Research Article

Totally Umbilical Lightlike Hypersurfaces in Robertson-Walker Spacetimes

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We study the problem of lightlike hypersurface immersed into Robertson-Walker (RW) spacetimes in this paper, where the screen bundle of the hypersurface has constant higher order mean curvature. We consider the following question: under what conditions is the compact lightlike hypersurface totally umbilical? Our approach is based on the relationship between the lightlike hypersurface with its screen bundle and the Minkowski formulae for the screen bundle.

1. Introduction

The mathematical interests for the study of spacelike hypersurfaces in spacetimes began in the seventies with the works of Cheng and Yau [1], Brill and Flaherty [2], Choquet-Bruhat [3], and later on with some other authors, such as in [4–6]. Moreover, the study of such hypersurfaces is also of interest from a physical point of view, because of its relation to several problems in general relativity. More recently, there has been an increasing interest in the study of spacelike hypersurfaces with constant higher order mean curvature, such as in [7–9]. At the same time, in [10, 11] there are system research works about the lightlike hypersurface of semi-Riemannian manifolds. In this paper, based on the previous result, we want to make an attempt to study the lightlike hypersurfaces of spacetimes. First of all, we are interested in the study of lightlike hypersurfaces in conformally stationary spacetimes, and the screen bundle with constant higher order mean curvature.

First, we use the Newton transformations as the main analytical tool and study the Minkowski-type integral formulas of the lightlike hypersurface. The use of these kinds of formulas in the Lorentzian manifold was first started by Montiel in [12] for the spacelike hypersurface with constant mean curvature in de Sitter spacetimes, and it was continued by Alías et al. [13, 14] for more general spacetimes. Higher-order Minkowski formula for hypersurface was first obtained by Hsiung in [15] in Euclidean space, and by Bivens in [16]

in the Euclidean sphere and hyperbolic space. In this paper, we obtain two Minkowski-type integral formulas about the higher-order mean curvature of the lightlike hypersurface as follows, which are called Minkowski formulas I and II.

The Minkowski Formula I. Let $(M, g, S(TM))$ be a compact lightlike hypersurface of a conformally stationary spacetime $\bar{M}$ of constant sectional curvature, on which the screen bundle $S(TM)$ is integrable and Ricci tensor of the induced connection ∇ is symmetric; then the following conclusion holds:

$$\int_M (\phi H_r + \langle \partial_t, \xi \rangle H'_{r+1} + \langle \partial_t, N \rangle H_{r+1}) dV = 0. \quad (1)$$

The Minkowski Formula II. Let $(M, g, S(TM))$ be a compact lightlike hypersurface immersed into a conformally stationary spacetime $(\bar{M}, g)$ of constant sectional curvature, where the screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric; then

$$\int_M (\phi H'_r + \langle \partial_t, \xi \rangle H'_{r+1}) dV = 0. \quad (2)$$

Second, by using the Minkowski-type integral formulas and divergence theorem, we get some sufficient conditions for lightlike hypersurfaces of conformally stationary spacetimes which are totally umbilical. The screen bundle of the lightlike hypersurface is spacelike; we study the lightlike hypersurfaces

of conformally stationary spacetimes via the screen bundle and get our main results (cf. Theorems 10–13).

This paper is organized as follows. In Section 2, we state some preliminary knowledge related to this paper. In Section 3, we introduce the shape operator and the corresponding Newton transformations and derive some formulas about the divergence of Newton transformations. In Section 4, based on the results in Section 3, we obtain the Minkowski-type integral formulas I and II of lightlike hypersurfaces. In Section 5, as the application of Minkowski formulas, we obtain some sufficient conditions for lightlike hypersurfaces totally umbilical when the screen bundle is with constant higher order mean curvature.

2. Preliminaries

Let F^{n+1} be an $(n+1)$ -dimensional ($n \geq 1$) Riemannian manifold and let I be a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$). We denote $\overline{M}^{n+2}$ by the $(n+2)$ -dimensional product manifold endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -dt^2 + f^2(t) \langle \cdot, \cdot \rangle_F, \quad (3)$$

where $f > 0$ is a positive smooth function on I , and $\langle \cdot, \cdot \rangle_F$ is the Riemannian metric on F^{n+1} ; then we have that $\overline{M}$ is a Lorentzian warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber F^{n+1} , $\langle \cdot, \cdot \rangle_F$, and warping function f , when the sectional curvature of Riemannian factor F^{n+1} is constant; then we call $-I \times F^{n+1}$ as Robertson-Walker (RW) spacetime.

Consider smooth immersion $\psi : M^{n+1} \hookrightarrow -I \times F^{n+1}$ of an $n+1$ -dimensional connected lightlike manifold M into a RW spacetime, where the induced metric on M is g ; then we have that M^{n+1} is a lightlike hypersurface.

Let $\partial_t = (\partial/\partial t)_{(t,x)}$, $(t, x) \in -I \times F^{n+1}$ be the unitary timelike conformal vector field globally defined on the RW spacetime $\overline{M}$; then consider lightlike hypersurface $(M, g, S(TM))$; there exists a pair lightlike vector fields $\{N, \xi\}$ defined on M which satisfied the following formulas:

$$\langle \xi, \xi \rangle = \langle N, N \rangle = 0, \quad \langle \xi, N \rangle = 1, \quad (4)$$

$$\langle \partial_t, \xi \rangle \neq 0, \quad \langle \partial_t, N \rangle \neq 0. \quad (5)$$

There exists a unitary timelike vector field $T = N - \xi$ defined on M which is the same time-orientation as ∂_t ; we will refer to T as the future-pointing Gauss map of the hypersurface; its opposite is the past-pointing Gauss map, where we can suppose $\langle \xi, \partial_t \rangle > 0$ on M in the future-pointing.

