Research Article

Wavelets Convergence and Unconditional Bases for $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$

Devendra Kumar

Department of Mathematics, Faculty of Science, Al-Baha University, P.O. Box 1988, Al-Aqiq, Al-Baha 65431, Saudi Arabia

Correspondence should be addressed to Devendra Kumar; d_kumar001@rediffmail.com

Received 25 October 2013; Accepted 16 December 2013; Published 6 February 2014

Academic Editors: Z. Wang and J. Zhang

Copyright © 2014 Devendra Kumar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove that reasonable nice wavelets form unconditional bases in function space other than $L^2(\mathbb{R}^n, X)$. Moreover, characterization of convergence of wavelet series in $L^p(\mathbb{R}^n, X)$ space and Hardy space $H^1(\mathbb{R}^n, X)$ has been obtained. Here, $X$ is a Banach space with boundedness of Riesz transform.

1. Introduction

Let $X$ be a separable Banach space and $(e_n)_{n \in \mathbb{N}}$ a sequence of vectors in $X$. We say that $(e_n)_{n \in \mathbb{N}}$ is an unconditional basis for $X$ if for every $x \in X$ there exists a unique sequence of scalars $(c_n)_{n \in \mathbb{N}}$ such that

$$x = \sum_{m \in \mathbb{N}} c_m e_m, \quad (1)$$

where the series converges in the norm and as the net of sums over finite subsets of $\mathbb{N}$. In other words we require that $\sum_{n \in \mathbb{N}} |c_\sigma(n)| e_\sigma(n)$ converges for every permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$. Again, a consequence of the uniform boundedness principle is that the coefficient functionals $f_n : x \mapsto c_n$ are continuous. If $X$ is reflexive, then $f_n$ is an unconditional basis for the dual space $X^*$.

Wavelet systems provide explicit unconditional bases for many function spaces. For instance, any orthonormal wavelet $\psi$ such that $\psi$ and $\psi'$ have a common radial decreasing $L^1$-majorant also generates an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$, and $H^1(\mathbb{R})$, although it has less regularity.

It can be shown [1] that if two coefficient spaces are equal (as subsets of $C^N$), then their norms are equivalent and corresponding Banach spaces $L^p(\mathbb{R})$, $1 < p < \infty$, and Hardy space $H^1(\mathbb{R})$ are (topologically) isomorphic through an isomorphism of one basis to another.

In this paper we will consider the spaces $L^p(\mathbb{R})$, $1 < p < \infty$, and $H^1(\mathbb{R}^n)$ and try to study that the reasonable nice wavelets form unconditional bases in these function spaces. Also, the convergence of wavelet series in these spaces will be characterized. There are several equivalent definitions of real Hardy space $H^1(\mathbb{R}^n)$. In the real variable theory, one finds (at least) four major types of characterizations of $H^1$: integrability of maximal functions, integrability of square functions, integrability of conjugate functions (the Riesz transform or its variants), and atomic decompositions.

Let $T$ be a classical Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with smooth kernel $K$ and let $T^*$ be the associated maximal singular integral

$$T^*(f)(x) = \sup_{\varepsilon > 0} |T^\varepsilon f(x)|, \quad (2)$$

where $T^\varepsilon f(x)$ is the truncation at level $\varepsilon$:

$$T^\varepsilon f(x) = \int_{|y-x| > \varepsilon} f(x-y) K(y) \, dy. \quad (3)$$

The $j$th Riesz transform, $1 \leq j \leq n$, is the singular integral operator

$$[\mathbb{R}_j f](x) = \text{P.V.} \int f(x-y) \frac{y_j}{|y|^{n+1}} \, dy \quad \equiv \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}} \, dy. \quad (4)$$
The principal value integral above exists for all $x$ if $f$ is a compactly supported smooth function and one shows for such functions the $L^p(\mathbb{R}^n)$ estimates:

$$\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

(5)

for some positive constant $C$ independent of $f$. Then a bounded operator $R_j$ can be defined on $L^p(\mathbb{R}^n)$ by the density argument. A subsurface arises when one tries to show that the principal value in (4) exists for almost all $x$ in $\mathbb{R}^n$ for each function $f$ in $L^p(\mathbb{R}^n)$. Following a well-known principle, one looks for $L^q(\mathbb{R}^n)$ estimates for the maximal Riesz transform $R_j^* f$ and one indeed proves that [2, 3]

$$\|R_j^* f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}, \quad 1 < p < \infty.$$  

(6)

The following results [4] improve inequality (6).

