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Research Article

Modified Theories of Gravitation behind the Spacetime Deformation

Gagik Ter-Kazarian

Ambartsumian Byurakan Astrophysical Observatory, Aragatsotn, 378433 Byurakan, Armenia

Correspondence should be addressed to Gagik Ter-Kazarian; gago_50@yahoo.com

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In the framework of proposed theory of spacetime deformation/distortion, we have a way to deform the spacetime through a nontrivial choice of the distortion-complex, displaying different connections, which may reveal different post-Riemannian spacetime structures as corollary. We extend this theory to address, in particular, the gauge model of the most general metric-affine gravity carrying both nontrivial torsion and nonmetricity. This model is constructed in the framework of the first order Lagrangian expressed in terms of the gauge potentials and their first derivatives. The equations of the standard theory, which have no propagating modes for torsion, can be equivalently replaced in modified framework by the modified equations, which in the limit of reducing the affine group leads to the modified Einstein-Cartan theory with dynamical torsion and beyond. In testing the modified framework for various particular cases, we use the Lagrange multipliers for extinguishing nonmetricity and torsion.

1. Introduction

We will not attempt a history of gauge theory of gravitation in general, but only of those that seem most relevant to the particular theory of this paper. From its historical development, the efforts in gauge treatment of gravity mainly focus on the quantum gravity and microphysics, with the recent interest, for example, in the theory of the quantum superstring or, in the very early universe, in the inflationary model. The papers on the gauge treatment of gravity provide a unified picture of gravity modified models based on several Lie groups [1–61]. However, currently no single theory has been uniquely accepted as the convincing gauge theory of gravitation which could lead to a consistent quantum theory of gravity. They have evoked the possibility that the treatment of spacetime might involve non-Riemannian features on the scale of the Planck length. This necessitates the study of dynamical theories involving post-Riemannian geometries. On the other hand, a general way to deform the spacetime metric with constant curvature has been explicitly posed by [62–66]. The problem was initially solved only for 3D spaces, but consequently it was solved also for spacetimes of any dimension. It was proved that any semi-Riemannian metric can be obtained as a deformation of constant curvature matric, with this deformation being parameterized by a 2-form. A novel definition of spacetime metric deformations, parameterized in terms of scalar field matrices, is proposed by [67]. However, without care of the historical justice and authenticity, it also should be emphasized that the continuous deformations within the space of connections were studied by [17, 18]. This concept of a deformation seems to be a special case of a prolongation. A main idea is as follows: since any tensor-valued one-form transforms homogenously with respect to linear gauge transformations, its subtraction from a connection can be regarded as a continuous deformation within the space of connections. A more general approach is suggested by [68], whereas the authors present the extension to 4D of an euclidean 2D model that exhibits spontaneous generation of a metric. In this model gravitons emerge as Goldstone bosons of a global $SO(D) \times GL(D)$ symmetry broken down to $SO(D)$. They formulated microscopic theory without having to appeal to any particular spacetime metric and only assume the preexistence of a manifold endowed with an affine connection. Moreover, here it seems that not even a flat metric needs to be assumed. In this sense the microscopic theory is quasitopological. The vierbein appears as a condensate of the fundamental fermions. In spite of having nonstandard characteristics, the microscopic theory

appears to be renormalizable. Recently, some clarifications are also collected in the book [69], where the authors have tried to collect the established results and to focus new investigations on the real loopholes of the theory. Only time will tell whether any of these intriguing theories is correct and which of the hypothesized hidden symmetries is actually realized in nature.