Now we denote ∇, ∇^* to be the connection of M and $S(TM)$, respectively, P the projection morphism of TM onto $S(TM)$; then the Gauss and Weingarten formula for the hypersurface in $\overline{M}$ are given as follows:

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) = \nabla_X Y + B(X, Y)N, \\ \overline{\nabla}_X N &= -A_N X + \nabla_X^t N = -A_N X + \tau(X)N, \\ \nabla_X^* PY &= \nabla_X^* PY + h^*(X, PY) = \nabla_X^* PY + C(X, PY)\xi, \\ \nabla_X \xi &= -A_\xi^* X - \tau(X)\xi, \quad A_\xi^* \xi = 0, \end{aligned} \quad (6)$$

where $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(\text{Rad}TM)$, $N \in \Gamma(\text{tr}(TM))$, h and h^* are the second fundamental forms of M and $S(TM)$, B and C are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, A_N and A_ξ^* are the local shape operator on M and $S(TM)$, respectively, and τ is the 1-form on TM .

Definition 1. $(M, g, S(TM))$ is a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. A point $u \in M$ is said to be umbilical if

$$B_u(X_u, Y_u) = kg_u(X_u, Y_u), \quad X_u, Y_u \in T_u M, \quad (7)$$

where B is the local second fundamental form of M defined around u and $k \in \mathbb{R}$. One says M is totally umbilical if any point of M is umbilical.

Then it is easy to see that M is totally umbilical if and only if on M there exists a smooth function ρ on M such that

$$B(X, Y) = \rho g(X, Y), \quad (8)$$

for all vector fields $X, Y \in \Gamma(TM)$.

Definition 2. Let $(M, g, S(TM))$ be a lightlike hypersurface; then the screen bundle is totally umbilical if on any coordinate neighbourhood $U \subset M$ there exist a smooth function λ such that

$$C(X, PY) = \lambda g(X, PY), \quad X, Y \in \Gamma(TM). \quad (9)$$

In case $\lambda = 0$ ($\lambda \neq 0$) on U one says $S(TM)$ is totally geodesic (properly totally umbilical).

3. The Shape Operator and Newton Transformation

In this section we will introduce the shape operators and the corresponding Newton transformations T_r .

From [10] we have the relationship between two local second fundamental forms with the shape operators

$$B(X, Y) = \langle A_\xi^* X, Y \rangle, \quad C(X, PY) = \langle A_N X, PY \rangle, \quad (10)$$

$$B(X, \xi) = \langle A_\xi^* X, N \rangle = 0, \quad \langle A_N X, N \rangle = 0, \quad (11)$$

where $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$, and $N \in \Gamma(\text{tr}TM)$. From (11) we have that $A_N X$ belong to screen bundle $S(TM)$, if $A_N \xi = 0$ and $S(TM)$ is an integrable distribution; then it is easy to know A_N is self-adjoint linear operator in each tangent space $T_p M$; its eigenvalues $k_1, k_2, \dots, k_n$ and k_{n+1} are the principal curvature of the lightlike hypersurface. If $\{e_1, \dots, e_n, \xi\}$ are the eigenvector of A_N corresponding to the eigenvalue $k_1(p), k_2(p), \dots, k_n(p)$ and $k_{n+1}(p)$ on $T_p M$, respectively, where $A_N(e_i)_p = k_i(p)e_i$, $1 \leq i \leq n$ and $k_{n+1} = 0$, then the matrix of A_N with respect to the orthogonal basis $\{e_1, \dots, e_n, \xi\}$ is given by

$$\begin{pmatrix} k_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & k_n & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (12)$$

Now we restrict A_N on $S(TM)$; then it can be denoted by

$$\begin{pmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{pmatrix}. \quad (13)$$

Proposition 3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$; then the screen bundle is totally umbilical if and only if there exist a smooth function λ such that*

$$A_N X = \lambda X, \quad A_N \xi = 0, \quad X \in \Gamma(S(TM)). \quad (14)$$

Proposition 4 (see [10]). *$(M^{n+1}, g, S(TM))$ is a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant sectional curvature c and totally umbilical screen bundle $S(TM)$; that is, there exists a smooth function λ making $A_N X = \lambda X$ and $A_N \xi = 0$. Then M is totally umbilical and immersed into $\bar{M}$ if $\lambda \neq 0$. Under this condition, we say M is totally geodesic immersed into $\bar{M}$ if and only if λ is a solution of the partial differential equation $\xi(\lambda) - \lambda\tau(\xi) + c = 0$.*

The proof of Proposition 4 has been given by Duggal and Bejancu in [10].

From Proposition 4 and the characters of totally umbilical lightlike hypersurface, we can obtain the following corollary.

Corollary 5. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}(c), \bar{g})$, where the sectional curvature of ambient space $\bar{M}$ is constant c . Now one supposes the screen bundle $S(TM)$ is integrable; $A_N \xi = 0$ and $A_N X = \lambda X (\lambda \neq 0)$, where λ is a smooth function. Then M is totally umbilical; that is, $A_\xi Y = \rho Y$, where $X, Y \in \Gamma(S(TM))$. Besides ρ, λ , and c satisfied the following equation:*

$$\xi(\lambda) - \lambda\tau(\xi) + c = \rho\lambda, \quad \xi \in \Gamma(TM^\perp). \quad (15)$$

From Corollary 5 we know that M^{n+1} is a totally umbilical hypersurface in $\bar{M}(c)$; if there exists a smooth function λ such that $A_N = \lambda I$ on screen bundle and $A_N \xi = 0$, where I denotes the identity in $S(TM)$.

Associated with the shape operator A_N there are n algebraic invariants given by S_r as follows:

$$S_r(p) = \sigma_r(k_1(p), k_2(p), \dots, k_n(p)), \quad 1 \leq r \leq n, \quad (16)$$

where $\sigma : R^n \rightarrow R$ are the elementary symmetric functions in R^n given by

$$\sigma_r(x_1, x_2, \dots, x_n) = \sum_{i_1 < \dots < i_r} x_{i_1} \cdots x_{i_r}. \quad (17)$$

Then the characteristic polynomial of A_N restrict on $S(TM)$ can be denoted by

$$\det(tI - A_N) = \sum_{r=0}^n (-1)^r S_r t^{n-r}, \quad (18)$$

where $S_0 = 1$. Now we denote H_r by $\binom{n}{r} H_r = (-1)^r S_r$; H_r is the r -mean curvature of the screen bundle with respect

to the shape operator A_N . In particular, when $r = 1$, $H_1 = (1/n) \text{tr}(A_N)$ is the main extrinsic mean curvature of the screen bundle with respect to A_N ; when $r = 2$, H_2 defines a 2-mean curvature which is intrinsic. Thus, if all k_i are nonzero, we can choose a proper orientation to make $H_1 > 0$ and H_2 unchanged.

