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Research Article

A Note on Taylor-Eddy and Kovasznay Solutions of NS-α-Deconvolution and Leray-α-Deconvolution Models

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We show that both the Taylor-eddy and Kovasznay exact solutions of the Navier-Stokes equations are also exact solutions of both the NS-α-deconvolution and Leray-α-deconvolution models, but with modified pressures that converge to the Navier-Stokes pressure solution as $\alpha \rightarrow 0$ or the order of deconvolution tends to infinity. The existence of these exact model solutions will provide for better benchmark testing and validation of numerical codes and also shows that the models preserve these special structures.

1. Introduction

The Leray-α and NS-α models and variations thereof have become of significant interest in both the mathematical and engineering communities interested in reduced-order fluid flow modeling. It is the purpose of this paper to derive exact solutions for these models, specifically those of Taylor-eddy and Kovasznay type, both for the purpose of providing better benchmark solutions for computational testing and to show that these models preserve some of the special structures of Navier-Stokes solutions. Solutions of Taylor-eddy type have been shown to exist for the Stolz-Adams approximate deconvolution model by Layton in [1] and for the Rational model by Berselli in [2], thus showing that existence of such solutions for α models is important for model comparisons. To our knowledge, no other model has been shown to admit exact Kovasznay solutions.

Denoting by *overbar* the α-filter $\overline{\phi} := (-\alpha^2\Delta + I)^{-1}\phi$, the models are defined by the following:

Leray-α

$$\begin{gathered} v_t + \overline{v} \cdot \nabla v + \nabla q - \mathrm{Re}^{-1}\Delta v = f \\ \nabla \cdot \overline{v} = 0, \end{gathered} \tag{1}$$

NS-α

$$\begin{gathered} v_t + (\nabla \times v) \times \overline{v} + \nabla Q - \mathrm{Re}^{-1}\Delta v = f \\ \nabla \cdot \overline{v} = 0. \end{gathered} \tag{2}$$

In this work, we will consider these models in $\mathbb{R}^d$, $d = 2$ or 3. The solutions we develop will satisfy the models pointwise, and so the models could also be equipped with boundary conditions, provided they are consistent with the solutions.

The Leray model was developed by Leray in 1934 (but using a Gaussian filter instead of the α-filter) as a theoretical tool to better understand the Navier-Stokes equations [3]. The model was then revisited by Cheskidov et al. in [4] with the α-filter, and they proved fundamental properties of the model including well-posedness and agreement of the energy spectrum with that of true fluid flow on the large scales and an increased rate on the small scales (thus showing that the model is more computable). Work in [5] proved a microscale for the model, which better quantified its advantage in computability versus the Navier-Stokes equations. All of these properties are also valid for NS-α [6, 7], but NS-α also has several advantages over Leray-α from the theoretical point of view such as helicity conservation [6], frame invariance [8], and adherence to Kelvin's circulation theorem [6]. Numerous numerical studies of these models and their variants [9–16] can be found in the literature, which find these models to be successful, although to varying degrees depending on the test problems and choice of discretization.

It is well known that both of these models can improve their accuracy, without losing the desirable properties discussed above, by changing the regularization to include approximate deconvolution [14]. Though there are several ways to apply this idea, the most common is van Cittert deconvolution. Denoting the α-filter also by F (so that $F\phi := \overline{\phi}$), for an integer $N \geq 0$ van Cittert approximate deconvolution is defined by

$$D_N := \sum_{n=0}^{N} (I - F)^n. \tag{3}$$

The idea of using approximate deconvolution in fluid flow models was pioneered by Stolz et al. in [17–19], and it was proven in [20] that, for smooth ϕ,

$$\phi - D_N\overline{\phi} = (-1)^{N+1}\alpha^{2N+2}\Delta^{N+1}F^{N+1}\phi. \tag{4}$$