It is well known that the notions of space and connections should be separated; see, for example, [13, 70–72]. The curvature and torsion are in fact properties of a connection, and many different connections are allowed to exist in the same spacetime. Therefore, when considering several connections with different curvature and torsion, one takes spacetime simply as a manifold and connections as additional structures. Allowing minimal departure from semi-Riemannian geometry would consist in admitting torsion. The concept of a linear connection as an independent and primary structure of spacetime is the fundamental proposal put forward by Cartan's geometrical analysis [43–46], whereas he gave a beautiful geometrical interpretation of torsion representing a translational misfit. The Einstein-Cartan (EC) theory also called Einstein-Cartan-Sciama-Kibble theory is the minimal extension of the general relativity, which considers curvature and torsion as representing independent degrees of freedom. In the standard EC theory, the equation defining torsion is of algebraic type, and not a differential equation, and no propagation of torsion is allowed. Even at the textbook level it was common knowledge known from the weak interaction that the causality reasons do not respect a contact interaction. Therefore, many modifications of the EC theory have been proposed in recent years; see, for example, [37, 73–79], but all these approaches are subject to many uncertainties. In particular, in [80] the authors analyze the functional renormalization group flow of quantum gravity on the EC theory space. The latter consists of all action functionals depending on the spin connection and the vielbein field (coframe) which are invariant under both spacetime diffeomorphisms and local frame rotations. They find evidence for the existence of at least one non-Gaussian renormalization group fixed point which seems suitable for the asymptotic safety construction in a setting where the spin connection and the vielbein are the fundamental field variables. The difference between Einstein's general relativity and its Cartan extension is analyzed within the scenario of asymptotic safety of quantum gravity by [81], whereas it was found that the four-fermion interaction distinguishes the Einstein-Cartan theory from its Riemannian limit. However, modifications of EC theory are intricate. In a recent paper [82], we construct the *two-step spacetime deformation* (TSSD) theory which generalizes and, in particular cases, fully recovers the results of the conventional theory of spacetime deformation [62–66]. Conceptually and techniquewise the TSSD theory is versatile and powerful and manifests its practical and technical virtue in the fact that through a nontrivial choice of explicit form of a *world-deformation tensor*, which we have at our disposal, in general, we have a way to deform the spacetime displaying different connections, which may reveal different post-Riemannian spacetime structures as corollary: (1) the Weitzenböck spacetime structure—(W_4) underlying a teleparallelism theory of gravity, see, for example, [23, 24, 38, 83–93]; (2) the RC manifold—(U_4) underlying EC theory, for a comprehensive references, see, for example, [34–37, 51, 52, 94]; (3) or even the most general linear connection of metric-affine gravity (MAG) taking values in the Lie algebra of the 4D-affine group, $A(4,R) = R^4 \ltimes GL(4,R)$; see, for example, [15–18, 42–50, 55–57]. This represents the semidirect product of the group of 4D-translations and general linear 4D-transformations. Continuing along this line, in the present paper, we address the essential features of the gauge model of the most general MAG theory in context of TSSD-construction of post-Riemannian geometry, whereas to complete the TSSD theory, we build up the *distortion-complex* (DC) and show how it restores the *world-deformation tensor*, which still has been put in [82] by hand. A formulation of the major physical aspects of the modified gauge MAG theory will be given in the framework of the first order Lagrangian expressed in terms of the gauge potentials and their first derivatives. All the fundamental gravitational structures in fact, the metric as much as the coframes and connections, acquire a DC induced theoretical interpretation. There is another line of reasoning which supports the side of this model. We address the key problem of a *dynamical torsion* and show that the equations of the standard MAG theory can be equivalently replaced by the set of *modified MAG equations* in which the torsion, in general, is turned out to be *dynamical*. We define the physical conditions for the spacetime deformations when the spin-spin interaction becomes short-range propagating. As an application we have to test the general TSSD-MAG framework in some limit, namely, we have to put on Lagrange multipliers to recover the TSSD versions of different (sub)cases of Poincaré gauge theory (PG), Einstein-Cartan theory, teleparallel gravity ($GR_{||}$), and general relativity (GR). This allows amplifying and substantiating the assertions made in [82]. Moreover, imposing different physical constraints upon the spacetime deformations, in this modified framework we may reproduce the term in the well known Lagrangian of pseudoscalar-photon interaction theory, or terms in the Lagrangians of pseudoscalar theories [95–100]. This paper is organized as follows. To start with, in Section 2 we complete the spacetime deformation theory [82] by introducing DC complex and showing how it restores the *world-deformation tensor*. We also briefly revisit the theory of TSSD to make the rest of the paper understandable. An outline of the key points of TSSD theory of relevance to gauge MAG theory is stated in Section 3. The concluding remarks are presented in Section 4. The appendices discuss some relevant topics on the TSSD theory and algebraic operations in use.

2. Spacetime Deformations and Beyond: Model Building

Let us consider a smooth deformation map $\overline{\Omega} : M_4 \to \mathscr{M}_4$, written in terms of the *world-deformation* tensor $\overline{\Omega}$, general, $\mathscr{M}_4$, and flat, M_4, smooth differential 4D-manifolds. The tensor, $\overline{\Omega}$, can be written in the form $\overline{\Omega} := \overline{Df}$ $(\overline{\Omega}^m{}_l := \overline{D}^m_\mu \overline{f}^\mu_l)$, where the distortion-complex (DC) includes the invertible

distortion matrix $\overline{D}$ ($\overline{D}^m_\mu$) and the tensor $\overline{f}$ ($\overline{f}^\mu_l := \partial_l x^\mu$ and $\partial_l := \partial/\partial x^l$). Here we use Greek alphabet ($\mu, \nu, \rho, \ldots = 0, 1, 2, 3$) to denote the holonomic world indices related to $\mathscr{M}_4$ and the second half of Latin alphabet ($l, m, k, \ldots = 0, 1, 2, 3$) to stand for the world indices related to M_4. The principle foundation of a *world-deformation* tensor comprises the following two steps [101]: (1) the basis vectors e_m at given point ($p \in M_4$) undergo *distortion* transformations by means of the matrix $\overline{D}$; and (2) the diffeomorphism $x^\mu(x^l) : M_4 \to \mathscr{M}_4$ is constructed by seeking a new holonomic coordinates $x^\mu(x^l)$ as the solutions of the first-order partial differential equations. Namely,

$$e_\mu = \overline{D}^l_\mu e_l, \quad e_\mu \overline{f}^\mu_l = \overline{\Omega}^m{}_l e_m, \tag{1}$$

where the conditions of integrability, $\partial_k \overline{f}^\mu_l = \partial_l \overline{f}^\mu_k$, and nondegeneracy, $\|\overline{f}\| \neq 0$, necessarily hold [102, 103]. For reasons that will become clear in the sequel, we write the norm ds of infinitesimal displacement dx^μ on $\mathscr{M}_4$ in terms of the spacetime structures of M_4:

$$ds := e\vartheta = e_\mu \otimes \vartheta^\mu = \overline{\Omega}^m{}_l e_m \otimes \vartheta^l \in \mathscr{M}_4. \tag{2}$$

