15),

$$\begin{aligned} \left| \frac{k}{n} - \frac{[k]}{[n]} \right| &\leq \left| \frac{k}{n} - \frac{k}{[n]} \right| + \left| \frac{k}{[n]} - \frac{[k]}{[n]} \right| \\ &= \frac{k}{n} \frac{n - [n]}{[n]} + \frac{k - [k]}{[n]} \leq \frac{2}{[n]} \leq \frac{4}{[n+1]}. \end{aligned} \quad (32)$$

Further, for $x \in E_{[n+1]}$ we have

$$\begin{aligned} \varphi^2(x) &= x(1-x) \geq \frac{1}{[n+1]} \left(1 - \frac{1}{[n+1]} \right) \\ &= \frac{1}{[n+1]} \frac{q[n]}{[n+1]} \geq \frac{1}{[n+1]} \frac{1-C_0}{2}, \end{aligned} \quad (33)$$

taking into account (15) and $q \geq q^n > 1 - C_0 > 0$, because of $n^3(1-q^n) \leq C_0 < 1$.

Now let $l \geq 1$ and $x \in E_{[n+1]}$. By (15), we have $E_{[n+1]} \subseteq [1/(2n), 1 - 1/(2n)] \subseteq [1/(2n^2), 1 - 1/(2n^2)]$. Then, in view of (30), (17), [7, Lemma 9.4.4, page 128], (15) and (31), we obtain

$$\begin{aligned} \sum_{k \in D(l, q, n, x)} p_{n,k}(q; x) \left(\left| \frac{[k]}{[n]} - x \right| + \frac{1}{[n+1]} \right) \\ \leq C \sum_{k \in D(l, q, n, x)} p_{n,k}(1; x) \left(\left| \frac{k}{n} - x \right| + \frac{1}{[n+1]} \right) \\ \leq \frac{C[n+1]^2}{l^4 \varphi^4(x)} \sum_{k=0}^n p_{n,k}(1; x) \left(\left| \frac{k}{n} - x \right|^5 + \frac{1}{[n+1]} \left| \frac{k}{n} - x \right|^4 \right) \\ \leq \frac{C[n+1]^2}{l^4 \varphi^4(x)} \left(\frac{\varphi^5(x)}{n^{5/2}} + \frac{1}{[n+1]} \frac{\varphi^4(x)}{n^2} \right) \\ \leq \frac{C}{l^4} \left(\frac{[n+1]^2}{[n]^{5/2}} \varphi(x) + \frac{[n+1]^2}{[n]^2} \frac{1}{[n+1]} \right) \\ \leq \frac{C}{l^4} \left(4\sqrt{2}[n+1]^{-1/2} \varphi(x) + 4[n+1]^{-1/2} \sqrt{\frac{2}{1-C_0}} \varphi(x) \right) \\ \leq \frac{C}{(l+1)^4} \varphi(x) [n+1]^{-1/2}. \end{aligned} \quad (34)$$

For $x \in E_{[n+1]}$, by (10), (15) and (31), we have

$$\begin{aligned} \sum_{k \in D(0, q, n, x)} p_{n,k}(q; x) \left(\left| \frac{[k]}{[n]} - x \right| + \frac{1}{[n+1]} \right) \\ \leq \{B_{n,q}((t-x)^2, x)\}^{1/2} + \frac{1}{[n+1]} B_{n,q}(1, x) \\ \leq \sqrt{2} \varphi(x) [n+1]^{-1/2} + \sqrt{\frac{2}{1-C_0}} \varphi(x) [n+1]^{-1/2} \\ \leq C \varphi(x) [n+1]^{-1/2}. \end{aligned} \quad (35)$$

We now define

$$\begin{aligned} F(l, x) &= \{v : v \in [0, 1], \\ &\quad |v - x| \leq \sqrt{5}(l+1)\varphi(x) [n+1]^{-1/2} + 5[n+1]^{-1}\}, \\ G(l, v) &= \{v : x \in E_{[n+1]}, v \in F(l, x)\}. \end{aligned} \quad (36)$$

Then $[k^*]/[n+1] \in F(l, x)$, where $k \in D(l, q, n, x)$, because (23), (30) and the definition of $D(l, q, n, x)$ imply that

$$\begin{aligned} \left| \frac{[k^*]}{[n+1]} - x \right| &\leq \left| \frac{[k^*]}{[n+1]} - \frac{[k]}{[n]} \right| + \left| \frac{[k]}{[n]} - \frac{k}{n} \right| + \left| \frac{k}{n} - x \right| \\ &\leq \frac{1}{[n+1]} + \frac{4}{[n+1]} + (l+1)\varphi(x) [n+1]^{-1/2} \\ &\leq \sqrt{5}(l+1)\varphi(x) [n+1]^{-1/2} + 5[n+1]^{-1}. \end{aligned} \quad (37)$$