Hence for smooth ϕ, the operator D_N acts as an approximate inverse to the α-filter. The strategy for improving accuracy in the α models above is to simply replace the regularization F with $D_N F$, which yields the Leray-α-deconvolution and NS-α-deconvolution models, respectively, as follows:

Leray-α-deconvolution model

$$\begin{gathered} v_t + D_N\overline{v}\cdot\nabla v + \nabla q - \mathrm{Re}^{-1}\Delta v = f \\ \nabla\cdot D_N\overline{v} = 0, \end{gathered} \tag{5}$$

NS-α-deconvolution model

$$\begin{gathered} v_t + (\nabla\times v)\times D_N\overline{v} + \nabla Q - \nu\Delta v = f \\ \nabla\cdot D_N\overline{v} = 0. \end{gathered} \tag{6}$$

Note that, since $D_0 = I$, considering these α-deconvolution models includes the "usual" α models as well.

This paper will show the existence of exact solutions of both Taylor-eddy and Kovasznay type for these models. We begin with Taylor-eddy solutions in Section 2 and then Kovasznay solutions in Section 3.

2. Taylor-Eddy Solutions

Taylor-eddy solutions are defined to be Navier-Stokes solutions of the form

$$\begin{gathered} u(x,t) = e^{-\mathrm{Re}^{-1}\lambda t}\phi(x), \\ \nabla p = -u\cdot\nabla u, \end{gathered} \tag{7}$$

where

$$-\Delta\phi = \lambda\phi, \qquad \nabla\cdot\phi = 0. \tag{8}$$

It is easy to verify that this provides a Navier-Stokes solution, since $\nabla\times(u\cdot\nabla u) = 0$ which guarantees the existence of p, and a simple calculation shows that $u_t = \mathrm{Re}^{-1}\Delta u$, thus providing

$$u_t + u\cdot\nabla u + \nabla p - \mathrm{Re}^{-1}\Delta u = 0, \qquad \nabla\cdot u = 0. \tag{9}$$

These solutions include well-known cases of Navier-Stokes solutions such as the Chorin solution [21] and the Ethier-Steinman solution [22]. The model solutions below will be defined in terms of this Navier-Stokes solution (u, p).

We begin with the Leray-α-deconvolution model solution.

Lemma 1. *The functions (v, q) defined by $v = u$ and $\nabla q = -D_N\overline{u}\cdot\nabla u$ are exact solutions of the Leray-α-deconvolution model with $f = 0$.*

Proof. Since $v = u$, it is already established that $v_t = u_t = \mathrm{Re}^{-1}\Delta u = \mathrm{Re}^{-1}\Delta v$; it remains only to show existence of q. Starting from the definition of ϕ from (8), we filter both sides with the α-filter to get

$$\lambda\overline{\phi} = -\Delta\overline{\phi}. \tag{10}$$

Next, multiply both sides by α^2 and use the definition of the filter to find that

$$\lambda\alpha^2\overline{\phi} = -\alpha^2\Delta\overline{\phi} = \phi - \overline{\phi}, \tag{11}$$

and so

$$\overline{\phi} = \frac{1}{1+\lambda\alpha^2}\phi. \tag{12}$$

Hence $D_N\overline{\phi}$ is a constant multiple of ϕ, and thus $D_N\overline{v}$ is a constant multiple of v, which gives us that $\nabla\times(D_N\overline{v}\cdot\nabla v) = 0$ since it is established from Navier-Stokes case that

$$\nabla\times(v\cdot\nabla v) = \nabla\times(u\cdot\nabla u) = 0. \tag{13}$$

This implies that $(D_N\overline{u}\cdot\nabla u)$ is potential and thus proves the existence of q. □

Remark 2. A calculation can verify that the pressure solution q has the explicit form

$$q = \left(1 - \left(\frac{\lambda\alpha^2}{1+\lambda\alpha^2}\right)^{N+1}\right)p. \tag{14}$$

Thus we observe that the velocity solution is independent of α and N, but the pressure solution does depend on these parameters. Moreover, as $\alpha \to 0$ or $N \to \infty$, the model pressure converges to the Navier-Stokes solution's pressure.