The curvature tensor $\bar{R}$ of the ambient space $\bar{M}$ can be written as follows:

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X \\ &\quad + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \end{aligned} \quad (19)$$

$$R(X, Y)Z = (\bar{R}(X, Y)Z)^T - A_{h(X, Z)}Y + A_{h(Y, Z)}X,$$

where $(\nabla_X h)(Y, Z) = \nabla_X^t(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, $(\bar{R}(X, Y)Z)^T$ is the tangential component of $\bar{R}(X, Y)Z$.

Now we denote $\{E_0, E_i, N\}$ to be the basis on M and Ric the Ricci tensor of M , where $1 \leq i \leq n$; that is, $\xi = E_0$, $\{E_i\}$ the basis of $S(TM)$; then from [10] we can obtain that

$$\begin{aligned} \text{Ric}(E_i, E_j) &= \bar{\text{Ric}}(E_i, E_j) - \bar{g}(\bar{R}(E_i, E_0)E_j, N) \\ &\quad - g^{kt}(C(E_k, E_t)B(E_i, E_j) \\ &\quad - C(E_i, E_t)B(E_j, E_k)). \end{aligned} \quad (20)$$

So we obtain

$$\begin{aligned} \text{Ric}(E_i, E_j) - \text{Ric}(E_j, E_i) &= g^{kt} \{C(E_i, E_t)B(E_j, E_k) - C(E_j, E_t)B(E_i, E_k)\} \\ &\quad + \bar{g}(\bar{R}(E_j, E_i)E_0, N) \\ &= A_j^k B_{ki} - A_i^k B_{kj} + \bar{R}_{0ij}^0 \\ &= 2d\tau \left(\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_i} \right), \end{aligned} \quad (21)$$

where $1 \leq k, t \leq n$, τ is the 1-form induced by $S(TM)$. When $i = 0$ we have

$$\text{Ric}(E_0, E_j) - \text{Ric}(E_j, E_0) = -A_0^k B_{kj} + \bar{R}_{00k}^0 = 2d\tau(E_0, E_j). \quad (22)$$

By using the equations above we obtain that $d\tau = 0$ when the Ricci tensor Ric is symmetric; then we have the following proposition.

Proposition 6. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a RW spacetime which is equipped with a conformal vector field ∂_t . If the Ricci tensor of the induced connection ∇ is symmetric, then there exists a pair $\{\xi, N\}$ on U such that the corresponding 1-form τ vanishes on any $U \subset M$; besides $\langle \partial_t, \xi \rangle$ and $\langle \partial_t, N \rangle$ are all nonzero.*

Proof. It is already known that there exist vector fields ξ and N such that $\langle \partial_t, \xi \rangle \neq 0$, $\langle \partial_t, N \rangle \neq 0$.

Since the Ricci tensor of ∇ is symmetric, then we have that the 1-form τ induced by $S(TM)$ is closed; that is, $d\tau \equiv 0$; so we have $\tau = \alpha$.

If $\tau = \alpha = 0$, then the theorem have been proved.

If $\tau = \alpha \neq 0$, we take $\bar{\xi} = \alpha\xi$, $\bar{N} = (1/\alpha)N$, the corresponding 1-form is $\bar{\tau}$, where

$$\begin{aligned}\bar{\tau}(X) &= \bar{g}(\nabla_X^t \bar{N}, \bar{\xi}) = \alpha X\left(\frac{1}{\alpha}\right) + \tau(X), \\ \tau(X) &= \bar{\tau}(X) - \alpha X\left(\frac{1}{\alpha}\right) = \bar{\tau}(X) - X(\log \alpha).\end{aligned}\quad (23)$$

By using Poincaré lemma we obtain $\tau(X) = X(f)$, where $f \in C^\infty(U)$. Then we can let $\alpha = e^f$, so $\bar{\tau}(X) = 0$ for any $X \in \Gamma(TM_U)$. Since $\langle \partial_t, \bar{\xi} \rangle = \alpha \langle \partial_t, \xi \rangle \neq 0$, $\langle \partial_t, \bar{N} \rangle = (1/\alpha) \langle \partial_t, N \rangle \neq 0$, then the $\{\bar{\xi}, \bar{N}\}$ is the vector fields satisfying the proposition. $\square$

From Proposition 6 we obtain that if the Ricci tensor of ∇ is symmetric, then we have

$$\begin{aligned}\langle \bar{R}(X, Y)PZ, N \rangle &= \langle (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ), \xi \rangle \\ &= \langle (\nabla_X A_N)Y - (\nabla_Y A_N)X, PZ \rangle,\end{aligned}\quad (24)$$

where $(\nabla_X C)(Y, PZ) = X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla_X^* PZ)$, $\nabla_X A_N$ is the covariant derivative of A_N .

Because the ambient space is RW spacetime, the sectional curvature is constant; then when $X, Y \in \Gamma(S(TM))$, we have that $\bar{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}$ has no component in $\Gamma(TM^\perp)$. Now (24) becomes

$$(\nabla_X A_N)Y = (\nabla_Y A_N)X. \quad (25)$$

Now we introduce the Newton transformations

$$T_r : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM)), \quad 0 \leq r \leq n, \quad (26)$$

which arises from the shape operator A_N . According to the definition of the r -mean curvature of screen bundle $S(TM)$, the Newton transformations are given by

$$T_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A_N + \cdots + \binom{n}{1} H_1 A_N^{r-1} + A_N^r, \quad (27)$$

where I denotes the identify in $\Gamma(S(TM))$, $1 \leq r \leq n$ and $T_0 = I$. Besides,

$$T_r = \binom{n}{r} H_r I + A_N T_{r-1}. \quad (28)$$

According to the Cayley-Hamilton theorem, we have $T_n = 0$. When r is even, the definition of T_r does not depend on the chosen Gauss map; when k is odd there is a change of sign in the definition of T_r .

