Research Article

Marshall-Olkin Discrete Uniform Distribution

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We introduce and characterize a new family of distributions, Marshall-Olkin discrete uniform distribution. The natures of hazard rate, entropy, and distribution of minimum of sequence of i.i.d. random variables are derived. First order autoregressive (AR (1)) model with this distribution for marginals is considered. The maximum likelihood estimates for the parameters are found out. Also, the goodness of the distribution is tested with real data.

1. Introduction

Marshall and Olkin [1] introduced a new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Jose and Alice [2] discussed Marshall-Olkin family of distributions and their applications in time series modeling and reliability. Jose and Krishna [3] have developed Marshall-Olkin extended uniform distribution. These works and most of the references there in, deal with continuous distribution. Not much work is seen in the discrete case. The reason behind this may be that it is difficult to obtain compact mathematical expressions for moments, reliability, and estimation in the discrete set up.

If $\bar{F}(x)$ is the survival function of a distribution, then, by Marshall-Olkin method, we get another survival function $\bar{G}(x)$, by adding a new parameter θ to it. That is,

$$\bar{G}(x, \theta) = \frac{\theta \bar{F}(x)}{(1 - (1 - \theta) \bar{F}(x))}, \quad -\infty < x < \infty, \theta > 0. \quad (1)$$

Then the corresponding distribution function is $G(x, \theta) = 1 - \bar{G}(x, \theta) = F(x)/(1 - (1 - \theta)\bar{F}(x))$.

Now its probability mass function (p.m.f.) is

$$g(x, \theta) = G(x, \theta) - G(x - 1, \theta) \\ = \frac{\theta f(x)}{[1 - (1 - \theta) \bar{F}(x)][1 - (1 - \theta) \bar{F}(x - 1)]}, \quad (2)$$

where $f(x)$ is the p.m.f. corresponding to $F(x)$.

The hazard rate of X is

$$\gamma_G(x) = \frac{g(x)}{\bar{G}(x)}. \quad (3)$$

We consider a new family of distributions by adding an additional parameter by using the Marshall-Olkin scheme (Marshall and Olkin [1] in Section 2). The nature of hazard rate, entropy and expectation are derived. In the third section, distributions of minimum sequence of i.i.d. random variables are found out. An AR (1) model with new Marshall-Olkin discrete uniform distribution is discussed in the next section. In the fifth section, the maximum likelihood estimate (m.l.e.) for the parameters is found out and, in the last section, the goodness of the distribution is tested with a real life data.

FIGURE 1: P.m.f. of $X \sim \text{MODU}$.

2. Marshall-Olkin Discrete Uniform Distribution

Let X be a discrete random variable following uniform distribution with p.m.f. $f(x) = 1/a$, $x = 1, 2, 3, \dots, a$. We have the distribution function $F(x) = x/a$ and the survival function $\bar{F}(x) = (a-x)/a$. By Marshall-Olkin method, we can form another survival function of Marshall-Olkin discrete uniform (MODU) distribution by substituting $\bar{F}(x) = (a-x)/a$ in $\bar{G}(x, \theta)$, in (1), and we get

$$\bar{G}(x, \theta) = \frac{\theta(a-x)}{[a\theta + (1-\theta)x]}, \quad 0 < \theta < \infty. \quad (4)$$

We write $X \sim \text{MODU}(a, \theta)$ for a random variable (r.v.) with $\bar{G}(x, \theta)$ given in (4).

2.1. Stability Property of the New Family. If we apply the same method again into the new family, by adding a new parameter “ α ” ($\alpha > 0$) to the reliability function, we get $\bar{G}_1(x) = (\alpha\theta)(a-x)/[a - (1-\alpha\theta)(a-x)]$, which is the reliability function of the new Marshall-Olkin family with parameters $(a, \alpha\theta)$.

2.2. Probability Mass Function. From (2), consider the probability mass function p.m.f.

$$g(x, \theta) = \frac{\theta f(x)}{[1 - (1-\theta)\bar{F}(x)][1 - (1-\theta)\bar{F}(x-1)]}, \quad (5)$$

where $f(x)$ is the p.m.f. corresponding to $F(x)$. That is,

$$g(x, \theta) = \frac{a\theta}{\{[a\theta + (1-\theta)x][a\theta + (1-\theta)(x-1)]\}}. \quad (6)$$

Now we plot the p.m.f. of $X \sim \text{MODU}$ for different values of a and θ (Figure 1).

Remark 1. If X follows $\text{MODU}(a, \theta)$, then $g(x, a, \theta) = g(a-x+1, a, 1/\theta)$, $x = 1, 2, \dots, a$, $\theta > 0$.