For $1 < p \leq \infty$ there exists a constant $C = C_{p,n}$ such that

$$\|R_j^* f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}$$

(7)

for each function $f$ belonging to some $L^q(\mathbb{R}^n)$, $1 \leq q \leq \infty$.

Inequality (7) still holds in the limiting case $p = 1$ and the answer is provided by next result.

Given $j, 1 \leq j \leq n$, and a positive constant $C$, there exists a function $f$ in $L^1(\mathbb{R}^n)$ such that

$$\|R_j^* f\|_{L^q(\mathbb{R}^n)} \geq C\|f\|_{L^q(\mathbb{R}^n)}$$

(8)

**Remark 1.** Inequality (8) does not hold even for the Hilbert transform, which is the case $n = 1$.

Stein and Weiss originally defined the real Hardy space $H^1(\mathbb{R})$ to be

$$H^1(\mathbb{R}) = \{ f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R}) \},$$

(9)

$$\|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})},$$

where $H$ denotes the Hilbert transform $(Hf)(x) = \text{P.V.} \int_{\mathbb{R}} (f(x-t)/\pi t)dt$ for $n > 1$, and we use Riesz transform.

In order to define real Hardy space $H^1(\mathbb{R}^n)$, set $b = (1, 0, \ldots, 0)$ and $a = (-1, 0, \ldots, 0)$ and let $\mu$ be the length measure on the segment joining $a$ and $b$. For an appropriate constant $C_n$ we have

$$\mu = C_n \left( \frac{1}{|x|^{n-1}} \ast \sum_{j=1}^{n} R_j (\partial_j \mu) \right).$$

(10)

Taking the Fourier transforms on both sides, we obtain

$$\delta_a - \delta_b = \partial_j \mu = R_j \left( \sum_{j=1}^{n} R_j (\partial_j \mu) \right).$$

(11)

Let $\varphi$ be a nonnegative continuously differentiable and compactly supported function in the unit ball such that $\varphi(0) = 1$, and set the approximate identity as

$$\varphi_x(x) = \mathbb{E}^x \varphi \left( \frac{x}{\varepsilon} \right).$$

(12)

Convolving identity (10) by approximate identity $\varphi_x$, we get

$$\varphi_x(x-a) - \varphi_x(x-b) = R_1 \left( C_n \sum_{j=1}^{n} R_j (\partial_j \mu) \ast \varphi_x \right).$$

(13)

Set $f_x = C_n \sum_{j=1}^{n} R_j (\partial_j \mu) \ast \varphi_x = C_n \sum_{j=1}^{n} R_j (\mu \ast \partial_j \varphi_x).$ Since $\mu \ast \partial_j \varphi_x$ is a compactly supported function in $L^\infty(\mathbb{R}^n)$ with zero integral, thus $\mu \ast \partial_j \varphi_x$ is a function in the Hardy space $H^1(\mathbb{R}^n)$ and so $f_x \in L^1(\mathbb{R}^n)$. From (13) we see that

$$\|(R_1)(f_x)\|_1 \leq 2.$$  

(14)

Now we define

$$H^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : R_j f \in L^1(\mathbb{R}^n) \},$$

(15)

$$\|f\|_{H^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \|R_j f\|_{L^1(\mathbb{R}^n)},$$

where $R_j$ denotes the Riesz transform.

The Hardy space $H^1(\mathbb{R}^n, X)$ is initially defined in terms of the atomic decomposition as follows.

Atoms are those measurable functions $a_j : \mathbb{R}^n \to \mathbb{C}$ for which there exist balls $B_i$ in $\mathbb{R}^n$ such that

$$\text{supp}(a_j) \subset B_i, \quad \int a_j(x) \, dx = 0,$$

(16)

$$\sum_{j=1}^{\infty} \|a_j\|_{L^p(\mathbb{R}^n)} |B_i|^{1/p'} < \infty$$

for some value of $p \in [1, \infty]$ being fixed and $p'$ denotes the conjugate exponent, $1/p + 1/p' = 1$.