It is easy to know that $T_r(p)$ is also a self-adjoint linear operator with respect to $A_N(p)$; besides, $A_N(p)$ and

$T_r(p)$ can be a simultaneous diagonalization. If $\{e_1, \dots, e_n\}$ is the orthogonal frame on $S(TM)$ which diagonalizes A_N , $A_N(e_i) = k_i(p)e_i$, T_r are all self-adjoint operators with respect to A_N , then we have

$$T_r(e_i) = \lambda_{i,r}(e_i), \quad (29)$$

where

$$\lambda_{i,r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} k_{i_1} \cdots k_{i_r}. \quad (30)$$

Then we obtain

$$\begin{aligned}\text{tr}(T_r) &= c_r H_r, \\ \text{tr}(A_N \circ T_r) &= -c_r H_{r+1},\end{aligned}\quad (31)$$

where $c_r = (n-r) \binom{n}{r}$.

The divergence of T_r on screen bundle is defined by

$$\text{div}_{S(TM)} T_r = \text{tr}(\nabla T_r) = \sum_{i=1}^n \langle \nabla_{e_i} T_r, e_i \rangle. \quad (32)$$

Proposition 7. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a RW spacetime in which the screen bundle $S(TM)$ is integrable and Ricci tensor of the induced connection ∇ is symmetric; when the sectional curvature of the ambient spacetime is constant, then the divergence of Newton transformations T_r is given by the following inductive formula:*

$$\text{div}_{S(TM)} T_0 = 0, \quad (33)$$

$$\text{div}_{S(TM)} T_r = A_N(\text{div}_{S(TM)} T_{r-1}) + \sum_{i=1}^n (\bar{R}(N, T_{r-1}e_i)e_i)^*, \quad (34)$$

where $(\bar{R}(N, X)Y)^*$ denotes the screen bundle component of $\bar{R}(N, X)Y$. Besides, for any vector field $X \in \Gamma(S(TM))$ it satisfies

$$\langle \text{div}_{S(TM)} T_r, X \rangle = \sum_{j=1}^r \sum_{i=1}^n (-1)^j \langle \bar{R}(\xi, T_{r-j}e_i)e_i, A_\xi^{j-1}X \rangle. \quad (35)$$

Proof. It is easy to know $\text{div}_{S(TM)} T_0 = \text{div } I = 0$, when $r \geq 1$, $X, Y \in \Gamma(S(TM))$; we have

$$\begin{aligned}(\nabla_X T_r)Y &= \binom{n}{r} \langle \nabla H_r, X \rangle Y + \nabla_X (A_N T_{r-1})Y \\ &= \binom{n}{r} \langle \nabla H_r, X \rangle Y + (\nabla_X A_N)T_{r-1}Y + A_N((\nabla_X T_{r-1})Y).\end{aligned}\quad (36)$$

Then

$$\begin{aligned}
 \operatorname{div}_{S(TM)} T_r &= \sum_{i=1}^n (\nabla_{e_i} T_r)(e_i) \\
 &= \binom{n}{r} \nabla H_r + \sum_{i=1}^n (\nabla_{e_i} A_N)(T_{r-1} e_i) + A_N (\operatorname{div}_{S(TM)} T_{r-1}).
 \end{aligned} \quad (37)$$

Using (24) we get for any vector field $X \in \Gamma(S(TM))$; we have

$$\begin{aligned}
 \langle (\nabla_{e_i} A_N)(T_{r-1} e_i), X \rangle &= \langle (\nabla_X A_N) e_i, T_{r-1} e_i \rangle \\
 &\quad + \langle \bar{R}(e_i, X) T_{r-1} e_i, N \rangle \\
 &= \langle T_{r-1} ((\nabla_{e_i} A_N) e_i), e_i \rangle \\
 &\quad - \langle \bar{R}(N, T_{r-1} e_i) e_i, X \rangle. \\
 \langle \operatorname{div}_{S(TM)} T_r, X \rangle &= \binom{n}{r} \langle \nabla H_r, X \rangle + \operatorname{tr}(T_{r-1} \nabla_X A_N) \\
 &\quad - \sum_{i=1}^n \langle \bar{R}(N, T_{r-1} e_i) e_i, X \rangle \\
 &\quad + \langle A_N (\operatorname{div}_{S(TM)} T_{r-1}), X \rangle.
 \end{aligned} \quad (38)$$

By using equation (4.4) in [7], we have

$$\operatorname{tr}(T_{r-1} \nabla_X A_N) = \binom{n}{r} \langle \nabla H_r, X \rangle. \quad (39)$$

From (24) and the above formula, we have

$$\begin{aligned}
 \langle \operatorname{div}_{S(TM)} T_r, X \rangle &= -\langle \bar{R}(N, T_{r-1} e_i) e_i, X \rangle + \langle A_N (\operatorname{div}_{S(TM)} T_{r-1}), X \rangle \\
 &= \langle -\bar{R}(N, T_{r-1} e_i) e_i + A_N (\operatorname{div}_{S(TM)} T_{r-1}), X \rangle \\
 &= -\sum_{j=1}^r \sum_{i=1}^n \langle \bar{R}(N, T_{r-j} e_i) e_i, A_N^{j-1} X \rangle.
 \end{aligned} \quad (40)$$

□

Since the sectional curvature of the RW spacetime is constant, then it is easy to know $\langle \bar{R}(N, X)Y, Z \rangle = \langle \bar{R}(N, X)Y, Z \rangle = 0$, for all vector fields $X, Y, Z \in \Gamma(S(TM))$. From (25) and (34) we have $\operatorname{div}_{S(TM)} T_r = 0$, $0 \leq r \leq n$; thus we have the corollary below.