Now, our next interest is to check whether (6) characterizes MODU. If X satisfies the condition $g(x, a, \theta) = g(a-x+1, a, 1/\theta)$, $x = 1, 2, \dots, a$, $\theta > 0$, this should be true for $\theta = 1$ also. For $\theta = 1$, it gives the parent distribution itself before reparameterization. Thus, Remark 1 implies that $P(x) = P(a-x+1)$, $x = 1, 2, 3, \dots, a$, where $P(x)$ is the p.m.f. of discrete uniform distribution with parameter “ a ”. That is, $P(1) = P(a)$, $P(2) = P(a-1)$, $\dots$, $P(a) = P(1)$. This relation need not characterize a discrete uniform distribution, when $a > 2$. For example, let $a = 3$; then we get $P(1) = P(3)$. Since $P(1) + P(2) + P(3) = 1$, we get $P(2) = 1 - P(1) - P(3)$. That is $P(2) = 1 - 2P(1)$. Let $P(1) = 0.1 = P(3)$ imply $P(2) = 0.8$, which is not discrete uniform. Thus we have the following.

Theorem 2. A random variable X follows $\text{MODU}(2, \theta)$, $x = 1, 2$ if and only if its p.m.f. satisfies

$$g(x, a, \theta) = g\left(a-x+1, a, \frac{1}{\theta}\right). \quad (7)$$

TABLE 1: Expectation and standard deviation (sd) with different “ a ” and “ θ ” ($\theta < 1$) for $X \sim \text{MODU}(a, \theta)$.

a	$\theta = 0.133$		$\theta = 0.25$		$\theta = 0.5$		$\theta = 0.75$	
	$E(X)$	$\text{sd}(X)$	$E(X)$	$\text{sd}(X)$	$E(X)$	$\text{sd}(X)$	$E(X)$	$\text{sd}(X)$
10	2.597426	2.186733	3.358938	2.519965	4.375428	2.778938	5.026709	2.855936
12	2.995745	2.638189	3.919444	3.033456	5.145940	3.341060	5.930269	3.432611
15	3.597637	3.312588	4.762709	3.801608	6.302744	4.182802	7.286014	4.296371
20	4.606958	4.43258	6.171538	5.079024	8.232135	5.583787	9.546128	5.734303
25	5.620115	5.550024	7.582418	6.354713	10.16236	6.983619	11.80657	7.171237
30	6.635235	6.666144	8.994330	7.629531	12.09300	8.382876	14.06717	8.607673
50	10.70370	11.12519	14.64612	12.72530	19.81722	13.97760	23.11022	14.35143
75	15.79535	16.69483	21.71398	19.09236	29.47374	20.96928	34.41451	21.52963
100	20.88903	22.26307	28.78287	25.45854	39.13069	27.96038	45.71897	28.70733

Proof. Suppose that $g(x, a, \theta) = g(a - x + 1, a, 1/\theta)$ is true for Marshall-Olkin (MO) scheme, then its parent distribution is discrete uniform on $\{1, 2\}$. That is,

$$g(x, 2, \theta) = g\left(2 - x + 1, 2, \frac{1}{\theta}\right) = \frac{1}{2}, \quad (8)$$

that is,

$$g(1, 2, \theta) = g\left(2, 2, \frac{1}{\theta}\right). \quad (9)$$

We know the p.m.f. $g(x, a, \theta) = \theta f(x) / [1 - (1 - \theta)\bar{F}(x - 1)][1 - (1 - \theta)\bar{F}(x)]$. That is,

$$\begin{aligned} g(1, 2, \theta) &= \frac{\theta P(1)}{[1 - (1 - \theta)\bar{F}(0)][1 - (1 - \theta)\bar{F}(1)]} \\ &= \frac{\theta P(1)}{[1 - (1 - \theta)][1 - (1 - \theta)\bar{F}(1)]} = \frac{1}{2}, \end{aligned} \quad (10)$$

by (8); that is,

$$2\theta P(1) = [1 - (1 - \theta)][1 - (1 - \theta)\bar{F}(1)]. \quad (11)$$

$$\begin{aligned} g\left(2, 2, \frac{1}{\theta}\right) &= \frac{P(2)}{\theta [1 - (1 - 1/\theta)\bar{F}(1)][1 - (1 - 1/\theta)\bar{F}(2)]} \\ &= \frac{P(2)}{\theta [1 - (1 - 1/\theta)\bar{F}(1)]} \\ &= \frac{P(2)}{\theta [1 - ((\theta - 1)/\theta)\bar{F}(1)]}. \end{aligned} \quad (12)$$

That is, $g(2, 2, 1/\theta) = P(2)/[\theta - (\theta - 1)\bar{F}(1)] = 1/2$, by (8).