The holonomic metric on the space $\mathscr{M}_4$ can be recast in the form $g = g_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu = g(e_\mu, e_\nu)\vartheta^\mu \otimes \vartheta^\nu$, with the components $g_{\mu\nu} = g(e_\mu, e_\nu)$ in dual holonomic basis $\{\vartheta^\mu \equiv dx^\mu\}$. In order to relate local Lorentz symmetry to more general deformed spacetime, there is, however, a need to introduce the soldering tools, which are the linear frames and forms in tangent fiber bundles to the external general smooth differential manifold, whose components are the so-called tetrad (vierbein) fields. The $\mathscr{M}_4$ has at each point a tangent space, $T_x\mathscr{M}_4$, spanned by the anholonomic orthonormal frame field, e, as a shorthand for the collection of the 4-tuplet $(e_0, \ldots, e_3)$, where $e_a = e_a{}^\mu \partial_\mu$. We use the first half of Latin alphabet ($a, b, c, \ldots = 0, 1, 2, 3$) to denote the anholonomic indices related to the tangent space. The frame field, e, then defines a dual vector, ϑ, of differential forms, $\vartheta = \begin{pmatrix} \vartheta^0 \\ \vdots \\ \vartheta^3 \end{pmatrix}$, as a shorthand for the collection of the $\vartheta^b = e^b{}_\mu dx^\mu$, whose values at every point form the dual basis, such that $e_a \rfloor \vartheta^b = \delta^b_a$, where $\rfloor$ denotes the interior product; namely, this is a C^∞-bilinear map $\rfloor : \Omega^1 \to \Omega^0$ with Ω^p denoting the C^∞-modulo of differential p-forms on $\mathscr{M}_4$. In components $e_a{}^\mu e^b{}_\mu = \delta^b_a$. On the manifold, $\mathscr{M}_4$, the tautological tensor field, id, of type $(1, 1)$ is defined which assigns to each tangent space the identity linear transformation. Thus for any point $x \in \mathscr{M}_4$ and any vector $\xi \in T_x\mathscr{M}_4$, one has $id(\xi) = \xi$. In terms of the frame field, the ϑ^a give the expression for id as $id := e\vartheta = e_0 \otimes \vartheta^0 + \cdots e_3 \otimes \vartheta^3$, in sense that both sides yield ξ when applied to any tangent vector ξ in the domain of definition of the frame field. One can also consider general transformations of the linear group, $GL(4, R)$, taking any basis into any other set of four linearly independent fields. The notation, $\{e_a, \vartheta^b\}$, will be used below for general linear frames.

Let us introduce the so-called *first deformation matrices*, $\overline{\pi}^m{}_k$ and $\pi^a{}_l$ ($\in GL(4, \mathscr{M})\ \forall x^\mu$), as follows:

$$\begin{gathered} \overline{D}^m_\mu = e_\mu{}^k \overline{\pi}^m{}_k, \qquad \overline{f}^\mu_l = e^\mu{}_k \overline{\pi}^k{}_l, \qquad e_\mu{}^k e^\mu{}_m = \delta^k_m, \\ \pi_a{}^m = e_a{}^\mu \overline{D}^m_\mu, \qquad \pi^a{}_l = e^a{}_\mu \overline{f}^\mu_l, \end{gathered} \tag{3}$$

where $g_{\mu\nu} e_k{}^\mu e_s{}^\nu = \eta_{ks}$ and η_{ks} is the metric on M_4. With this provision, we build up a *world-deformation* tensor, $\overline{\Omega}^m{}_l = \overline{\pi}^m{}_k \overline{\pi}^k{}_l = \pi^m{}_a \pi^a{}_l$, which yields local tetrad deformations

$$\begin{gathered} e_a = \pi_a{}^m e_m, \qquad \vartheta^a = \pi^a{}_l \vartheta^l, \\ \overline{e}_k = \overline{\pi}^m{}_k e_m, \qquad \overline{\vartheta}^k = \overline{\pi}^k{}_l \vartheta^l, \end{gathered} \tag{4}$$

and $ds = id = e_a \otimes \vartheta^a = \overline{e}_k \otimes \overline{\vartheta}^k \in \mathscr{M}_4$. The *first deformation matrices* π, in general, are the elements of the quotient group $GL(4, \mathscr{M})/SO(3, 1)$. Here $GL(4, \mathscr{M})/SO(3, 1)$ is the manifold of left cosets of the group $GL(4, \mathscr{M})/SO(3, 1)$ over its subgroup $SO(3, 1)$. If we deform the cotetrad according to (4), we have two choices to recast metric as follows: either writing the deformation of the metric in the space of tetrads or deforming the tetrad field:

$$g = o_{ab}\vartheta^a \otimes \vartheta^b = o_{ab}\pi^a{}_l \pi^b{}_m \vartheta^l \otimes \vartheta^m = \gamma_{lm}\vartheta^l \otimes \vartheta^m, \tag{5}$$

where γ_{lm}, following [82], is called *second deformation matrix* and reads $\gamma_{lm} = o_{ab}\pi^a{}_l \pi^b{}_m$. The deformed metric splits as

$$g_{\mu\nu} = \Upsilon^2 \eta_{\mu\nu} + \gamma_{\mu\nu}, \tag{6}$$

where $\Upsilon = \pi^a{}_a = \overline{\pi}^k{}_k$, and

$$\gamma_{\mu\nu} = \left(\gamma_{al} - \Upsilon^2 o_{al}\right) e^a{}_\mu e^l{}_\nu = \left(\gamma_{ks} - \Upsilon^2 \eta_{ks}\right) e^k{}_\mu e^s{}_\nu. \tag{7}$$

The anholonomic orthonormal frame field, e, relates g to the tangent space metric, $o_{ab} = \mathrm{diag}(+---)$, by $o_{ab} = g(e_a, e_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu$, which has the converse $g_{\mu\nu} = o_{ab} e^a{}_\mu e^b{}_\nu$ because $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$. The γ_{lm} can be decomposed in terms of symmetric, $\pi_{(al)}$, and antisymmetric, $\pi_{[al]}$, parts of the matrix $\pi_{al} = o_{ac}\pi^c{}_l$ (or, respectively, in terms of $\overline{\pi}_{(kl)}$ and $\overline{\pi}_{[kl]}$, where $\overline{\pi}_{kl} = \eta_{ks}\overline{\pi}^s{}_l$) as

$$\begin{aligned} \gamma_{al} = {} & \Upsilon^2 o_{al} + 2\Upsilon\Theta_{al} + o_{cd}\Theta^c{}_a \Theta^d{}_l \\ & + o_{cd}\left(\Theta^c{}_a \varphi^d{}_l + \varphi^c{}_a \Theta^d{}_l\right) + o_{cd}\varphi^c{}_a \varphi^d{}_l, \end{aligned} \tag{8}$$

where

$$\pi_{al} = \Upsilon o_{al} + \Theta_{al} + \varphi_{al}, \tag{9}$$

provided Θ_{al} is the traceless symmetric part and φ_{al} is the skew symmetric part of the first deformation matrix. The anholonomy objects defined on the tangent space, $T_x\mathscr{M}_4$, read

$$C^a := d\vartheta^a = \frac{1}{2} C^a{}_{bc} \vartheta^b \wedge \vartheta^c, \tag{10}$$

where the anholonomy coefficients, $C^a{}_{bc}$, which represent the curls of the base members, are

$$\begin{aligned} C^c{}_{ab} &= -\vartheta^c([e_a, e_b]) = e_a{}^\mu e_b{}^\nu \left(\partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu\right) \\ &= -e^c{}_\mu \left[e_a\left(e_b{}^\mu\right) - e_b\left(e_a{}^\mu\right)\right] \\ &= 2\pi^c{}_l e^\mu_m \left(\pi^{-1m}{}_{[a}\partial_\mu \pi^{-1l}{}_{b]}\right). \end{aligned} \tag{11}$$

In particular case of constant metric in the tetradic space, the deformed connection can be written as

$$\Gamma^a{}_{bc} = \frac{1}{2}\left(C^a{}_{bc} - o^{aa'} o_{bb'} C^{b'}{}_{a'c} - o^{aa'} o_{cc'} C^{c'}{}_{a'b}\right). \tag{12}$$

The deformation $\overline{\Omega} : M_4 \to \mathscr{M}_4$ comprises the following two 4D deformations $\breve{\Omega} : M_4 \to V_4$ and $\Omega : V_4 \to \mathscr{M}_4$, where V_4 is the semi-Riemannian space and $\breve{\Omega}$ and Ω are the corresponding *world-deformation* tensors. All magnitudes related to the space, V_4, will be denoted with an over "$\breve{}$". According to (1), now we have $\breve{\Omega}^m{}_l = \breve{D}^m_\mu \breve{f}^\mu_l$ and $\Omega^\mu{}_\nu = D^\mu_\rho f^\rho_\nu$, provided

$$\begin{aligned} \breve{e}_\mu &= \breve{D}^l_\mu e_l, \qquad \breve{e}_\mu \breve{f}^\mu_l = \breve{\Omega}^m{}_l e_m, \\ e_\rho &= D^\mu_\rho \breve{e}_\mu, \qquad e_\rho f^\rho_\nu = \Omega^\mu{}_\nu \breve{e}_\mu. \end{aligned} \tag{13}$$