Corollary 8. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a conformally stationary spacetime in which the screen bundle $S(TM)$ is integrable and Ricci tensor of the induced connection ∇ is symmetric; then we have that the Newton transformations on $S(TM)$ are divergence-free: $\operatorname{div}_{S(TM)} T_r = 0$, $0 \leq r \leq n$.*

4. Minkowski Integral Formula for Lightlike Hypersurface in RW Spacetime

In this section, by using divergence theorem we derive some general integral formulae for compact lightlike hypersurface in a RW spacetime $\bar{M}$. In order to get that, we consider the vector field $K(t, x) = f(t)(\partial/\partial t)_{(t,x)}$, $(t, x) \in -I \times_f F^{n+1}$; it determines a nonvanishing future-pointing closed conformal vector field on $-I \times_f F^{n+1}$; besides,

$$\bar{\nabla}_X K = f'(t) X \quad (41)$$

for any vector field X tangent to $-I \times_f F^{n+1}$ at a point (t, x) .

We denote the height function of M by h , which is the restriction of the projection $\pi_I(t, x) = t$ to M ; $h : M \rightarrow I$ is given by $h = \pi_I \circ \psi$. Observe the gradient of π_I on $-I \times_f F^{n+1}$ is given by

$$\bar{\nabla} \pi_I = -\langle \bar{\nabla} \pi_I, \partial_t \rangle \partial_t = -\partial_t; \quad (42)$$

then we can get the gradient of h on M is

$$\nabla h = (\bar{\nabla} \pi_I)^T = -\partial_t^T, \quad (43)$$

where $\partial_t^T = \partial_t - \langle \xi, \partial_t \rangle N$, ∂_t^T is the tangential component of ∂_t . Now we denote $g : I \rightarrow \mathbb{R}$ is an arbitrary primitive of f ; that is, $g' = f > 0$; so we can consider g as a reparametrization of the height function. Then we have

$$\nabla g(h) = f(h) \nabla h = -f(h) \partial_t^T = -K^T, \quad (44)$$

where $K^T = K - \langle K, \xi \rangle N$. From (41) we get

$$\bar{\nabla}_X K = f'(h) X, \quad (45)$$

for any $X \in \Gamma(TM)$.

Since $\nabla_X K^T = f'(h) X + \langle K, \xi \rangle A_N X$ for every $X \in \Gamma(TM)$, then we have

$$\nabla_X (\nabla h) = -\nabla_X K^T = -f'(h) X - \langle K, \xi \rangle A_N X. \quad (46)$$

From $\nabla h = (1/f(h)) \nabla g(h)$ we have

$$\begin{aligned}
 \nabla_X (\nabla h) &= \nabla_X \left(\frac{1}{f(h)} \nabla g(h) \right) \\
 &= -\frac{f'(h)}{f(h)} \nabla h + \frac{1}{f(h)} (-f'(h) X - \langle K, \xi \rangle A_N X) \\
 &= -(\log f)'(h) (\langle \nabla h, X \rangle \nabla h + X) - \langle \partial_t, \xi \rangle A_N X.
 \end{aligned} \quad (47)$$

If we take K^T as the tangential component of K along the hypersurface, we have

$$K^T = K - \langle K, \xi \rangle N, \quad (48)$$

where $\xi \in \Gamma(TM^\perp)$ and $\langle \xi, N \rangle = 1$; then we have

$$\begin{aligned} \nabla_X K^T &= \bar{\nabla}_X (K - \langle K, \xi \rangle N) - B(X, K^T) N \\ &= f'(h) X - \bar{\nabla}_X (\langle K, \xi \rangle N) - B(X, K^T) N \\ &= f'(h) X + \langle K, \xi \rangle A_N X + \langle K, A_\xi^* X \rangle N \\ &\quad - \langle K^T, A_\xi^* X \rangle N \\ &= f'(h) X + \langle K, \xi \rangle A_N X. \end{aligned} \quad (49)$$

Now we have

$$\begin{aligned} \nabla_{e_i} K^T &= \langle f'(h) e_i + \langle K, \xi \rangle A_N e_i, T_r e_i \rangle \\ &= f'(h) \langle e_i, T_r e_i \rangle + \langle K, \xi \rangle \langle A_N T_r e_i, e_i \rangle. \end{aligned} \quad (50)$$

In order to compute the divergence of K^T , we have to expand the definition of operator T_r to the tangential bundle. Now let $A_N \xi = 0$ and $\{e_1, \dots, e_n, \xi\}$ be the local orthogonal frame on M ; then A_N with respect to the basis can be denoted by

$$\begin{pmatrix} k_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & k_n & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}; \quad (51)$$

in this case, we denote it by $\bar{A}_N$ to distinguish it from A_N restricted on $S(TM)$. By a direct calculation we obtain

$$\det(tI - \bar{A}_N) = t \cdot \det(tI - A_N) = \sum_{r=0}^n \binom{n}{r} T_r t^{n-r+1}. \quad (52)$$

Then $\bar{T}_r = \begin{pmatrix} T_r & 0 \\ 0 & \binom{n}{r} H_r \end{pmatrix}$, $\text{div}_M \bar{T}_r = \text{div}_M \binom{n}{r} H_r \cdot \text{div}_M T_r$, $\bar{T}_r e_i = \lambda_{i,r} e_i$. From all the above, we have the theorem below.

Theorem 9. Let $(M^{n+1}, g, S(TM))$ be a compact lightlike hypersurface immersed into a RW spacetime $(\bar{M}, g)$, where the screen bundle $S(TM)$ is integrable, $A_N \xi = 0$, and the Ricci tensor of induced connection ∇ is symmetric. Then the following formula holds:

$$\begin{aligned} \int_M \text{div}_M (\bar{T}_r K^T) dV \\ = \int_M c_r (f'(h) H_r - \langle K, \xi \rangle H_{r+1}) dV = 0. \end{aligned} \quad (53)$$

Proof. Since the sectional curvature of RW spacetime is constant, then we have the calculation about the divergence of $\bar{T}_r K^T$ as follows:

$$\begin{aligned} \text{div}_M (\bar{T}_r K^T) &= \text{div}_{S(TM)} (\bar{T}_r K^T) + \langle \xi, \xi \rangle \langle \bar{\nabla}_\xi \bar{T}_r K^T, \xi \rangle \\ &= \text{div}_{S(TM)} (\bar{T}_r K^T) \\ &= \langle \text{div}_{S(TM)} \bar{T}_r, K^T \rangle + \sum_{i=1}^n \langle \nabla_{e_i} K^T, \bar{T}_r e_i \rangle \\ &= \left\langle \text{div}_{S(TM)} T'_r \cdot \text{div}_{S(TM)} \binom{n}{r} H_r, K^T \right\rangle \\ &\quad + f'(h) \text{tr}(T_r) + \langle K, \xi \rangle \text{tr}(A_N T_r) \\ &= \left\langle \text{div}_{S(TM)} T_r \cdot \text{div}_{S(TM)} \binom{n}{r} H_r, K^T \right\rangle \\ &\quad + c_r (f'(h) H_r - \langle K, \xi \rangle H_{r+1}). \end{aligned} \quad (54)$$