That is, $2P(2) = [\theta - (\theta - 1)\bar{F}(1)]$ or $2P(2) - \theta = -(\theta - 1)\bar{F}(1)$. We get,

$$\bar{F}(1) = \frac{(2P(2) - \theta)}{1 - \theta}. \quad (13)$$

By substituting this in (11),

$$\begin{aligned} 2\theta P(1) &= [1 - (1 - \theta)] \left[1 - (1 - \theta) \left(\frac{(2P(2) - \theta)}{(1 - \theta)} \right) \right] \\ &= [1 - (1 - \theta)] [1 - (2P(2) - \theta)], \\ 2\theta P(1) &= \theta [1 - (2P(2) - \theta)]. \end{aligned} \quad (14)$$

That is, $2P(1) = 1 - 2P(2) + \theta$ or $2(P(1) + P(2)) = 1 + \theta$. So $2 = 1 + \theta$, since $P(1) + P(2) = 1$.

That is, $\theta = 1$. But, when $\theta = 1$, it corresponds to discrete uniform distribution on $\{1, 2\}$. The converse is straight forward. $\square$

Remark 3. It is clear that MODU distribution does not possess additive property.

Remark 4. MODU is not infinitely divisible (i.d.), since its support is on $\{1, 2, 3, \dots, a\}$, a finite range. It is not log convex, since the class of log convex distribution forms a subclass of the class of i.d. distributions.

Corollary 5. If $X \sim \text{MODU}(a, \theta)$, then $\bar{F}(x) \leq \bar{G}(x, \theta) \leq \theta \bar{F}(x)$, if $\theta \leq 1$, and $\theta \bar{F}(x) < \bar{G}(x, \theta) < \bar{F}(x)$, if $0 < \theta < 1$.

Proof. Let $X \sim \text{MODU}(a, \theta)$, $\bar{F}(x) = (a - x)/a$ and $\bar{G}(x, \theta) = \theta(a - x)/[a\theta + (1 - \theta)x]$. So, $\bar{F}(x) \leq \bar{G}(x, \theta)$ implies that $(a - x)/a \leq \theta(a - x)/[a\theta + (1 - \theta)x]$. That is $1/a - \theta/[a\theta + (1 - \theta)x] \leq 0$, $a\theta + (1 - \theta)x - a\theta \leq 0$, or $(1 - \theta)x \leq 0$. This happens only when $\theta \geq 1$. Now assume that $\bar{G}(x, \theta) \leq \theta \bar{F}(x)$. That is, $\theta(a - x)/[a\theta + (1 - \theta)x] \leq \theta(a - x)/a$. So $(1 - \theta)a - (1 - \theta)x \leq 0$; that is, $(a - x)(1 - \theta) \leq 0$, which is true only when $\theta \geq 1$, since $(a - x) \geq 0$. Hence, the proof is completed. $\square$

2.3. Expectation, Standard Deviation, and Entropy of MODU Distribution. We numerically compute the expectation and standard deviation (Tables 1 and 2) and entropy (Table 3) of the MODU distribution with different a and θ , since compact

TABLE 2: Expectation and standard deviation with different “ a ” and “ θ ” ($\theta > 1$) for $X \sim \text{MODU}(a, \theta)$.

a	$\theta = 1.33$		$\theta = 2$		$\theta = 4$		$\theta = 7.5$	
	$E(X)$	$\text{sd}(X)$	$E(x)$	$\text{sd}(x)$	$E(x)$	$\text{sd}(x)$	$E(x)$	$\text{sd}(x)$
10	5.97329	2.855936	6.62457	2.77893	7.64106	2.51996	8.40257	2.18673
12	7.06973	3.432611	7.85406	3.34106	9.08055	3.03345	10.0042	2.63818
15	8.71398	4.296371	9.69725	4.18280	11.2372	3.80160	12.4023	3.31258
20	11.4538	5.734303	12.7678	5.58378	14.8284	5.07902	16.3930	4.43258
25	14.1934	7.171237	15.8376	6.98361	18.4175	6.35471	20.3798	5.55002
30	16.9328	8.607673	18.9070	8.38287	22.0056	7.62953	24.3647	6.66614
50	27.8897	14.35143	31.1827	13.9776	36.3538	12.7253	40.2963	11.1251
75	41.5854	21.52963	46.5262	20.9692	54.2860	19.0923	60.2046	16.6948
100	55.2810	28.70733	61.86931	27.96038	72.21713	25.45854	80.11097	22.26307

TABLE 3: Entropy with different “ a ” and “ θ ” for $X \sim \text{MODU}(a, \theta)$.