In analogy with (3), the following relations hold:

$$\begin{aligned} \breve{D}^m_\mu &= \breve{e}_\mu{}^k \breve{\pi}^m_k, \qquad \breve{f}^\mu_l = \breve{e}^\mu{}_k \breve{\pi}^k{}_l, \qquad \breve{e}_\mu{}^k \breve{e}^\mu{}_m = \delta^k_m, \\ \breve{\pi}_a{}^m &= \breve{e}_a{}^\mu \breve{D}^m_\mu, \qquad \breve{\pi}^a{}_l = \breve{e}^a{}_\mu \breve{f}^\mu_l, \end{aligned} \tag{14}$$

where $\breve{\Omega}^m{}_l = \breve{\pi}^m{}_\rho \breve{\pi}^\rho{}_l$, $\Omega^\mu{}_\nu = \pi^\mu{}_\rho \pi^\rho{}_\nu$. So, $\breve{g}_{\mu\nu}\breve{e}_k{}^\mu \breve{e}_s{}^\nu = \eta_{ks}$, and

$$\begin{aligned} D^\mu_\rho &= e_\nu{}^\mu \pi^\nu{}_\rho, \qquad f^\rho_\nu = e^\rho{}_\mu \pi^\mu{}_\nu, \qquad e_\nu{}^\mu e^\nu{}_\rho = \delta^\mu_\rho, \\ \pi_a{}^\mu &= e_a{}^\rho D^\mu_\rho, \qquad \pi^a{}_\nu = e^a{}_\rho f^\rho_\nu. \end{aligned} \tag{15}$$

The norm $d\breve{s} = i\breve{d}$ of the infinitesimal displacement $d\breve{x}^\mu$ on the V_4 can be written in terms of the spacetime structures of M_4 as

$$i\breve{d} := \breve{e}\breve{\vartheta} = \breve{\Omega}^m{}_l e_m \otimes \vartheta^l \in V_4. \tag{16}$$

The holonomic metric can be recast in the form

$$\breve{g} = \breve{g}_{\mu\nu}\breve{\vartheta}^\mu \otimes \breve{\vartheta}^\nu = \breve{g}\left(\breve{e}_\mu, \breve{e}_\nu\right)\breve{\vartheta}^\mu \otimes \breve{\vartheta}^\nu. \tag{17}$$

The anholonomy objects defined on the tangent space, $\breve{T}_{\breve{x}}V_4$, read

$$\breve{C}^a := d\breve{\vartheta}^a = \frac{1}{2}\breve{C}^a{}_{bc}\,\breve{\vartheta}^b \wedge \breve{\vartheta}^c, \tag{18}$$

where the anholonomy coefficients, $\breve{C}^a{}_{bc}$, which represent the curls of the base members, are

$$\begin{aligned} \breve{C}^c{}_{ab} &= -\breve{\vartheta}^c([\breve{e}_a, \breve{e}_b]) = \breve{e}_a{}^\mu \breve{e}_b{}^\nu \left(\breve{\partial}_\mu \breve{e}^c{}_\nu - \breve{\partial}_\nu \breve{e}^c{}_\mu\right) \\ &= -\breve{e}^c{}_\mu \left[\breve{e}_a\left(\breve{e}_b{}^\mu\right) - \breve{e}_b\left(\breve{e}_a{}^\mu\right)\right]. \end{aligned} \tag{19}$$

The connection form in terms of the frame field is then determined by

$$d\breve{\vartheta} + \breve{\Gamma} \wedge \breve{\vartheta} = 0, \tag{20}$$

and the curvature by

$$\breve{R} = d\breve{\Gamma} + \breve{\Gamma} \wedge \breve{\Gamma}, \tag{21}$$

where the (anholonomic) Levi-Civita (or Christoffel) connection can be written as

$$\breve{\Gamma}_{ab} := \breve{e}_{[a}\rfloor d\breve{\vartheta}_{b]} - \frac{1}{2}\left(\breve{e}_a\rfloor \breve{e}_b\rfloor d\breve{\vartheta}_c\right) \wedge \breve{\vartheta}^c \tag{22}$$

and $\breve{\vartheta}_c$ is understood as the downindexed 1-form $\breve{\vartheta}_c = o_{cb}\breve{\vartheta}^b$. In the usual Riemannian language involving the holonomic metric $\breve{g}$ (17) defined from the tetrads, the anholonomic Christoffel connection (22) transforms into

$$\breve{\Gamma}_{ab} := -\,d\breve{x}^\mu \breve{e}_{[a}{}^\nu \left(\partial_\mu \breve{e}_{\nu b]} - \breve{\Gamma}^\sigma{}_{\mu\nu}\breve{e}_{\sigma b]}\right), \tag{23}$$

where $\breve{\Gamma}^\sigma{}_{\mu\nu}$ is the ordinary holonomic Levi-Civita connection. The corresponding curvature reduces to

$$\breve{R}_{ab} = \frac{1}{2} d\breve{x}^\mu \wedge d\breve{x}^\nu \breve{e}_{[a}{}^\rho \breve{e}_{\sigma b]} \breve{R}_{\mu\nu\rho}{}^\sigma, \tag{24}$$