Since the sectional curvature of RW spacetime is constant, by using Proposition 7 we have $\text{div}_{S(TM)} T_r = 0$; then

$$\text{div}_M (\bar{T}_r K^T) = c_r (f'(h) H_r - \langle K, \xi \rangle H_{r+1}). \quad (55)$$

From divergence theorem we have

$$\int_M (f'(h) H_r - \langle K, \xi \rangle H_{r+1}) dV = 0. \quad (56)$$

□

We call the formula (53) as Minkowski-type integral formula.

5. Umbility of the Compact Lightlike Hypersurface in RW Spacetime

In this section, we will derive some application of Minkowski formulae for the case of lightlike hypersurface immersed into a spatially closed RW spacetime, where the screen bundle of the hypersurface has constant r -mean curvature; under this condition by using (35), it is easy to know that if M is totally umbilical, then H_r must be constant.

Theorem 10. Let $\bar{M} = -I \times_f F^{n+1}$ be spatially closed RW spacetime; then every compact lightlike hypersurface immersed into $\bar{M}$, where $A_N \xi = 0$, screen bundle $S(TM)$ is integrable and Ricci tensor of the induced connection is symmetric. If H_1 of the screen bundle with respect to A_N is constant, then the hypersurface must be totally umbilical.

Proof. Using the Theorem 9, It is easy to see that if H'_1 is constant, we can consider $r = 0$, then multiplying the Minkowski-type formula by H_1 we have

$$\int_M (f'(h) H_1 - \langle K, \xi \rangle H_1^2) dV = 0. \quad (57)$$

When $r = 1$, we obtain that the Minkowski formula is

$$\int_M (f'(h) H_1 + \langle K, \xi \rangle H_2) dV = 0. \quad (58)$$

Subtracting the two formulas above we get

$$\int_M \langle K, \xi \rangle (H_1^2 - H_2) dV = 0; \quad (59)$$

from the Cauchy-Schwarz inequality we have

$$H_1^2 - H_2 = \frac{1}{n(n-1)} \left(\sum_{i=1}^n k_i^2 - \frac{1}{n} \left(\sum_{i=1}^n k_i \right)^2 \right) \geq 0, \quad (60)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$. From $\langle K, \xi \rangle > 0$ we have that the equality holds; then there exists constant λ such that $A_N = \lambda I$ on screen bundle. Besides, $A_N \xi = 0$; then by using Corollary 5 we have that the hypersurface is totally umbilical. $\square$

Theorem 11. Let $\bar{M} = -I \times_f F^{n+1}$ be a spatially closed RW spacetime. Assume that M is a compact lightlike hypersurface immersed into $\bar{M}$ with $A_N \xi = 0$; screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. If H_2 is a positive constant, then M is a totally umbilical lightlike hypersurface of $\bar{M}$.

Proof. Using the Theorem 9 and multiplying the formula (53) by H_2' we have

$$\int_M (f'(h) H_2 - \langle K, \xi \rangle H_1 H_2) dV = 0. \quad (61)$$

When $r = 2$, we obtain that the Minkowski formula (53) is

$$\int_M (f'(h) H_2 - \langle K, \xi \rangle H_3) dV = 0. \quad (62)$$

Subtracting the two formulas above we get

$$\int_M \langle K, \xi \rangle (H_1 H_2 - H_3) dV = 0 \quad (63)$$

Since $H_2 > 0$, from (60) we have $H_1^2 \geq H_2 > 0$, so H_1 does not vanish on M . Since H_2 is an intrinsic quantity and it does not depend on the chosen orientation, then we can choose an appropriate orientation such that $H_1 > 0$. From the generalization Cauchy-Schwarz inequality in [17] we have $H_2^2 - H_1 H_3 \geq 0$; then

$$H_1 H_2 - H_3 \geq H_1 H_2 - \frac{H_2^2}{H_1} = \frac{H_2}{H_1} (H_1^2 - H_2) \geq 0, \quad (64)$$

where the equality holds only at the point when $k_1' = k_2' = \dots = k_n'$. Besides, $\langle K, \xi \rangle > 0$, so we have $H_1 H_2 - H_3 \equiv 0$; then it is easy to know that there exists constant λ making that $A_N = \lambda I$ on screen bundle. Since $A_N \xi = 0$, then from Corollary 5 we have that M is totally umbilical. $\square$

Theorem 12. Let $\bar{M} = -I \times_f F^{n+1}$ be a spatially closed RW spacetime. Assume that M is a compact lightlike hypersurface immersed into $\bar{M}$ with $A_N \xi = 0$; screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. If H_r and H_{r-1} are both constants ($1 \leq r \leq n$), then the hypersurface M must be totally umbilical.

Proof. Using Theorem 9 and multiplying the formula (53) by constant H_r , then we obtain

$$\int_M (f'(h) H_{r-1} H_r + \langle \partial_t, \xi \rangle H_r'^2) dV = 0. \quad (65)$$

Multiplying the r -th formula by constant H_{r-1} , then we have

$$\int_M (f'(h) H_{r-1} H_r - \langle K, \xi \rangle H_{r-1} H_{r+1}) dV = 0. \quad (66)$$

Subtracting the two formulas above we get

$$\int_M \langle K, \xi \rangle (H_r^2 - H_{r-1} H_{r+1}) dV = 0. \quad (67)$$

By using the generalization of Cauchy-Schwarz inequality we have

$$H_r^2 - H_{r-1} H_{r+1} \geq 0, \quad (68)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$. Besides, $\langle K, \xi \rangle > 0$, so we have $H_r^2 - H_{r-1} H_{r+1} \equiv 0$; then there exists constant λ making that $A_N = \lambda I$ on screen bundle. Since $A_N \xi = 0$; thus by using the Corollary 5, we have that M is totally umbilical. $\square$

Theorem 13. Let $\bar{M} = -I \times_f F^{n+1}$ be a spatially closed RW spacetime. Assume that M is a compact lightlike hypersurface immersed into $\bar{M}$ with $A_N \xi = 0$; screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. If $H_1 > 0, \dots, H_r > 0$ and H_r is constant, where $2 \leq r \leq n-1$, then the hypersurface M is totally umbilical in $\bar{M}$.