Hence, by (29), (28) and (15), we get

$$\begin{aligned}
I_{q,n} &\leq \frac{C}{[n+1]} \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \\
&\quad \times \int_{E_{[n+1]}} \frac{[n+1]^{1/2}}{\varphi(x)} \int_{F(l,x)} \varphi^2(v) |g''(v)| dv dx \\
&\leq \frac{C}{[n+1]} \sum_{l \leq n^{1/4}} \frac{1}{(l+1)^4} \\
&\quad \times \int_0^1 \varphi^2(v) |g''(v)| [n+1]^{1/2} \left\{ \int_{G(l,v)} \varphi^{-1}(x) dx \right\} dv \\
&\quad + \frac{C}{[n+1]} \sum_{l \geq n^{1/4}} \frac{1}{(l+1)^4} \\
&\quad \times \int_0^1 \varphi^2(v) |g''(v)| n^{1/2} \times \left\{ \int_{G(l,v)} \varphi^{-1}(x) dx \right\} dv.
\end{aligned} \tag{38}$$

Applying the estimate (see [7, (9.6.14), page 147])

$$\int_{x: |v-x| \leq h\varphi(x)+h^2} \varphi^{-1}(x) dx \leq Mh \tag{39}$$

for $h = \sqrt{5}(l+1)[n+1]^{-1/2}$ and recalling that $h^2 \geq 5[n+1]^{-1}$, we have

$$\int_{G(l,v)} \varphi^{-1}(x) dx \leq \sqrt{5}M(l+1)[n+1]^{-1/2}. \tag{40}$$

Substituting this estimate in (38) for $l \leq n^{1/4}$ and following the procedure given in [7, page 147], we obtain the statement of our lemma, because of (25).

Case 2 ($1 < p < \infty$). Let $x \in E_{[n+1]}$. The function $\delta_{q,n}^2$ is concave on $[0, 1]$; therefore

$$\frac{|t-v|}{\delta_{q,n}^2(v)} \leq \frac{|t-x|}{\delta_{q,n}^2(x)} \tag{41}$$

(see [9, (4)]). Then, by Lemma 1,

$$\begin{aligned}
&\left| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right| \\
&\leq K_{n,q} \left(\frac{|t-x|}{\delta_{q,n}^2(x)} \left| \int_x^t \delta_{q,n}^2(v) |g''(v)| dv \right|, x \right) \\
&= K_{n,q} \left(\frac{(t-x)^2}{\delta_{q,n}^2(x)} \left| \frac{1}{t-x} \int_x^t \delta_{q,n}^2(v) |g''(v)| dv \right|, x \right) \tag{42} \\
&\leq K_{n,q} \left(\frac{(t-x)^2}{\delta_{q,n}^2(x)}, x \right) M(\delta_{q,n}^2 |g''|, x) \\
&\leq \frac{8}{[n+1]} M(\delta_{q,n}^2 |g''|, x),
\end{aligned}$$

where $M(\delta_{q,n}^2 |g''|, x) = \sup_t |(1/(t-x)) \int_x^t \delta_{q,n}^2(v) |g''(v)| dv|$ is the Hardy-Littlewood maximal function. It is known that $\|M(\delta_{q,n}^2 |g''|)\|_p \leq (p/(p-1)) \|\delta_{q,n}^2 g''\|_p$, $1 < p < \infty$. Hence, by (42),

$$\begin{aligned}
&\left\| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right\|_{L_p(E_{[n+1]})} \\
&\leq \frac{8}{[n+1]} \frac{p}{p-1} \|\delta_{q,n}^2 g''\|_p.
\end{aligned} \tag{43}$$

But $\delta_{q,n}^2(x) \leq 2(\varphi^2(x) + (1/[n+1])) \leq C\varphi^2(x)$, because of (31). Thus

$$\left\| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right\|_{L_p(E_{[n+1]})} \leq \frac{C}{[n+1]} \|\varphi^2 g''\|_p, \tag{44}$$

which completes the proof. $\square$

Lemma 4. If P_l is a polynomial of degree at most l , where $l < n$, then $K_{n,q} P_l$ is also a polynomial of degree at most l .