provided $\breve{R}_{\mu\nu\rho}{}^\sigma := \breve{\vartheta}^\sigma(R_{\breve{e}_\nu \breve{e}_\rho}\breve{e}_\mu)$ is the ordinary Riemann tensor:

$$\breve{R}_{\mu\nu\rho}{}^\sigma = 2\left(\breve{\partial}_{[\mu}\breve{\Gamma}^\sigma{}_{\nu]\rho} + \breve{\Gamma}^\sigma{}_{[\mu\lambda}\breve{\Gamma}^\lambda{}_{\nu]\rho}\right), \tag{25}$$

and hence, the Ricci tensor takes the form

$$\breve{R} = \breve{R}_{\breve{e}_\mu \breve{e}_\nu}\breve{\vartheta}^\mu \breve{\vartheta}^\nu = \breve{R}_{\mu\nu}\breve{\vartheta}^\mu \breve{\vartheta}^\nu = \breve{R}_{\mu\sigma\nu}{}^\sigma \breve{\vartheta}^\mu \breve{\vartheta}^\nu. \tag{26}$$

The norm id (2) can then be written in terms of the spacetime structures of V_4 and M_4:

$$\begin{aligned} id &:= e\vartheta = e_\rho \otimes \vartheta^\rho = e_a \otimes \vartheta^a = \Omega^\mu{}_\nu \breve{e}_\mu \otimes \breve{\vartheta}^\nu \\ &= \Omega^a{}_b e_a \vartheta^b = \overline{\Omega}^m_l e_m \otimes \vartheta^l \in \mathscr{M}_4, \end{aligned} \tag{27}$$

provided

$$\begin{aligned} \Omega^a{}_b &= \pi^a{}_c \pi^c{}_b = \Omega^\mu_\nu \breve{e}^a{}_\mu \breve{e}_b{}^\nu, \qquad e_\rho = \pi^\nu{}_\rho \breve{e}_\nu, \\ \vartheta^\rho &= \pi_\mu{}^\rho \breve{\vartheta}^\mu, \qquad e_c = \pi_c{}^a \breve{e}_a, \qquad \vartheta^c = \pi^c{}_b \breve{\vartheta}^b. \end{aligned} \tag{28}$$

Under a local tetrad deformation (28), a general spin connection transforms according to

$$\omega^a{}_{b\mu} = \pi_c{}^a \breve{\omega}^c{}_{d\mu} \pi^d{}_b + \pi_c{}^a \partial_\mu \pi^c{}_b = \pi_l{}^a \partial_\mu \pi^l{}_b. \tag{29}$$

We have then two choices to recast metric as follows:

$$g = o_{ab}\vartheta^a \otimes \vartheta^b = o_{ab}\pi^a{}_c \pi^b{}_d \breve{\vartheta}^c \otimes \breve{\vartheta}^d = \gamma_{cd}\breve{\vartheta}^c \otimes \breve{\vartheta}^d. \tag{30}$$

In the first case, the contribution of the Christoffel symbols constructed by the metric $\gamma_{ab} = o_{cd}\pi^c{}_a \pi^d{}_b$ reads

$$\begin{aligned} \Gamma^a{}_{bc} &= \frac{1}{2}\left(\breve{C}^a{}_{bc} - \gamma^{aa'}\gamma_{bb'}\breve{C}^{b'}{}_{a'c} - \gamma^{aa'}\gamma_{cc'}\breve{C}^{c'}{}_{a'b}\right) \\ &\quad + \frac{1}{2}\gamma^{aa'}\left(\breve{e}_c\rfloor d\gamma_{ba'} - \breve{e}_b\rfloor d\gamma_{ca'} - \breve{e}_{a'}\rfloor d\gamma_{bc}\right). \end{aligned} \tag{31}$$

As before, the second deformation matrix, γ_{ab}, can be decomposed in terms of symmetric, $\pi_{(ab)}$, and antisymmetric, $\pi_{[ab]}$, parts of the matrix $\pi_{ab} = o_{ac}\pi^{c}{}_{b}$. So,

$$\pi_{ab} = \Upsilon o_{ab} + \Theta_{ab} + \varphi_{ab}, \tag{32}$$

where $\Upsilon = \pi^{a}{}_{a}$, Θ_{ab} is the traceless symmetric part, and φ_{ab} is the skew symmetric part of the first deformation matrix. In analogy with (6), the deformed metric can then be split as

$$g_{\mu\nu}(\pi) = \Upsilon^{2}(\pi)\,\breve{g}_{\mu\nu} + \gamma_{\mu\nu}(\pi), \tag{33}$$

where

$$\gamma_{\mu\nu}(\pi) = \left[\gamma_{ab} - \Upsilon^{2} o_{ab}\right]\breve{e}^{a}{}_{\mu}\breve{e}^{b}{}_{\nu}. \tag{34}$$