We say that $f \in H^1(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$ has an expansion of the form $f(x) = \sum_{i=1}^{\infty} a_i(x)$, and the norm is defined as

$$\|f\|_{H^1(\mathbb{R}^n)} = \inf \left\{ \sum_{i=1}^{\infty} \|a_i\|_{L^p(\mathbb{R}^n)} |B_i|^{1/p'} \right\}.$$  

(17)

Atomic Hardy space is a Banach space

$$H^1(\mathbb{R}^n) = \left\{ \sum_{i=1}^{\infty} a_i x : a_i \text{are atoms} \right\}.$$  

(18)

Notice that $H^1(\mathbb{R}^n) \subseteq \{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f = 0 \}$, but the converse inclusion is false.

If in the definition of atoms we consider only dyadic intervals in place of balls in $\mathbb{R}^n$, we obtain a "smaller" space called the dyadic real Hardy space $H^1_{\text{dyadic}}(\mathbb{R}).$ The spaces $H^1(\mathbb{R})$ and $H^1_{\text{dyadic}}(\mathbb{R})$ are (topologically) isomorphic Banach spaces. This isomorphism leads to a nonconstructive proof of the existence of an unconditional basis for $H^1(\mathbb{R})$. Namely, the Haar system provides a perfect unconditional basis for $H^1_{\text{dyadic}}(\mathbb{R})$, because no regularity is needed in the dyadic setting. In this paper we approach is different and based on the fact that smoother wavelets will directly provide an unconditional basis for $H^1(\mathbb{R}^n)$; we will characterize the coefficient space by taking the wavelet basis in $H^1(\mathbb{R}^n)$. 
It should be significant to mention here that it would be better to obtain constants depending only on $p$ and regularity of wavelet, but no other information about it. For that reason, a better approach is to prove square function estimates using Calderón-Zygmund decomposition and real interpolation [5].

A Calderón-Zygmund operator is a bounded linear operator $T^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with operator norm at most $k$ and such that there exists a $C^1$-function $k(x, y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|k(x, y)| \leq k|x - y|^{-n},$$

$$|\nabla_x k(x, y)| + |\nabla_y k(x, y)| \leq k|x - y|^{-n-1},$$

$$T^* f(x) = \int_{\mathbb{R}^n} k(x, y) f(y) \, dy,$$

for $f \in L^2(\mathbb{R}^n), \quad x \notin \text{supp}(f).$

A function with these properties is called a Calderón-Zygmund kernel or a standard singular kernel.

### 2. Some Definitions and Auxiliary Results

**Definition 2.** A Banach space $X$ is UMD if for some (and then all, cf. [6]) $1 < p < \infty$ there is a constant $C$ so that

$$\left\| \sum_{k=1}^{n} \alpha_k d_k \right\|_{L^p(\Omega, X)} \leq C \left\| \sum_{k=1}^{n} \alpha_k \right\|_{L^p(\Omega)}$$

(20)

for all $n \in \mathbb{Z}_+$, whenever $(\alpha_k)_{k=1}^{n} \in \{-1, 1\}^n$ and $(d_k)_{k=1}^{n} \in L^p(\Omega, X)$, a martingale difference sequence or an arbitrary probability space $(\Omega, A, \mu)$ (i.e., there are sub-$\sigma$-algebras $A_0 \subset A_1 \subset \cdots \subset A_n \subset A$), such that, for all $k = 1, \ldots, n$, the function $d_k$ is $A_k$-measurable and $\int_A d_k \, d\mu = 0$ for all $A \in A_{k-1}$.

**Remark 3.** The validity of Theorem 1.9 of [7] on $H^1(\mathbb{R}^n, X)$ actually characterizes the UMD-property of $X$. Indeed, let $X$ be any complex Banach space and let conclusion of Theorem 1.9 of [7] be satisfied. Since $\|\mathcal{R}_j f\|_{L^p(\mathbb{R}^n, X)} \leq C \|f\|_{L^p(\mathbb{R}^n, X)}$, this is equivalent to the UMD-property of $X$. We also have $\|\mathcal{R}_j f\| \leq C \|f\|_{L^p(\mathbb{R}^n, X)}$. This implies that $\mathcal{R}_j$ is bounded on $H^1(\mathbb{R}^n, X)$.

In view of above remark we can get without modification in [8] the following.