Proof. In the same way as the proof of Theorem 11, we have

$$\int_M \langle K, \xi \rangle (H_1 H_r - H_{r+1}) dV = 0, \quad (69)$$

From the generalization of Cauchy-Schwarz inequality we have for any $1 \leq i \leq n$

$$H_i^2 - H_{i-1} H_{i+1} \geq 0, \quad (70)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$. Since $H_k > 0$, $1 \leq k \leq r$, then we have

$$H_1 \geq \frac{H_2}{H_1} \geq \dots \geq \frac{H_r}{H_{r-1}}, \quad (71)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$. From $\langle K, \xi \rangle > 0$ we have $H_1 H_r - H_{r+1} \equiv 0$; then there exists constant λ making that $A_N = \lambda I$ on screen bundle. Since $A_N \xi = 0$, thus by using the Corollary 5, we have that M is totally umbilical. $\square$

6. Compact Lightlike Hypersurface with Elliptic Point

In this section, an elliptic point we mean is a point where all the principal curvature $k_i(p)$ of A_N restricted on the screen bundle has the same sign. Now we will derive the result about the existence of elliptic point lightlike hypersurface.

Lemma 14. *Let $M^{n+1} \rightarrow -I \times_f F^{n+1}$ be a compact lightlike hypersurface immersed into a spatially closed RW spacetime, where $A_N \xi = 0$, screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. When $f'(h)$ does not vanish on M , then (i). If $f'(h) > 0$ on M , then there exists an elliptic point of M with respect to its future-pointing Gauss map. (ii) If $f'(h) < 0$ on M , then there exists an elliptic point of M with respect to its past-pointing Gauss map.*

Proof. When $f'(h) > 0$, let us choose on M the future-pointing Gauss map T and let $p_0 \in M$ be the point where the height function h attains its minimum on M , so we have $\nabla h(p_0) = 0$, $\nabla^2 \rho(p_0) \geq 0$ and $\langle \xi, \partial_t \rangle > 0$. From $\nabla_X(\nabla h) = -(\log f)'(h)(\langle \nabla h, X \rangle \nabla h + X) - \langle \partial_t, \xi \rangle A_N X$ we have

$$\nabla^2 h_{p_0}(e_i, e_i) = -(\log f)'(h)(p_0) - \langle \partial_t, \xi \rangle k_i(p_0) \geq 0, \quad (72)$$

where $\{e_1, \dots, e_n\}$ is a basis of principal direction at p_0 , $i = 1, \dots, n$, so we have

$$k_i(p_0) \leq \frac{1}{\langle \partial_t, \xi \rangle} (-(\log f)'(h)(p_0)) < 0; \quad (73)$$

then we obtain the result.

In a similar way, when $f'(h) < 0$ we choose the past-pointing Gauss map on M such that $\langle \partial_t, \xi \rangle < 0$; consider $p_0 \in M$ to be the point where the height function h attains its maximum on M . Now we have $\nabla h(p_0) = 0$,

$$\nabla^2 h_{p_0}(e_i, e_i) = -(\log f)'(h)(p_0) - \langle \partial_t, \xi \rangle k_i(p_0) \leq 0. \quad (74)$$

From $f'(h) < 0$ we have

$$k_i(p_0) \leq -\frac{1}{\langle \partial_t, \xi \rangle} (\log f)'(h)(p_0) < 0. \quad (75)$$

Then we finish the proof. $\square$

Theorem 15. *Let $\overline{M} = -I \times_f F^{n+1}$ ($n \geq 4$) be a spatially closed RW spacetime. Assume that M is a compact lightlike hypersurface immersed into $\overline{M}$ which contains an elliptic point, $A_N \xi = 0$; screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. If H_k is constant, where $3 \leq k \leq n-1$, then the hypersurface M is totally umbilical in $\overline{M}$.*

Proof. From Lemma 14 we have that there exists elliptic point p_0 with respect to the future-pointing Gauss map T , such that all the principle curvature of A_N restricts on $S(TM)$; then constant $H_k = H_k(p_0)$ is positive. By using Garding inequality we have

$$H_1 \geq H_2^{1/2} \geq \dots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} > 0, \quad (76)$$

where the equality holds only at the point where $k_1 = k_2 = \dots = k_n$. From $f'(h) > 0$ on M we have that $f'(h)H_k \geq f'(h)H_k^{(k-1)/k}$. Integrating this inequality and using the Minkowski formula we obtain

$$\begin{aligned} H_k \int_M \langle K, \xi \rangle dV &= \int_M f'(h) H_{k-1} dV \\ &\geq H_k^{(k-1)/k} \int_M f'(h) dV \\ &= H_k^{(k-1)/k} \int_M \langle K, \xi \rangle dV. \end{aligned} \quad (77)$$

So we have

$$\int_M (H_1 - H_k^{1/k}) \langle K, \xi \rangle dV \leq 0. \quad (78)$$

Besides, we already know that $\langle K, \xi \rangle > 0$ and $H_1 - H_k^{1/k} \geq 0$, so we have $H_1 - H_k^{1/k} \equiv 0$; then $k_1 = k_2 = \dots = k_n$; that is, there exists constant λ making that $A_N = \lambda I$ on screen bundle. Since $A_N \xi = 0$, thus by using the Corollary 5 we know that M is totally umbilical. $\square$

When the hypersurface immersed into the RW spacetime with linearly related r -mean curvature, then we have the following totally umbilical result.