Next, we consider the NS-α-deconvolution model.

Lemma 3. *The functions (v, q) defined by $v = u$ and $\nabla Q = -(\nabla\times u)\times D_N\overline{u}$ are exact solutions of the NS-α-deconvolution model with $f = 0$.*

Proof. Similar to the Leray case, $v_t = \mathrm{Re}^{-1}\Delta v$; so we need only to establish the existence of q to prove the lemma. From the vector identity

$$v\cdot\nabla v = (\nabla\times v)\times v + \frac{1}{2}\nabla|v|^2, \tag{15}$$

we have that

$$\nabla \times ((\nabla \times v) \times v) = \nabla \times (v \cdot \nabla v) - \nabla \times \left(\frac{1}{2}\nabla |v|^2\right) = 0, \tag{16}$$

since from the Navier-Stokes case the identity $\nabla \times (u \cdot \nabla u) = 0$ and the second term vanishes due to symmetry of mixed partials. Combining this result with the fact established in the Leray solution proof that, for such a v and ϕ, $D_N \overline{v}$ is a scalar multiple of v, we immediately get that $(\nabla \times v) \times D_N \overline{v}$ is potential, which proves the existence of Q. □

Remark 4. A calculation can verify that the pressure solution Q has the explicit form

$$Q = \left(1 - \left(\frac{\lambda\alpha^2}{1 + \lambda\alpha^2}\right)^{N+1}\right)\left(p + \frac{1}{2}|u|^2\right). \tag{17}$$

Thus we observe that the velocity solution is independent of α and N, but the pressure solution does depend on these parameters. Moreover, as $\alpha \to 0$ or $N \to \infty$, the model pressure converges to the Navier-Stokes solution's Bernoulli pressure.

3. Kovasznay Solutions

Next, we consider exact model solutions for Kovasznay flow [23]. The exact solution for Navier-Stokes Kovasznay flow is the following steady solution defined in terms of a fixed constant λ which depends on the Reynolds number:

$$\begin{gathered} u_1 = 1 - e^{\lambda x}\cos(2\pi y), \\ u_2 = \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y), \\ p = p_0 - \frac{1}{2}e^{2\lambda x}, \quad \text{where } p_0 \text{ is an arbitrary constant}, \\ \lambda = \frac{\mathrm{Re}}{2} - \sqrt{\frac{\mathrm{Re}^2}{4} + 4\pi^2}. \end{gathered} \tag{18}$$

It will be helpful for the proofs that follow to demonstrate how to verify that (18) is an exact solution for the Navier-Stokes equations with $f = 0$. Since the solution is independent of time, $u_t = 0$ is obvious. Calculations, which use the identity $\lambda^2 - 4\pi^2 = \lambda\,\mathrm{Re}$, provide

$$\begin{gathered} u \cdot \nabla u = \lambda \begin{pmatrix} e^{2\lambda x} + 1 + u_1 \\ u_2 \end{pmatrix}, \\ -\mathrm{Re}^{-1}\Delta u = -\lambda \begin{pmatrix} 1 + u_1 \\ u_2 \end{pmatrix}. \end{gathered} \tag{19}$$

Thus, we have

$$u \cdot \nabla u - \mathrm{Re}^{-1}\Delta u = \begin{pmatrix} \lambda e^{2\lambda x} \\ 0 \end{pmatrix} = -\nabla p - u_t, \tag{20}$$

which shows that (18) satisfies the Navier-Stokes equations exactly.

Next, we consider Leray-α-deconvolution solutions.