**Proposition 4.** Let $k(x, y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the standard estimates

$$|k(x, y)| \leq k|x - y|^{-n},$$

$$|\nabla_x k(x, y)| + |\nabla_y k(x, y)| \leq k|x - y|^{-n-1},$$

(21)

Assume, moreover, that $T$ is bounded on $L^2(\mathbb{R}^n)$ with operator norm at most $k$. Then $T$ is also bounded on $L^2(\mathbb{R}^n, X)$, where

$$\|\mathcal{R}_j f\|_{L^p(\mathbb{R}^n, X)} \leq C_{j,p} \|f\|_{L^p(\mathbb{R}^n, X)}$$

(22)

for all $p \in [1, \infty[$, and it is bounded from $H^1(\mathbb{R}^n, X)$ to $L^1(\mathbb{R}^n, X)$ with norm $C_{j,n}(X) k$. If, in addition,

$$[T^1](y) = \int_{\mathbb{R}^n} k(x, y) \, dx \equiv 0,$$

then $T$ is bounded on $H^1(\mathbb{R}^n, X)$ with norm $\leq C_{0}(X) k$.

Let us note that atomic definition of $H^1(\mathbb{R}^n, X)$ is known to agree with the one given in terms of various maximal functions, even for an arbitrary Banach space $X$. One can check that the proof of this fact in the scalar case, as given, for example, in Stein’s book [9], goes through word by word in the general setting for $n = 1$; the conjugate Hardy space defined as the domain of the Hilbert transform on $L^1(\mathbb{R}, X)$ with the graph norm is always contained in the atomic Hardy space and agrees with it exactly when $X$ is a UMD-space. For $n > 1$, we can replace Hilbert transform by Riesz transform and the condition of UMD-space for $X$ by the boundedness of Riesz transform in $L^2(\mathbb{R}^n)$-space.

It is well-known fact that an integral operator satisfying the standard estimates and bounded on $L^2(\mathbb{R}^n, X)$ is also bounded from $H^1(\mathbb{R}^n, X)$ to $L^1(\mathbb{R}^n, X)$. The square function description of $H^1(\mathbb{R}^n, X)$ that we have in mind involves the wave expansion of a function in $H^1(\mathbb{R}^n)$. Recall the definition of the wavelet [10–12]. Let $\{\psi^j : l = 1, 2, \ldots, 2^n - 1\}$ be a set of functions belonging to $L^2(\mathbb{R}^n)$.

Consider

$$\phi^j(x) = 2^{in/2} \frac{\psi(2^jx - k)}{2^jx_k},$$

(24)

then $T$ is bounded on $H^1(\mathbb{R}^n, X)$ with norm $\leq C_{0}(X) k$.

The aim of the present paper is to study that reasonable nice wavelets form unconditional bases in function space other than $L^2(\mathbb{R}^n)$ and characterization of convergence of wavelets series in $L^2(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$ has been obtained. For this purpose it will be enough to assume that $(\varphi_{i})_{i \in \Lambda}$, where $\Lambda$ is the set of dyadic cube of the form

$$Q_{j,k} = \prod_{i=1}^{n} \left[ 2^{-j-1} k_2 - 2^{-j}(k_1 + 1) \right],$$

(25)

is an orthonormal wavelet basis such that $\varphi \in C^1(\mathbb{R}^n)$ and

$$|\varphi(x)| + |\varphi'(x)| \leq C(1 + |x|)^{-n-2}.$$

(26)

This is certainly not the weakest possible assumption on $\varphi$ for somewhat less restrictive; see [10] (where the class $\mathcal{R}^0(\mathbb{R}^n)$ is introduced).
3. Main Results

Theorem 5. Let $1 < p < \infty$ and let $\varphi_A$ be a wavelet such that $\varphi \in C^1(\mathbb{R}^n)$ and satisfies condition (26). Then the system $(\varphi_A)_{A \in \Lambda}$ is an unconditional basis for $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$.

Proof. For arbitrary family of $\varphi_A \in C, A \in \Lambda$ such that $|\varphi_A| \leq 1$, we define $T_{\varphi_A}$ on the orthonormal basis by

$$T_{\varphi_A} : L^2(\mathbb{R}^n, X) \rightarrow L^2(\mathbb{R}^n, X), \quad T_{\varphi_A} : \varphi_A \varphi_A_A. \quad (27)$$

Here $T_{\varphi_A}$ is not necessarily unitary, but it is bounded and

$$\|T_{\varphi_A}\|_{L^2 \rightarrow L^2} \leq 1. \quad (28)$$

Now we can write

$$(T_{\varphi_A} f)(x) = \sum_{A \in \Lambda} \langle f, \varphi_A \rangle \varphi_A(x) = \int_{\mathbb{R}^n} k_{\varphi_A}(x, y) f(y) dy, \quad (29)$$

where

$$k_{\varphi_A}(x, y) = \sum_{A \in F} \langle \varphi_A, \varphi_A \rangle \varphi_A(x) \varphi_A(y) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} ^{2^j} \varphi \left(2^j x - k \right) \varphi \left(2^j y - k \right), \quad (30)$$

where $F \subset \Lambda$ is any finite set.