a	$\theta = 0.1$	$\theta = 0.133$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1.33$	$\theta = 2$	$\theta = 4$	$\theta = 7.5$	$\theta = 10$
10	1.5310	1.6923	1.9985	2.2241	2.2889	2.2889	2.2241	1.9985	1.6923	1.5310
15	1.9142	2.0846	2.4004	2.6290	2.6943	2.6943	2.6290	2.4004	2.0846	1.9142
20	2.1933	2.3674	2.6868	2.9165	2.9819	2.9819	2.9165	2.6868	2.3674	2.1933
30	2.5923	2.7692	3.0914	3.3218	3.3874	3.3874	3.3218	3.0914	2.7692	2.5923
50	3.0997	3.2782	3.6017	3.8326	3.8982	3.8982	3.8326	3.6017	3.2782	3.0997
100	3.7913	3.9705	4.2947	4.5257	4.5913	4.5913	4.5257	4.2947	3.9705	3.7913

expressions are not available for calculating the same. We have

Shannon’s Entropy

$$\begin{aligned}
 &= -\sum_I p(x_i) \log p(x_i), \quad i \in I = \{1, 2, \dots\} \\
 &= -\sum \frac{a\theta}{[a\theta + (1-\theta)x_i][a\theta + (1-\theta)(x_i-1)]} \\
 &\quad \times \log \frac{a\theta}{[a\theta + (1-\theta)x_i][a\theta + (1-\theta)(x_i-1)]}. \quad (15)
 \end{aligned}$$

Remark 6. If $X \sim \text{MODU}(a, \theta)$, then the $\text{sd}(X)$ is decreasing with increasing value of θ when $\theta > 1$ and the $\text{sd}(X)$ increasing with increasing value of θ when $\theta < 1$.

From Remark 1 and Tables 1 and 2, we have the following.

Remark 7. The $\text{sd}(X)$ for $X \sim \text{MODU}(a, \theta)$ is equal to $\text{sd}(X)$ for $X \sim \text{MODU}(a, 1/\theta)$.

From Remark 1, Table 3, and Figure 2, the following is observed.

Remark 8. The entropy of $X \sim \text{MODU}(a, \theta)$ is equal to the entropy of $X \sim \text{MODU}(a, 1/\theta)$.

TABLE 4: Mean, median, and mode of the MODU distribution with different a and θ .

a	θ	Mean	Median	Mode
10	0.2	3.06	2	1
	0.5	4.40	3	1
	2	6.60	7	10
	5	7.90	9	10
20	0.2	5.57	4	1
	0.5	8.23	7	1
	2	12.76	14	20
	5	15.40	17	20
100	0.2	25.70	17	1
	0.5	39.10	34	1
	2	61.80	67	100
	5	75.20	84	100

Let $X \sim \text{MODU}(a, \theta)$; then the mean, median, and mode of the distribution (in Figure 1 and Tables 1 and 2) for different a and θ are computed in Table 4.

From Figure 1 and from Table 4, the following is clear.

Remark 9. The MODU distribution is positively skewed, when $\theta < 1$, since mode < median < mean and the distribution is negatively skewed, when $\theta > 1$, since mode > median > mean. Also, it is to be noted that the distribution is

FIGURE 2: Entropy with different “ a ” and “ θ ” for $X \sim \text{MODU}(a, \theta)$.TABLE 5: Hazard function $\gamma_G(x)$ of $X \sim \text{MODU}(a, \theta)$ with $a = 10$.

X	θ						
	0.75	0.5	0.2	0.1	2	5	7.5
1	0.1481481	0.2222222	0.5555556	1.111111	0.05555556	0.02222222	0.01481481
2	0.1612903	0.2272727	0.4464286	0.6578947	0.06578947	0.02717391	0.01824818
3	0.1785714	0.2380952	0.3968254	0.5102041	0.07936508	0.03401361	0.02304147
4	0.2020202	0.2564103	0.3787879	0.4504505	0.09803922	0.04385965	0.03003003
5	0.2352941	0.2857143	0.3846154	0.4347826	0.1250000	0.05882353	0.04081633
6	0.2857143	0.3333333	0.4166667	0.4545455	0.1666667	0.08333333	0.05882353
7	0.3703704	0.4166667	0.4901961	0.5208333	0.2380952	0.1282051	0.09259259
8	0.5405405	0.5882353	0.6578947	0.6849315	0.3846154	0.2272727	0.1694915
9	1.052632	1.111111	1.190476	1.219512	0.8333333	0.5555556	0.4347826

unimodal; that is, when $\theta < 1$, the mode = 1 and, when $\theta > 1$, the mode = a .