Lemma 5. *Given $\alpha > 0$, $N \geq 0$, define the velocity v and pressure q by*

$$\begin{gathered} v_1 = 1 - e^{\lambda x}\cos(2\pi y), \\ v_2 = \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y), \\ q = q_0 - \frac{1 - \left(-\alpha^2\lambda\,\mathrm{Re} / \left(1 - \alpha^2\lambda\,\mathrm{Re}\right)\right)^{N+1}}{2} e^{2\lambda x} \\ \text{where } q_0 \text{ is an arbitrary constant}, \\ \lambda = \frac{\mathrm{Re}}{2} - \sqrt{\frac{\mathrm{Re}^2}{4} + 4\pi^2}. \end{gathered} \tag{21}$$

Then v and q are solutions to the Leray-α-deconvolution model with $f = 0$.

Proof. We first observe that

$$\overline{v} = \begin{pmatrix} 1 - \dfrac{1}{1 - \alpha^2\lambda\,\mathrm{Re}} e^{\lambda x}\cos(2\pi y) \\ \dfrac{\lambda}{2\pi}\dfrac{1}{1 - \alpha^2\lambda\,\mathrm{Re}} e^{\lambda x}\sin(2\pi y) \end{pmatrix} \tag{22}$$

satisfies the filter equation, $-\alpha^2\Delta\overline{v} + \overline{v} = v$.

For notational convenience, denote the filtering operation also by F, so that $F\phi := \overline{\phi}$, and also observe that

$$F^N v = \begin{pmatrix} 1 - \left(\dfrac{1}{1 - \alpha^2\lambda\,\mathrm{Re}}\right)^N e^{\lambda x}\cos(2\pi y) \\ \dfrac{\lambda}{2\pi}\left(\dfrac{1}{1 - \alpha^2\lambda\,\mathrm{Re}}\right)^N e^{\lambda x}\sin(2\pi y) \end{pmatrix}. \tag{23}$$

Using these filtering solutions and the identity (4), we get that

$$D_N \overline{v} = \begin{pmatrix} 1 - \left(1 - \left(\dfrac{-\alpha^2\lambda\,\mathrm{Re}}{1 - \alpha^2\lambda\,\mathrm{Re}}\right)^{N+1}\right) e^{\lambda x}\cos(2\pi y) \\ \dfrac{\lambda}{2\pi}\left(1 - \left(\dfrac{-\alpha^2\lambda\,\mathrm{Re}}{1 - \alpha^2\lambda\,\mathrm{Re}}\right)^{N+1}\right) e^{\lambda x}\sin(2\pi y) \end{pmatrix}. \tag{24}$$

From here, since $(1 - (-\alpha^2\lambda\,\mathrm{Re}/(1 - \alpha^2\lambda\,\mathrm{Re}))^{N+1})$ is a fixed constant with respect to x and y, the same calculation that shows (18) is a Navier-Stokes solution will verify that (21) is a Leray-α-deconvolution solution. □

Remark 6. Hence we have proven that the velocity solution for Leray-α-deconvolution is the same as for Navier-Stokes. As $\alpha \to 0$ or as $N \to \infty$, we observe that the Leray-α-deconvolution pressure solution converges to the Navier-Stokes pressure solution.

We now consider NS-α-deconvolution solutions for Kovasznay flow. We note here that the pressure solution should be considered an approximation of the Bernoulli pressure, since the model is written in rotational form.

Lemma 7. *Given $\alpha > 0$, $N \geq 0$, define the velocity v and pressure Q by*

$$v_1 = 1 - e^{\lambda x}\cos(2\pi y),$$

$$v_2 = \frac{\lambda}{2\pi}e^{\lambda x}\sin(2\pi y),$$

$$\begin{aligned} Q = q_0 &- \frac{1 - \left(-\alpha^2\lambda\,\mathrm{Re}/\left(1-\alpha^2\lambda\,\mathrm{Re}\right)\right)^{N+1}}{2}e^{2\lambda x} \\ &- \frac{1}{2}\left(1 - \left(\frac{-\alpha^2\lambda\,\mathrm{Re}}{1-\alpha^2\lambda\,\mathrm{Re}}\right)^{N+1}\right)|v|^2 + D_N\bar{v}\cdot v, \end{aligned} \tag{25}$$

$$\lambda = \frac{\mathrm{Re}}{2} - \sqrt{\frac{\mathrm{Re}^2}{4} + 4\pi^2},$$

where q_0 is an arbitrary constant, and $D_N\bar{v}$ is given explicitly in (24). *Then v and Q satisfy the NS-α-deconvolution model equations.*