We claim that $T_{\varphi_A}$ are Calderón-Zygmund operators with constants uniform in $\varphi_A$ and for that we have to show that $k_{\varphi_A}$ satisfy standard estimates uniformly in $\varphi_A$.

Consider

$$|k_{\varphi_A}(x, y)| \leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} 2^{nj} \left(1 + |2^j x - k| \right)^{-n-2} \times \left(1 + |2^j y - k| \right)^{-n-2} \leq \sum_{j \in \mathbb{Z}} ^{2^j} \left(1 + 2^{j} |x - y| \right)^{-n-2} \leq C_n |x - y|^{-n}. \quad (31)$$

Also for derivatives

$$\nabla_x k_{\varphi_A}(x, y) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} v_j \varphi \left(2^j x - k \right) \varphi \left(2^j y - k \right), \quad (32)$$

Condition $\int_{\mathbb{R}^n} f = 0 \Rightarrow \int_{\mathbb{R}^n} T_{\varphi_A} f = 0$ follows from $\int_{\mathbb{R}^n} k_{\varphi_A}(x, y) dy = 0$, which in turn follows from $\int_{\mathbb{R}^n} \varphi(x) = 0$. Therefore from Proposition 4, we know that operators $T_{\varphi_A}$ extend continuously to $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$, and we get inequalities

$$\|T_{\varphi_A} f\|_{L^p(\mathbb{R}^n, X)} \leq C_{n, p} \|f\|_{L^p(\mathbb{R}^n, X)}, \quad (33)$$

and particularly for $f = \sum_{A \in F} \varphi_A \varphi_A A, F$ is a finite subset of $\Lambda, C_A \in C$, can be written as

$$\left\| \sum_{A \in F} \varphi_A \varphi_A A \varphi_A \right\|_{L^p(\mathbb{R}^n, X)} \leq C_{n, p} \left\| \sum_{A \in F} \varphi_A \varphi_A A \varphi_A \right\|_{L^p(\mathbb{R}^n, X)}, \quad (34)$$

These inequalities imply that $(\varphi_A)_{A \in \Lambda}$ is an unconditional basis for

$$\text{span}_{L^p(\mathbb{R}^n, X)} \left( \{\varphi_A : A \in \Lambda \} \right), \quad (35)$$

$$\text{span}_{H^1(\mathbb{R}^n, X)} \left( \{\varphi_A : A \in \Lambda \} \right). \quad (36)$$

However, it is still not clear that $(\varphi_A)_{A \in \Lambda}$ is complete in $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$.

In particular, if we take $(e_A)_{A \in \Lambda} \in \{\pm 1\}^\Lambda$, then (34) and $T_{e_A}^2$ imply that

$$c_{n, p} \|f\|_{L^p(\mathbb{R}^n, X)} \leq \|T_{e_A} f\|_{L^p(\mathbb{R}^n, X)} \leq C_{n, p} \|f\|_{L^p(\mathbb{R}^n, X)}, \quad (37)$$

$$c_n \|f\|_{H^1(\mathbb{R}^n, X)} \leq \|T_{e_A} f\|_{H^1(\mathbb{R}^n, X)} \leq C_n \|f\|_{H^1(\mathbb{R}^n, X)}. \quad (38)$$
Let us assume that $Ω = \{±1\}^Λ$ with the product probability measure. First we consider the function of the form $f = \sum_{A∈Λ} c_Aφ_A$ for some finite $F ⊆ Λ$ and $c_A ∈ C$. Integrating (38) over $Ω$, interchanging the order of integration, and using Khintchine's inequality, we get

\[
\left( \int_Ω \|T_x f\|_{L^p(R^n,ξ)}^p dξ \right)^{1/p} \\
× \left( \int_Ω \left| \sum_{A∈Λ} e_ξ c_Aφ_A \right|^p dξ \right)^{1/p} \leq c_{n,p} \left( \sum_{A∈Λ} |c_A|^2 |φ_A|^2 \right)^{1/2} \left\| f \right\|_{L^p(R^n,ξ)},
\]