Therefore, the MODU distribution can be applied for the data showing positive skewness when $\theta > 1$ and the data showing negative skewness when $\theta < 1$.

2.4. Hazard Function. The hazard function of $X \sim \text{MODU}(a, \theta)$ is

$$\begin{aligned} \gamma_G(x) &= \frac{g(x)}{G(x)} \\ &= \frac{\theta(a-x+1) / [a\theta + (1-\theta)(x-1)]}{\theta(a-x) / [a\theta + (1-\theta)x]} - 1. \end{aligned} \quad (16)$$

That is,

$$\gamma_G(x) = \frac{a}{(a-x)(a\theta + (1-\theta)(x-1))}. \quad (17)$$

Theorem 10. Let $X \sim \text{MODU}(a, \theta)$; then the distribution is with Increasing Failure Rate (IFR) when $\theta > (a-2x)/(2a-2x)$, Decreasing Failure Rate (DFR) when $\theta < (a-2x)/(2a-2x)$, and constant FR when $\theta = (a-2x)/(2a-2x)$.

Proof. From (17), the hazard function of $X \sim \text{MODU}(a, \theta)$ is $\gamma_G(x) = a / \{(a-x)(a\theta + (1-\theta)(x-1))\}$.

Then $\gamma_G(x+1) = a / \{(a-x-1)(a\theta + (1-\theta)x)\}$. If $\gamma_G(x+1) - \gamma_G(x) > 0$, the distribution is with IFR.

That is, if $\gamma_G(x+1) / \gamma_G(x) > 1$ or

$$\begin{aligned} \frac{(a-x)(a\theta + (1-\theta)(x-1))}{(a-x-1)(a\theta + (1-\theta)x)} &> 1, \\ (a-x)(a\theta + (1-\theta)(x-1)) & > (a-x-1)(a\theta + (1-\theta)x), \\ - (a-x-1)(a\theta + (1-\theta)x) &> 0, \\ (-a + 2a\theta + 2x - 2x\theta) &> 0, \\ \theta(2a-2x) &> a-2x \text{ or} \end{aligned} \quad (18)$$

if $\theta > (a-2x)/(2a-2x)$, the distribution is with IFR.

Similarly, if $\theta < (a-2x)/(2a-2x)$, the distribution is with DFR, and if $\theta = (a-2x)/(2a-2x)$, the distribution is with a constant FR.

We illustrate these results graphically (Figure 3) and numerically (Table 5).

From Figure 3, also, it is clear that the failure rate is IFR, when $\theta > (a-2x)/(2a-2x)$, and DFR, when $\theta < (a-2x)/(2a-2x)$. $\square$

3. MODU Distribution as the Distribution of Minimum of a Sequence of i.i.d. Random Variables

The following theorem gives a characterization of minimum of a sequence of i.i.d. random variables following MODU distribution.

TABLE 6: Initial observations.

N	a	$\hat{a}$	θ	$\hat{\theta}$	Mean $\hat{\theta}$	SE($\hat{\theta}$)
1				0.104118		
2				0.111474		
3				0.095902		
4				0.101358		
5	2500	20	0.1	0.098971	0.1016693	0.1016693
6				0.095781		
7				0.101448		
8				0.098904		
9				0.103342		
10				0.105396		

Theorem 11. Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with common survival function $\bar{F}(x)$. Let N be a geometric random variable independent of $\{X_i, i \geq 1\}$ such that $P(N = n) = pq^{n-1}$, $n = 1, 2, 3, \dots$; $0 < p = \theta < 1$, and $q = 1 - p$. Let $U_N = \min_{1 \leq i \leq N} (X_i)$. Then $\{U_N\}$ is distributed as MODU distribution if and only if $\{X_i\}$ follows discrete uniform distribution with $x = 1, 2, 3, \dots, a$.

Proof. Proof follows as in the same lines as given in Satheesh et al. [4–6] for a similar characterization of minimum of sequence of i.i.d. random variables.