Proof. It is sufficient to prove that $K_{n,q}(t^l, x)$ is a polynomial of degree at most l . By (4), we have

$$\begin{aligned}
K_{n,q}(t^l, x) &= [n+1] \sum_{k=0}^n p_{n,k}(q; x) q^{-k} \int_{[k]/[n+1]}^{[k+1]/[n+1]} u^l du \\
&= \frac{1}{l+1} \sum_{k=0}^n p_{n,k}(q; x) \\
&\quad \times \left\{ \left(\frac{[k+1]}{[n+1]} \right)^l + \left(\frac{[k+1]}{[n+1]} \right)^{l-1} \frac{[k]}{[n+1]} \right. \\
&\quad \left. + \cdots + \left(\frac{[k]}{[n+1]} \right)^l \right\} \\
&= \frac{1}{l+1} \left(\frac{[n]}{[n+1]} \right)^l \sum_{k=0}^n p_{n,k}(q; x) \\
&\quad \times \left\{ \left(\frac{1+q[k]}{[n]} \right)^l + \left(\frac{1+q[k]}{[n]} \right)^{l-1} \frac{[k]}{[n]} \right. \\
&\quad \left. + \cdots + \left(\frac{[k]}{[n]} \right)^l \right\} \\
&= \frac{1}{l+1} \left(\frac{[n]}{[n+1]} \right)^l \sum_{k=0}^n p_{n,k}(q; x)
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ a_0(l, q, n) \right. \\
& \quad \left. + a_1(l, q, n) \frac{[k]}{[n]} + \cdots + a_l(l, q, n) \left(\frac{[k]}{[n]} \right)^l \right\} \\
& = \frac{1}{l+1} \left(\frac{[n]}{[n+1]} \right)^l \left\{ a_0(l, q, n) B_{n,q}(1, x) \right. \\
& \quad + a_1(l, q, n) B_{n,q}(t, x) \\
& \quad \left. + \cdots + a_l(l, q, n) B_{n,q}(t^l, x) \right\}. \quad (45)
\end{aligned}$$

Since $B_{n,q}(t^m, x)$ is a polynomial of degree $\min\{m, n\}$ (see [1]), we obtain the assertion of the lemma. $\square$

3. Main Results

The direct approximation results are included in the following theorem.

Theorem 5. Let $f \in L_p[0, 1]$ be arbitrary and let $K_{n,q}f$ be given by (4).

(i) If $1 \leq p < \infty$, $q = q_n \in (0, 1)$, and $C_0 > 0$ satisfies that $n^3(1 - q^n) \leq C_0 < 1$ for all $n = 1, 2, \dots$, then

$$\|K_{n,q}f - f\|_p \leq C \left\{ \omega_\varphi^2(f, [n+1]^{-1/2})_p + [n+1]^{-1} \|f\|_p \right\}. \quad (46)$$

(ii) If $q = q_n \in (0, 1)$ satisfies that $q_n \rightarrow 1$ as $n \rightarrow \infty$, then

$$\begin{aligned}
& \|K_{n,q}f - f\|_\infty \\
& \leq C \left\{ \omega_\varphi^2(f, [n+1]^{-1/2})_\infty + \omega\left(f, \frac{1}{2}[n+1]^{-1}\right) \right\}. \quad (47)
\end{aligned}$$

Proof. (i) Let $g \in W_p^2$. By Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v) g''(v) dv \quad (48)$$

and Lemma 1, we obtain

$$\begin{aligned}
& |K_{n,q}(g, x) - g(x)| \\
& \leq |g'(x)| |K_{n,q}(t-x, x)| + \left| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right| \\
& \leq \frac{2}{[n+1]} |g'(x)| + \left| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right|. \quad (49)
\end{aligned}$$

Hence, for $x \in E_{[n+1]}$, we have

$$\begin{aligned}
& \|K_{n,q}g - g\|_{L_p(E_{[n+1]})} \\
& \leq \frac{2}{[n+1]} \|g'\|_p + \left\| K_{n,q} \left(\int_x^t (t-v) g''(v) dv, x \right) \right\|_{L_p(E_{[n+1]})}. \quad (50)
\end{aligned}$$

Using the estimate $\|g'\|_p \leq C(\|\varphi^2 g''\|_p + \|g\|_p)$, $1 \leq p < \infty$ (see [7, (a), page 135]), and Lemma 3, we get, in view of (50), that

$$\|K_{n,q}g - g\|_{L_p(E_{[n+1]})} \leq C[n+1]^{-1} (\|\varphi^2 g''\|_p + \|g\|_p). \quad (51)$$