(40)

Similarly, we obtain

\[
c_{n,p} \left\| f \right\|_{L^p(R^n,ξ)} \leq \left( \sum_{A∈Λ} |c_A|^2 |φ_A|^2 \right)^{1/2} \left\| f \right\|_{L^p(R^n,ξ)} \leq C_{n,p} \left\| f \right\|_{L^p(R^n,ξ)}.
\]

(41)

By virtue of monotone convergence theorem as finite sets $F$ exhaust $Λ$, we get

\[
\left( \sum_{A∈Λ} \left| \langle f, φ_A \rangle \right|^2 \right)^{1/2} \leq C_{n,p} \left\| f \right\|_{L^p(R^n,ξ)}.
\]

(44)

In order to prove the completeness of $(φ_A)_{A∈Λ}$ in $L^p(R^n,ξ)$, consider $f \in L^p(R^n,ξ) ∩ L^2(R^n,ξ)$ such that $f = \sum_{i=1}^∞ \langle f, φ_{j,k} \rangle φ_{j,k}$ with convergence in $L^2(R^n,ξ)$ and there exists a subsequence of partial sums $(\sum_{i=1}^{N_n} \langle f, φ_{j,k} \rangle φ_{j,k})_{n∈N}$ that converge almost everywhere on $R^n$. By virtue of Fatou’s lemma and (41) we obtain

\[
\left\| f - \sum_{i=1}^N \langle f, φ_{j,k} \rangle φ_{j,k} \right\|_{L^p(R^n,ξ)} \leq \liminf_{m→∞} \left\| \sum_{i=1}^m \langle f, φ_{j,k} \rangle φ_{j,k} \right\|_{L^p(R^n,ξ)} \leq C_{n,p} \left( \sum_{i=1}^{∞} \left| \langle f, φ_{j,k} \rangle \right|^2 \right)^{1/2} \left\| f \right\|_{L^p(R^n,ξ)}.
\]

(45)

In view of (44) we have

\[
\left( \sum_{A∈Λ} \left| \langle f, φ_A \rangle \right|^2 \right)^{1/2} \left\| f \right\|_{L^p(R^n,ξ)} < ∞.
\]

(46)

Using dominated convergence theorem, we obtain

\[
f = \sum_{i=1}^∞ \langle f, φ_{j,k} \rangle φ_{j,k},
\]

(47)

with convergence in $L^p(R^n,ξ)$. Therefore $\text{span}_{Λ} L^p(R^n,ξ)(φ_A : A ∈ Λ)$ is dense in $L^p(R^n,ξ) ∩ L^2(R^n,ξ)$ since $L^p(R^n,ξ) ∩ L^2(R^n,ξ)$ is dense in $L^p(R^n,ξ)$. These facts lead to the conclusion that $(φ_A)_{A∈Λ}$ is complete in $L^p(R^n,ξ)$. By virtue of (35) this proves that $(φ_A)_{A∈Λ}$ is an unconditional basis for $L^p(R^n,ξ)$.

Similarly we can prove the result for $H^1(R^n,ξ)$ using (42). Hence the proof is completed.

**Theorem 6.** Let $1 < p < ∞$ and let $φ_A$ be a wavelet such that $φ ∈ C^1(R^n)$ and satisfies condition (26). Then for $f ∈ L^p(R^n,ξ)$ one has

\[
c_{n,p} \left\| f \right\|_{L^p(R^n,ξ)} \leq \left( \sum_{A∈Λ} \left| \langle f, φ_A \rangle \right|^2 \right)^{1/2} \left\| f \right\|_{L^p(R^n,ξ)}
\]

(48)

and for $f ∈ H^1(R^n,ξ)$ one has

\[
c_{n,p} \left\| f \right\|_{H^1(R^n,ξ)} \leq \left( \sum_{A∈Λ} \left| \langle f, φ_A \rangle \right|^2 \right)^{1/2} \left\| f \right\|_{H^1(R^n,ξ)}.
\]

(49)