Theorem 16. *Let $\overline{M} = -I \times_f F^{n+1}$ ($n \geq 4$) be a spatially closed RW spacetime. Assume that M is a compact lightlike hypersurface immersed into $\overline{M}$ with $A_N \xi = 0$; screen bundle $S(TM)$ is integrable and the Ricci tensor of induced connection ∇ is symmetric. If $f'(h) \neq 0$, there are exists integers k and s , where $0 \leq r < s < n$ or $0 < r < s \leq n$ and the higher-order mean curvature restricted on $S(TM)$ is linearly related by*

$$H_s = a_k H_k + a_{k+1} H_{k+1} + \dots + a_s H_s, \quad (79)$$

where $a_k, \dots, a_s$ are nonnegative; then the hypersurface M is totally umbilical.

Proof. (i) When

$$0 \leq r < s < n, \quad (80)$$

from the formula (79) and the $s+1$ th Minkowski formula, we have

$$\begin{aligned} \int_M \langle K, \xi \rangle H_{s+1} dV &= \int_M f'(h) H_s dV = \sum_{i=r}^{s-1} a_i \int_M f'(h) H_i dV \\ &= \sum_{i=r}^{s-1} a_i \int_M \langle K, \xi \rangle H_{i+1} dV; \end{aligned} \quad (81)$$

then we have

$$\int_M \langle K, \xi \rangle \left(H_{s+1} - \sum_{i=r}^{s-1} a_i H_{i+1} \right) dV = 0. \quad (82)$$

Since $f'(h) \neq 0$, we suppose that $f'(h)$ is positive on M ; then from Lemma 14 we know that there exists a point $p_0 \in M$ where all the principal curvature of A_N restricted on $S(TM)$ is negative. Denote M_0 as the connected component of $\{p \in M : H_s(p) > 0\}$ containing the elliptic point p_0 , so M_0 is a nonempty subset of M . If M_0 is also closed, then $M_0 = M$; now we will prove that it is closed. From $H_s(p_0) > 0$ we have that there exists at least one positive coefficient a_k , such that $a_k > 0$; then by using Garding inequalities we have for any point $p \in M_0$,

$$H_j^{s/j} \geq H_s(p) > 0, \quad (83)$$

where $1 \leq j \leq s-1$, $a_j \geq 0$; then we obtain $H_s(p) \geq a_k H_k(p)$.

If $k = 0$, then $H_s \geq a_0 > 0$ on M_0 , so M_0 is closed.

If $k \geq 1$, then on M_0 we have $H_k^{s/k} \geq H_s \geq a_k H_k > 0$, so $H_k^{(s-k)/k} \geq a_k$ on M_0 ; that is, $H_s \geq a_k a_k^{k/(s-k)} = a_k^{s/(s-k)} > 0$; then M_0 also is closed.

Thus $M_0 = M$ and (82) always holds on M .

From formula (83) and Cauchy-Schwarz inequality we have $H_j > 0$ and

$$H_j^2 - H_{j-1}H_{j+1} \geq 0, \quad (84)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$, $1 \leq j \leq s$. Then it equals

$$\frac{H_{s+1}}{H_s} \leq \frac{H_s}{H_{s-1}} \leq \dots \leq \frac{H_{j+1}}{H_j} \leq \dots \leq \frac{H_2}{H_1} \leq H_1, \quad (85)$$

where the equality holds only at the point $k_1 = k_2 = \dots = k_n$, $1 \leq j \leq s$.

From (79) and (85) we get that

$$\frac{H_{s+1}}{H_s} \leq \frac{H_s}{H_{s-1}} = \sum_{j=r}^{s-1} a_j \frac{H_j}{H_{s-1}} \leq \sum_{j=r}^{s-1} a_j \frac{H_{j+1}}{H_s}; \quad (86)$$

then we obtain $H_{s+1} \leq \sum_{j=r}^{s-1} a_j H_{j+1}$. By using $\langle K, \xi \rangle > 0$ and (82) we have

$$H_{s+1} = \sum_{j=r}^{s-1} a_j H_{j+1}, \quad (87)$$

It means that $k_1 = k_2 = \dots = k_n$. Since $A_N = 0$ then by using Corollary 5 we have that M is totally umbilical.

(ii) When $0 < r < s \leq n$, now we just need to prove the case $s = n$ and $r > 0$; others are the same with (i); by using the Minkowski formula we have that

$$\begin{aligned} \int_M f'(h) H_{n-1} dV &= \sum_{i=r}^{n-1} a_j \int_M \langle K, \xi \rangle H_j dV \\ &= \sum_{j=r}^{n-1} a_i \int_M f'(h) H_{j-1} dV, \end{aligned} \quad (88)$$

so we have

$$\int_M f'(h) \left(H_{n-1} - \sum_{j=r}^{n-1} a_i H_{j-1} \right) dV = 0. \quad (89)$$

In a similar way, we assume $f'(h)$ is positive on M , then there exists a point $p_0 \in M$ where all the principal curvatures of A_N restricted on screen bundle are negative. In the same way we obtain that $M_n = \{p \in M : H_n(p) > 0\} = M$ and $H_j^{n/j}(p) \geq H_n(p) > 0$ at each point $p \in M$. From Newton inequality we have

$$\frac{H_n}{H_{n-1}} \leq \frac{H_{n-1}}{H_{n-2}} \leq \dots \leq \frac{H_{j+1}}{H_j} \leq \dots \leq \frac{H_2}{H_1} \leq H_1, \quad (90)$$

where the equality holds only at the points $k_1 = k_2 = \dots = k_n$. From (79) we have

$$\frac{H_{n-1}}{H_{n-2}} \geq \frac{H_n}{H_{n-1}} = \sum_{j=r}^{n-1} a_j \frac{H_j}{H_{n-1}} \geq \sum_{j=r}^{n-1} a_j \frac{H_{j-1}}{H_{n-2}}; \quad (91)$$

then from (89) we obtain that

$$H_{n-1} \geq \sum_{j=r}^{n-1} a_j H_{j-1}, \quad (92)$$

where the equality holds only at the points $k_1 = k_2 = \dots = k_n$. By using Corollary 5 we have that M is totally umbilical. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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