On the other hand, the condition $n^3(1 - q^n) \leq C_0 < 1$, $n = 1, 2, \dots$, implies that $0 \leq n - [n] = (1 - q) + (1 - q^2) + \cdots + (1 - q^{n-1}) \leq (n-1)(1 - q^{n-1}) \leq n^3(1 - q^n) \leq C_0 < 1$ and $q \geq q^n \geq 1 - C_0 n^{-3} \geq 1 - C_0 > 0$. Then $n < 1 + [n] = (1/q)(q + q[n]) < (1/q)[n+1] \leq (1/(1 - C_0))[n+1]$. In conclusion

$$(1 - C_0)n < [n+1]. \quad (52)$$

Finally, we follow the procedure given in [7, pages 118–119]. The definition of $K_{2,\varphi}(f, [n+1]^{-1})_p$ implies the existence of $g_n \in W_p^2$ such that

$$\|f - g_n\|_p \leq 2K_{2,\varphi}(f, [n+1]^{-1})_p, \quad (53)$$

$$\|\varphi^2 g_n''\|_p \leq 2[n+1] K_{2,\varphi}(f, [n+1]^{-1})_p. \quad (54)$$

Then, by (51), (53), (54) and (6), we get

$$\begin{aligned}
& \|K_{n,q}g_n - g_n\|_{L_p(E_{[n+1]})} \\
& \leq C[n+1]^{-1} (\|\varphi^2 g_n''\|_p + \|g_n\|_p) \\
& \leq C[n+1]^{-1} (\|\varphi^2 g_n''\|_p + \|f - g_n\|_p + \|f\|_p) \quad (55) \\
& \leq C \left\{ K_{2,\varphi}(f, [n+1]^{-1})_p + [n+1]^{-1} \|f\|_p \right\} \\
& \leq C \left\{ \omega_\varphi^2(f, [n+1]^{-1/2})_p + [n+1]^{-1} \|f\|_p \right\}.
\end{aligned}$$

Further, due to (54), (15) and (52), we have $\|\varphi^2 g_n''\|_p \leq CnK_{2,\varphi}(f, n^{-1})_p$. Then, by [7, Theorem 7.3.1, page 84], we can choose g_n to be $P_{[\sqrt{n}]}$ the best $[\sqrt{n}]$ th degree polynomial approximation in $L_p[0, 1]$. Therefore, by (52) and (55), we obtain

$$\begin{aligned}
& \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{L_p[1/((1-C_0)n), 1-1/((1-C_0)n)]} \\
& \leq \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{L_p(E_{[n+1]})} \quad (56) \\
& \leq C \left\{ \omega_\varphi^2(f, [n+1]^{-1/2})_p + [n+1]^{-1} \|f\|_p \right\}.
\end{aligned}$$

Taking into account Lemma 4 and Theorem 8.4.8 of [7, page 108] translated from $[-1, 1]$ to $[0, 1]$ and n to $[\sqrt{n}]$, we obtain, using (56), that

$$\begin{aligned}
& \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{L_p[0,1]} \\
& \leq C \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{L_p[1/((1-C_0)n), 1-1/((1-C_0)n)]} \quad (57) \\
& \leq C \left\{ \omega_\varphi^2(f, [n+1]^{-1/2})_p + [n+1]^{-1} \|f\|_p \right\}.
\end{aligned}$$

Hence, due to Lemma 2, (53) and (6), we find

$$\begin{aligned}
& \|K_{n,q}f - f\|_p \\
& \leq \|K_{n,q}(f - P_{[\sqrt{n}]}) - (f - P_{[\sqrt{n}]})\|_p \\
& \quad + \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_p \\
& \leq C\|f - P_{[\sqrt{n}]}\|_p + \|K_{n,q}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_p \\
& \leq C\left\{\omega_\varphi^2(f, [n+1]^{-1/2})_p + [n+1]^{-1}\|f\|_p\right\}.
\end{aligned} \tag{58}$$

(ii) We introduce the operator

$$\tilde{K}_{n,q}(f, x) = K_{n,q}(f, x) - f(a(q, n, x)) + f(x), \tag{59}$$

where $a(q, n, x) = ((1+q)/2)([n]/[n+1])x + (1/2)[n+1]$. By Lemmas 1 and 2, we have $\tilde{K}_{n,q}(1, x) = 1$ and

$$\begin{aligned}
|\tilde{K}_{n,q}(f, x)| & \leq |K_{n,q}(f, x)| + |f(a(q, n, x))| \\
& \quad + |f(x)| \leq 3\|f\|_\infty.
\end{aligned} \tag{60}$$