The survival function of U_N is $\bar{G}(x) = P(U_N > x) = \sum_{n=1}^{\infty} [\bar{F}(x)]^n P(N = n) = \sum_{n=1}^{\infty} [\bar{F}(x)]^n \theta(1 - \theta)^n = \theta \sum_{n=1}^{\infty} [\bar{F}(x)(1 - \theta)]^n = \theta \bar{F}(x) / [1 - (1 - \theta)\bar{F}(x)]$. Substituting $\bar{F}(x) = (a - x)/a$, the survival function of the discrete uniform distribution; that is,

$$\begin{aligned} \bar{G}(x) &= \frac{\theta[(a-x)/a]}{[1 - (1-\theta)((a-x)/a)]} \\ &= \frac{\theta(a-x)}{[a - (1-\theta)(a-x)]} = \frac{\theta(a-x)}{[a\theta + (1-\theta)x]}, \end{aligned} \quad (19)$$

which is the survival function of MODU(a, θ). By retracing the steps we can easily show the converse. $\square$

4. AR (1) Model with MODU Distribution as Innovating Distribution

Arnold and Robertson [7], Chernick [8], and Satheesh et al. [9] studied some properties of the AR (1) models. As Satheesh et al. [9] discussed, construct a first order autoregressive minification process with the following structure:

$$\begin{aligned} X_n &= \begin{cases} \epsilon_n & \text{with probability } \theta \\ \min(X_{n-1}, \epsilon_n) & \text{with probability } (1-\theta), \end{cases} \quad 0 \leq \theta \leq 1, \end{aligned} \quad (20)$$

where $\{\epsilon_n\}$ is a sequences of i.i.d. random variables following a discrete uniform distribution with parameter a independent of $\{X_{n-1}, X_{n-2}, \dots\}$. Then the process is stationary and is

FIGURE 3: Hazard function $y_G(x)$ of $X \sim \text{MODU}(a, \theta)$.TABLE 7: Mean of m.l.e. of θ from 10 repetitions. ($a = 20, \hat{a} = \max(x) = 20$, sample size = n).

θ	n			
	2500		5000	
	Mean $\hat{\theta}$	SE($\hat{\theta}$)	Mean $\hat{\theta}$	SE($\hat{\theta}$)
0.1	0.1016693	0.1020	0.09995393	0.0020
0.25	0.2487304	0.0020	0.2480182	0.0020
0.5	0.5020383	0.0030	0.4920417	0.0030
0.75	0.7512596	0.0023	0.7496541	0.0019
0.9	0.9015150	0.0060	0.8997414	0.0043
2	2.0007690	0.0191	1.9986450	0.0131
5	4.9308800	0.0507	5.0252140	0.0362
10	9.8801710	0.0900	9.9767230	0.0900
50	51.069840	0.4321	100.04960	0.4006
100	100.58670	1.3979	100.04960	0.4666

marginally distributed with $\text{MODU}(a, \theta)$ marginal distribution.

Theorem 12. In an AR (1) process with structure (20) $\{Y_{i,n}\}$ is stationary Markovian with $\text{MODU}(a, \theta)$ marginal if and only if $\{\epsilon_n\}$ is distributed as uniform with parameter $a, x = 1, 2, 3, \dots, a$.

Proof. From the structure (20), it follows that

$$\bar{F}_{X_n}(x) = \theta \bar{F}_{\epsilon_n}(x) + (1 - \theta) \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x). \quad (21)$$

Under stationary equilibrium, this implies

$$\bar{F}_X(x) = \frac{\theta \bar{F}_{\epsilon_n}(x)}{[1 - (1 - \theta) \bar{F}_{\epsilon_n}(x)]}. \quad (22)$$

If we take $\bar{F}_{\epsilon_n}(x) = (a - x)/a$, then,

$\bar{F}_X(x) = \theta(a - x)/[a - (1 - \theta)(a - x)]$, which is the survival function of $\text{MODU}(a, \theta)$. We can also show that $\bar{F}_{\epsilon_n}(x)$ follows uniform distribution with parameter a . By retracing the steps we can easily show the converse. $\square$

TABLE 8: Observed frequency of 50 students in 10 months from science and commerce classes.

Month	x																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
May	1	1	0	3	2	4	2	2	0	2	3	2	1	2	2	4	3	8	6	2
June	0	2	1	2	4	1	1	1	2	4	2	2	1	3	2	4	5	5	3	5
July	0	0	0	2	2	3	1	3	2	3	4	1	4	2	5	5	4	2	4	3
Aug	2	1	3	1	1	1	1	2	3	1	1	4	4	2	1	6	5	3	3	5
Sep	1	2	1	3	1	2	3	3	0	2	0	1	3	4	5	4	1	6	3	5
Oct	1	0	2	0	3	0	2	3	0	6	2	4	3	1	4	4	2	8	2	3
Nov	1	0	2	3	3	0	1	3	1	4	3	2	0	2	1	6	4	4	7	3
Dec	1	3	0	1	3	3	2	0	4	0	2	1	2	5	2	4	4	5	3	5
Jan	0	0	0	2	2	3	1	3	2	3	4	1	4	2	5	5	4	2	4	3
Feb	2	1	3	1	1	1	0	2	3	1	1	4	4	2	2	6	5	3	3	5

TABLE 9: The m.l.e. of θ (sample size = 50, m.l.e. of a , $\hat{a} = 20$).