Now we take $f ∈ L^p(R^n,ξ)$ for some $F$ finite subset of $Λ$ and define $(γ_A)_{A∈Λ}$ by $γ_A = 1$ if $A ∈ F$ and by $γ_A = 0$ if $A ∉ F$, so that $T_x f = \sum_{A∈Λ} (f, φ_A) φ_A$. Applying estimate (41) to $T_x f$ and using (34), we get

\[
c_{n,p} \left\| T_x f \right\|_{L^p(R^n,ξ)} \leq \left( \sum_{A∈Λ} \left| \langle f, φ_A \rangle \right|^2 \right)^{1/2} \left\| T_x f \right\|_{L^p(R^n,ξ)} \leq C_{n,p} \left\| f \right\|_{L^p(R^n,ξ)}.
\]

(43)
Proof. We have
\[
\|\langle f, \varphi_A \rangle \|_{L^p(R^n, X)} \| \varphi_A \|_{L^p(R^n, X)}' 
\]
where the space BMO($R^n$) is functions with bounded mean oscillation. $H^1(R^n)$ is the predual of BMO($R^n$). We see that \( \{\langle \cdot, \varphi_A \rangle : A \in \Lambda \} \) defines continuous linear functionals on $L^p(R^n, X), 1 < p < \infty$, and $H^1(R^n, X)$. If $f = \sum_{A \in \Lambda} C_A \varphi_A$ unconditionally in $L^p(R^n, X)$ or $H^1(R^n, X)$, then for every $A' \in \Lambda$ by continuity of $\langle \cdot, \varphi_{A'} \rangle$
\[
\langle f, \varphi_A \rangle = \sum_{A \in \Lambda} C_A \langle \varphi_A, \varphi_{A'} \rangle = \sum_{A \in \Lambda} C_A \delta_{AA'} = C_{A'}. \tag{51}
\]
Now for every $f \in L^p(R^n, X)$ we have $f = \sum_{i=1}^N \langle f, \psi_{j_k} \rangle \psi_{j_k}$ with convergence in $L^p(R^n, X)$. Using (41), we obtain
\[
\left\| \sum_{i=1}^N \langle f, \psi_{j_k} \rangle \varphi_{j_k} \right\|_{L^p(R^n, X)} \leq C_{n,p} \left( \sum_{A \in \Lambda} \left| \sum_{i=1}^N \langle f, \psi_{j_k} \rangle \varphi_{j_k} \right|^2 \right)^{1/2}_{L^p(R^n, X)}.
\]
Combining (44) and (52), we get (48). Similarly we would handle the case for $H^1(R^n, X)$. Hence the proof is completed.

Corollary 7. For every family of scalars $(C_A)_{A \in \Lambda}$ one has that
\[
\sum_{A \in \Lambda} C_A \varphi_A \text{ converges unconditionally in } L^p(R^n, X)
\]
\[
\iff \left( \sum_{A \in \Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} < \infty.
\]
\[
\sum_{A \in \Lambda} C_A \varphi_A \text{ converges unconditionally in } H^1(R^n, X)
\]
\[
\iff \left( \sum_{A \in \Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} < \infty.
\]

Proof. Suppose that $(C_A)_{A \in \Lambda}$ is a family of scalars such that $\left( \sum_{A \in \Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} < \infty$. For every $\varepsilon > 0$ we can find $F_0$ finite subset of $\Lambda$ such that $\left( \sum_{A \in \Lambda \setminus F_0} |C_A|^2 |\varphi_A|^2 \right)^{1/2} \leq \varepsilon$. Then for every $F$ finite subset of $\Lambda \setminus F_0$ by (44), we have
\[
\left\| \sum_{A \in F} C_A \varphi_A \right\|_{L^p(R^n, X)} \leq C_{n,p} \varepsilon. \tag{54}
\]
This implies that the series $\sum_{A \in \Lambda} C_A \varphi_A$ converges unconditionally in $L^p(R^n, X)$ to the same $L^p$-function. Moreover, we have proved that
\[
\sum_{A \in \Lambda} C_A \varphi_A \text{ converges unconditionally in } L^p(R^n, X) \iff \left( \sum_{A \in \Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} < \infty. \tag{53}
\]
This gives the characterization of the convergence of wavelet series in $L^p(R^n, X)$. Similarly we would characterize convergence in $H^1(R^n, X)$.

Hence the proof is completed.

Conflict of Interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment
This work is supported by University Grants Commission, New Delhi, under the major research Project by F. no. 41-792/2012 (SR).

References

Submit your manuscripts at http://www.hindawi.com