The inverse deformed metric reads

$$g^{\mu\nu}(\pi) = o^{cd}\pi^{-1a}{}_{c}\,\pi^{-1b}{}_{d}\,\breve{e}_{a}{}^{\mu}\,\breve{e}_{b}{}^{\nu}, \tag{35}$$

where $\pi^{-1a}{}_{c}\pi^{c}{}_{b} = \pi^{c}{}_{b}\pi^{-1a}{}_{c} = \delta^{a}_{b}$. Hence, the usual Levi-Civita connection is related to the original connection by the relation

$$\widetilde{\Gamma}^{\mu}{}_{\rho\sigma} = \breve{\Gamma}^{\mu}{}_{\rho\sigma} + \Pi^{\mu}{}_{\rho\sigma}, \tag{36}$$

provided

$$\begin{aligned}\Pi^{\mu}{}_{\rho\sigma} &= 2g^{\mu\nu}\breve{g}_{\nu(\rho}\breve{\nabla}_{\sigma)}\breve{\Upsilon} - \breve{g}_{\rho\sigma}g^{\mu\nu}\breve{\nabla}_{\nu}\breve{\Upsilon}\\ &\quad + \frac{1}{2}g^{\mu\nu}\left(\breve{\nabla}_{\rho}\breve{\gamma}_{\nu\sigma} + \breve{\nabla}_{\sigma}\breve{\gamma}_{\rho\nu} - \breve{\nabla}_{\nu}\breve{\gamma}_{\rho\sigma}\right),\end{aligned} \tag{37}$$

where $\breve{\nabla}$ is the covariant derivative. The contravariant deformed metric $g^{\nu\rho}$ is defined as the inverse of $g_{\mu\nu}$: $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}$. Hence, the connection deformation $\Pi^{\mu}{}_{\rho\sigma}$ acts like a force that deviates the test particles from the geodesic motion in the space, V_4. A metric-affine space $(\mathscr{M}_4, g, \Gamma)$ is defined to have a metric and a linear connection that need not be dependent on each other. In general, the lifting of the constraints of metric compatibility and symmetry yields the new geometrical property of the spacetime, which are the *nonmetricity* 1-form N_{ab} and the affine *torsion* 2-form T^{a} representing a translational misfit (for a comprehensive discussion see [35–37, 94]). These, together with the *curvature* 2-form $R_{a}{}^{b}$, symbolically can be presented as [8, 9]

$$\left(N_{ab}, T^{a}, R_{a}{}^{b}\right) \sim \mathscr{D}\left(g_{ab}, \vartheta^{a}, \Gamma_{a}{}^{b}\right), \tag{38}$$

where $\mathscr{D}$ is the *covariant exterior derivative*. If the nonmetricity tensor $N_{\lambda\mu\nu} = -\mathscr{D}_{\lambda}g_{\mu\nu} \equiv -g_{\mu\nu;\lambda}$ does not vanish, the general formula for the affine connection written in the spacetime components is

$$\Gamma^{\rho}{}_{\mu\nu} = \breve{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} - N^{\rho}{}_{\mu\nu} + \frac{1}{2}N_{(\mu\nu)}{}^{\rho}, \tag{39}$$

where $\breve{\Gamma}^{\rho}{}_{\mu\nu}$ is the Riemann part and $K^{\rho}{}_{\mu\nu} := 2Q_{(\mu\nu)}{}^{\rho} + Q^{\rho}{}_{\mu\nu}$ is the non-Riemann part—the affine *contortion tensor*. The torsion, $Q^{\rho}{}_{\mu\nu} = (1/2)T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{[\mu\nu]}$ given with respect to a holonomic frame, $d\vartheta^{\rho} = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components. In the presence of curvature and torsion, the coupling prescription of a general field carrying an arbitrary representation of the Lorentz group will be

$$\partial_{\mu} \longrightarrow \mathscr{D}_{\mu} = \partial_{\mu} - \frac{i}{2}\left(\omega^{ab}{}_{\mu} - K^{ab}{}_{\mu}\right)J_{ab}, \tag{40}$$

with J_{ab} denoting the corresponding Lorentz generator. The Riemann-Cartan manifold, U_4, is a particular case of the general metric-affine manifold $\mathscr{M}_4$, restricted by the metricity condition $N_{\lambda\mu\nu} = 0$, when a nonsymmetric linear connection is said to be metric compatible. The Lorentz and diffeomorphism invariant scalar curvature, R, becomes either a function of $e^{a}{}_{\mu}$ only or $g_{\mu\nu}$:

$$R(\omega) \equiv e_{a}{}^{\mu}\,e_{b}{}^{\nu}\,R_{\mu\nu}{}^{ab}(\omega) = R(g, \Gamma) \equiv g^{\rho\nu}R^{\mu}{}_{\rho\mu\nu}(\Gamma). \tag{41}$$