Month	May	June	July	Aug	Sept	Oct	Nov	Dec	Jan	Feb
m.l.e. $\hat{\theta}$	2.0347	2.0364	1.9247	2.0432	2.0347	1.9368	2.0743	1.9995	2.0039	1.995

5. Maximum Likelihood Estimates (m.l.e.) for the Parameters of MODU(a, θ)

Let $x_1, x_2, \dots, x_n$ be independent random samples from MODU(a, θ). Then, from the likelihood function of the distribution, we can write

$$\left(\frac{n}{\theta}\right) - \sum \frac{(a - x_i)}{[a\theta + (1 - \theta)x_i]} - \sum \frac{(a - x_i + 1)}{[a\theta + (1 - \theta)(x_i - 1)]} = 0, \quad (23)$$

$$\left(\frac{n}{a}\right) - \theta \sum \frac{1}{[a\theta + (1 - \theta)x_i]} - \theta \sum \frac{1}{[a\theta + (1 - \theta)(x_i - 1)]} = 0. \quad (24)$$

We calculated the m.l.e. of a and θ numerically (using Mathematica) as the solution of these two nonlinear equations. But, here, the maximum of the range of observation is a . So m.l.e. of the parameter $a(\hat{a})$ is the largest value of the observation. Then, from (23), we can find m.l.e. of θ .

Algorithm 13. Consider the following.

- (1) Through simulation, 2500 random samples were generated from inverse function of distribution function $G(x)$ for some given value of the parameters θ and a .
- (2) Then we iterate the m.l.e. for the parameters from the equations (23) and (24) (Table 6).
- (3) For accuracy, we repeat the same calculation ten times with same values of θ and a .
- (4) The mean and the standard error (SE) of these estimates are calculated.

TABLE 10: Mean m.l.e. of θ and mean Standard Error (SE) of θ for the samples of 10 months.

Sample size	$\hat{a} = \max(x)$	Mean $\hat{\theta}$	SE($\hat{\theta}$)
50	20	2.00832	0.04486918

TABLE 11: Goodness of fit of uniform distribution with $\alpha = 0.05$.

	May	August	October	February
m.l.e. of $\theta = \hat{\theta}$	2.0347	2.0432	1.9368	1.995
χ^2 statistics	14.800	9.4000	13.200	10.400
P value	0.0632	0.3096	0.1052	0.2381

TABLE 12: Mean, median, and mode of the data in May, August, October, and February.

a	Month	m.l.e. of $\theta = \hat{\theta}$	Mean	Median	Mode
20	May	2.0347	12.80	14	18
	August	2.0432	12.30	13	16
	October	1.9368	12.86	13	18
	February	1.9950	12.64	14	16

TABLE 13: Goodness of fit of MODU distribution with $\alpha = 0.05$.

	May	August	October	February
m.l.e. of $\theta = \hat{\theta}$	2.0347	2.04320	1.93680	1.99500
χ^2 statistics	9.4668	6.45238	1.80953	6.95238
P value	0.1489	0.37446	0.93640	0.325270

- (5) Then it is calculated with different values of the parameter θ and the same is found with an increase in sample size 5000 also.
- (6) The mean estimated values for the parameters are tabulated (Table 7).

TABLE 14: Observed frequency of 50 students in 10 months from the language classes.

Month	X																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
May	5	5	7	5	5	1	1	4	1	5	2	1	0	1	1	2	1	1	1	1
June	3	9	7	4	2	4	3	1	3	1	1	1	1	2	3	0	1	0	1	3
July	7	4	5	5	2	3	6	0	4	2	2	1	2	1	0	2	1	1	1	1
Aug	10	4	5	2	1	4	2	2	4	1	0	1	0	2	3	4	1	0	3	1
Sep	7	3	10	2	3	1	5	3	3	1	2	1	3	1	1	2	0	1	1	0
Oct	8	4	4	3	5	3	2	4	1	4	3	0	3	1	1	0	1	2	0	1
Nov	9	5	7	0	3	2	2	4	1	2	2	2	2	1	2	2	0	3	1	0
Dec	8	4	3	5	7	4	2	5	0	0	3	0	1	1	1	1	4	1	0	0
Jan	6	2	7	3	4	1	7	5	3	1	2	1	2	1	1	3	0	0	0	1
Feb	5	6	7	6	5	2	2	0	3	1	0	2	2	3	1	2	2	1	0	0

TABLE 15: The m.l.e. of θ and SE of θ (sample size = 50, m.l.e. of a , $\hat{a} = \max(x) = 20$).