Further, by (4) and (10), we obtain

$$\begin{aligned}
K_{n,q}(t, x) & = [n+1] \sum_{k=0}^n p_{n,k}(q; x) q^{-k} \frac{[k+1]^2 - [k]^2}{2[n+1]^2} \\
& = \frac{1}{2[n+1]} \sum_{k=0}^n p_{n,k}(q; x) (1 + (q+1)[k]) \\
& = \frac{1}{2[n+1]} (B_{n,q}(1, x) + (1+q)[n] B_{n,q}(t, x)) \\
& = a(n, q, x).
\end{aligned} \tag{61}$$

Thus $\tilde{K}_{n,q}(t, x) = x$. Hence, in view of Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v) g''(v) dv, \quad g \in W_\infty^2, \tag{62}$$

and (59), we get

$$\begin{aligned}
\tilde{K}_{n,q}(g, x) & = g(x) + \tilde{K}_{n,q}\left(\int_x^t (t-v) g''(v) dv, x\right) \\
& = g(x) + K_{n,q}\left(\int_x^t (t-v) g''(v) dv, x\right) \\
& \quad - \int_x^{a(q,n,x)} (a(q, n, x) - v) g''(v) dv.
\end{aligned} \tag{63}$$

But

$$\begin{aligned}
|a(q, n, x) - x| & = \left| \frac{(1+q)[n] - 2[n+1]}{2[n+1]}x + \frac{1}{2[n+1]} \right| \\
& = \left| \frac{(1-q)[n] - 2}{2[n+1]}x + \frac{1}{2[n+1]} \right| \\
& = \left| \frac{1 - (1+q^n)x}{2[n+1]} \right| \leq \frac{1}{2[n+1]}.
\end{aligned} \tag{64}$$

Hence, by (63), (41), Lemma 1 and $\delta_{q,n}^2(x) \leq 2(\varphi^2(x) + [n+1]^{-1})$, we find

$$\begin{aligned}
& |\tilde{K}_{n,q}(g, x) - g(x)| \\
& \leq K_{n,q}\left(|t-x|\delta_{q,n}^{-2}(x) \left| \int_x^t \delta_{q,n}^2(v) |g''(v)| dv \right|, x\right) \\
& \quad + (a(q, n, x) - x)^2 \|g''\|_\infty \\
& \leq \delta_{q,n}^{-2}(x) K_{n,q}((t-x)^2, x) \|\delta_{q,n}^2 g''\|_\infty \\
& \quad + \frac{1}{4[n+1]^2} \|g''\|_\infty \\
& \leq \frac{8}{[n+1]} \|\delta_{q,n}^2 g''\|_\infty + \frac{1}{4[n+1]^2} \|g''\|_\infty \\
& \leq 17 \left(\frac{1}{[n+1]} \|\varphi^2 g''\|_\infty + \frac{1}{[n+1]^2} \|g''\|_\infty \right).
\end{aligned} \tag{65}$$

Now, from (59), (60), (65) and (64), we have

$$\begin{aligned}
& |K_{n,q}(f, x) - f(x)| \\
& \leq |\tilde{K}_{n,q}(f, x) - f(x)| + |f(a(q, n, x)) - f(x)| \\
& \leq |\tilde{K}_{n,q}(f - g, x)| + |\tilde{K}_{n,q}(g, x) - g(x)| \\
& \quad + |g(x) - f(x)| + |f(a(q, n, x)) - f(x)| \\
& \leq 4\|f - g\|_\infty + 17 \left(\frac{1}{[n+1]} \|\varphi^2 g''\|_\infty + \frac{1}{[n+1]^2} \|g''\|_\infty \right) \\
& \quad + \omega(f, |a(q, n, x) - x|) \\
& \leq 17 \left(\|f - g\|_\infty + \frac{1}{[n+1]} \|\varphi^2 g''\|_\infty + \frac{1}{[n+1]^2} \|g''\|_\infty \right) \\
& \quad + \omega\left(f, \frac{1}{2}[n+1]^{-1}\right).
\end{aligned} \tag{66}$$

Taking the infimum on the right-hand side over all $g \in W_\infty^2$, and using the equivalence between $\bar{K}_{2,\varphi}(f, [n+1]^{-1})_\infty$ and $\omega_\varphi^2(f, [n+1]^{-1/2})_\infty$ (see (7)), we obtain the statement of the theorem. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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