Proof. We begin with the vector identity in $\mathbb{R}^3$ as follows:

$$(\nabla\times a)\times b = b\cdot\nabla a + (\nabla b)^T a - \nabla(a\cdot b). \tag{26}$$

Taking $b = D_N\bar{v}$ and $a = v$, we have that

$$(\nabla\times v)\times D_N\bar{v} = D_N\bar{v}\cdot\nabla v + \left(\nabla D_N\bar{v}\right)^T v - \nabla\left(D_N\bar{v}\cdot v\right). \tag{27}$$

Setting $\gamma := (1 - (-\alpha^2\lambda\,\mathrm{Re}/(1-\alpha^2\lambda\,\mathrm{Re}))^{N+1})$, it is clear to observe from (24) that

$$\left(\nabla D_N\bar{v}\right) = \gamma\nabla v, \tag{28}$$

and thus

$$\left(\nabla D_N\bar{v}\right)^T v = \gamma(\nabla v)^T v = \frac{\gamma}{2}\nabla|v|^2, \tag{29}$$

with the last inequality being a vector identity. Setting

$$q = q_0 - \frac{1 - \left(-\alpha^2\lambda\,\mathrm{Re}/\left(1-\alpha^2\lambda\,\mathrm{Re}\right)\right)^{N+1}}{2}e^{2\lambda x}, \tag{30}$$

we have that

$$\begin{aligned} &v_t + (\nabla\times v)\times D_N\bar{v} + \nabla q - \mathrm{Re}^{-1}\Delta v \\ &\quad = 0 + D_N\bar{v}\cdot\nabla v + \frac{\gamma}{2}\nabla|v|^2 - \nabla\left(D_N\bar{v}\cdot v\right) \\ &\qquad + \nabla q - \mathrm{Re}^{-1}\Delta v \\ &\quad = \frac{\gamma}{2}\nabla|v|^2 - \nabla\left(D_N\bar{v}\cdot v\right), \end{aligned} \tag{31}$$

with the last equality holding by using that (v, q) is a Leray-α-deconvolution solution. Thus setting $Q = q-(\gamma/2)|v|^2+D_N\bar{v}\cdot v$ verifies that (v, Q) is a NS-α-deconvolution solution. □

Remark 8. Hence we have proven that the velocity solution for NS-α-deconvolution is the same as for Navier-Stokes. As $\alpha \rightarrow 0$ or $N \rightarrow \infty$, the pressure Q converges to the Navier-Stokes Bernoulli pressure $p + (1/2)|u|^2$. Since NS-α-deconvolution is written in rotational form, this is the correct pressure to converge to.

4. Conclusions

We have shown that the velocity solutions of the well-known Taylor-eddy and Kovasznay exact Navier-Stokes solutions are also exact solutions for the Leray-α-deconvolution and NS-α-deconvolution models, for arbitrary filtering radius $\alpha > 0$ and deconvolution order $N \geq 0$. The pressure (or Bernoulli pressure) solutions of the models differ from the Navier-Stokes pressure but converge to the Navier-Stokes pressure as either $\alpha \rightarrow 0$ or $N \rightarrow \infty$. Such solutions are useful from the numerical viewpoint, as they will provide better testing of numerical codes. From the theoretical viewpoint, that these models admit these solutions shows that they preserve some of the Navier-Stokes solution structure.

An extension of this work to cases of flows with boundaries will be pursued. Recent work of Holm et al. in [24] showed that NS-α admits exact solutions in certain settings, and thus important extensions of that work will be to study what effect deconvolution may have and also to study the Leray models in such settings.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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