Month	May	June	July	Aug	Sept	Oct	Nov	Dec	Jan	Feb
m.l.e. $\hat{\theta}$	0.3511	0.3544	0.3536	0.3802	0.3619	0.3522	0.3720	0.3707	0.3745	0.3594

It is clear that the SE of m.l.e. of the parameters “ a ” and θ is decreasing with increase in sample size.

6. An Application of MODU

Example 14. The data was collected from the daily attendance register of the science and commerce (nonlanguage) supplementary class (20 working days in a month) of a higher secondary school in Palakkad district, Kerala. We took a sample of 50 students (out of 360) and let X be the number of days these students attended in the class for the whole year 2012-2013 (an academic year is 10 months) (see Tables 8, 9, and 10).

We arbitrarily fix 4 months, and the m.l.e.s are computed. Initially we fit discrete uniform to the data (see Tables 11 and 12).

Since mode > median > mean, the data exhibits negative skewness with m.l.e. of $\theta > 1$ and it is observed that the distribution is unimodal; hence, MODU distribution is supposed to give a better fit than uniform distribution (see Table 13).

Result. From the P values, it is seen that MODU distribution is better fit than uniform distribution.

Example 15. The data was collected from the daily attendance register of the supplementary language class (20 working days in a month) from the same higher secondary school in Palakkad district, Kerala. We took a sample of 50 students and let X be the number of days these students attended in the same for the whole year 2012-2013 (an academic year is 10 months) (see Tables 14, 15, and 16).

We arbitrarily fix 3 months, and the m.l.e.s are computed. Fit discrete uniform to the above data. We have the results as shown in Tables 17 and 18.

TABLE 16: Mean m.l.e. of θ and mean SE of θ for the samples of 10 months.

Sample size	$\hat{a} = \max(x)$	Mean $\hat{\theta}$	SE ($\hat{\theta}$)
50	20	0.3630059	0.0033407

TABLE 17: Goodness of fit of uniform distribution with $\alpha = 0.05$.

	May	September	February
m.l.e. of $\theta = \hat{\theta}$	0.3511	0.3630	0.3594
m.l.e. of $a = \hat{a}$	20	19	18
χ^2 statistics	23.60	25.20	35.60
P value	0.0027	0.0014	0.00002

TABLE 18: Mean median and mode of the data in May, September, and February.

a	Month	m.l.e. of $\theta = \hat{\theta}$	Mean	Median	Mode
20	May	0.3510	7.80	5	3
	September	0.3619	6.78	5	1
	February	0.3594	6.82	5	3

TABLE 19: Goodness of fit of MODU distribution with $\alpha = 0.05$.

	May	September	February
m.l.e. of $\theta = \hat{\theta}$	0.3511	0.3630	0.3594
$\hat{a}$	20	19	18
χ^2 statistics	9.80007	7.23453	7.79048
P value	0.1333	0.2997	0.2538

Since mode < median < mean, the data exhibits positive skewness with m.l.e. of $\theta < 1$ and it is observed that the distribution is unimodal; hence, MODU distribution is supposed to give a better fit than uniform distribution (see Table 19).

Result. From the P values it is seen that, here, MODU distribution is a better fit than uniform distribution.

7. Conclusions

The p.m.f.s of MODU distribution $g(x, \theta)$ satisfies the relation $g(x, \theta) = g(a - x + 1, 1/\theta)$, $x = 1, 2, \dots, a$, $\theta > 0$. The failure rate of $\text{MODU}(a, \theta)$ is IFR, when $\theta > (a - 2x)/(2a - 2x)$, DFR, when $\theta < (a - 2x)/(2a - 2x)$, and constant, when $\theta = (a - 2x)/(2a - 2x)$. Like discrete uniform distribution, the probability function degenerates when a is large. In the case of $\text{MODU}(a, \theta)$, the standard deviation is decreasing with the increasing value of θ , when $\theta > 1$, and is increasing with increasing value of θ , when $\theta < 1$. For $X \sim \text{MODU}(a, \theta)$ and $X \sim \text{MODU}(a, 1/\theta)$, the $\text{sd}(X)$ is equal in both cases. It is also observed that $\text{entropy}(a, \theta) = \text{entropy}(a, 1/\theta)$. The MODU distribution is positively skewed, when $\theta < 1$, since, here, $\text{mode} < \text{median} < \text{mean}$ and the distribution is negatively skewed, when $\theta > 1$, since then $\text{mode} > \text{median} > \text{mean}$. It is to be noted that the distribution is unimodal; that is, when $\theta < 1$, the $\text{mode} = 1$ and, when $\theta > 1$, $\text{mode} = a$. An application of MODU distribution is also discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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