2.1. Determination of $\overline{D}$ and $\overline{f}$. Let $\omega^{ab} = \omega^{ab}_{\mu} \wedge dx^{\mu}$ be the 1-forms of corresponding connections assuming values in the Lorentz Lie algebra. The action for gravitational field can be written in the form

$$S_{g} = \breve{S} + S_{Q}, \tag{42}$$

where the integral,

$$\begin{aligned}\breve{S} &= -\frac{1}{4æ}\int \star\breve{R} = -\frac{1}{4æ}\int \star\breve{R}_{cd} \wedge \vartheta^{c} \wedge \vartheta^{d}\\ &= -\frac{1}{2æ}\int \breve{R}\sqrt{-g}\,d\Omega,\end{aligned} \tag{43}$$

is the usual Einstein action, with the coupling constant relating to the Newton gravitational constant $æ = 8\pi G_N/c^4$, S_Q is the phenomenological action of the spin-torsion interaction, and $\star$ denotes the Hodge dual (Appendix B). The variation of the connection 1-form ω^{ab} yields

$$\delta S_{Q} = \frac{1}{æ}\int \star\mathscr{T}_{ab} \wedge \delta\omega^{ab}, \tag{44}$$

where

$$\begin{aligned}\star\mathscr{T}_{ab} &:= \frac{1}{2}\star\left(Q_{a} \wedge e_{b}\right) = Q^{c} \wedge \vartheta^{d}\varepsilon_{cdab}\\ &= \frac{1}{2}Q^{c}{}_{\mu\nu} \wedge e^{d}{}_{\alpha}\,\varepsilon_{abcd}\vartheta^{\mu\nu\alpha};\end{aligned} \tag{45}$$

here we used the abbreviated notations for the wedge product monomials, $\vartheta^{\mu\nu\alpha\cdots} = \vartheta^{\mu} \wedge \vartheta^{\nu} \wedge \vartheta^{\alpha} \wedge \cdots$, defined on the U_4 space and

$$Q^{a} = D\vartheta^{a} = d\vartheta^{a} + \omega^{a}{}_{b} \wedge \vartheta^{b}. \tag{46}$$

The variation of the action describing the macroscopic matter sources S_m with respect to the coframe ϑ^{a} and connection 1-form ω^{ab} reads

$$\delta S_{m} = \int \delta L_{m} = \int\left(-\star\theta_{a} \wedge \delta\vartheta^{a} + \frac{1}{2}\star\Sigma_{ab} \wedge \delta\omega^{ab}\right), \tag{47}$$

where $\star\theta_a$ is the dual 3-form relating to the canonical energy-momentum tensor, θ_a^μ, by

$$\star\theta_a = \frac{1}{3!}\theta_a^\mu \varepsilon_{\mu\nu\alpha\beta}\vartheta^{\nu\alpha\beta}. \tag{48}$$

And $\star\Sigma_{ab} = -\star\Sigma_{ba}$ is the dual 3-form corresponding to the canonical spin tensor, which is identical with the dynamical spin tensor S_{abc}, namely,

$$\star\Sigma_{ab} = S^\mu{}_{ab}\varepsilon_{\mu\nu\alpha\beta}\vartheta^{\nu\alpha\beta}. \tag{49}$$

The variation of the total action, $S = S_g + S_m$, with respect to the e_a, ω^{ab}, and Φ, gives the field equations as follows:

$$\begin{aligned} &(1)\ \frac{1}{2}\breve{R}_{cd}\wedge\vartheta^c = æ\theta_d = 0,\\ &(2)\ \star\mathcal{T}_{ab} = -\frac{1}{2}æ\star\Sigma_{ab},\\ &(3)\ \frac{\delta L_m}{\delta\Phi} = 0, \qquad \frac{\delta L_m}{\delta\overline{\Phi}} = 0. \end{aligned} \tag{50}$$

The DC-members $\overline{D}$ and $\overline{f}$ can then be determined as follows:

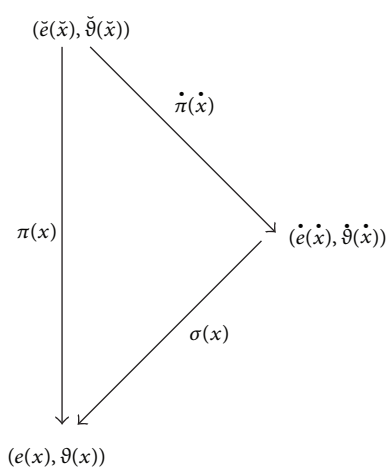
$$\overline{D}_a^l = \eta^{lm}\langle e_a, e_m\rangle, \qquad \overline{f}_l^a = \eta_{lm}\vartheta^a\left(\vartheta^{-1}\right)^m. \tag{51}$$

However, it should be emphasized that the standard Riemannian (and its extensions) space interacting quantum field theory cannot be a satisfactory ground for addressing the most important processes of rearrangement of vacuum state and gauge symmetry breaking in gravity at huge energies. The difficulties here arise because the Riemannian geometry, in general, does not admit a group of isometries and it is impossible to define energy-momentum as Noether local currents related to exact symmetries. This, in turn, posed severe problem of nonuniqueness of the physical vacuum and the associated Fock space. A definition of positive frequency modes cannot, in general, be unambiguously fixed in the past and future, which leads to $|\text{in}\rangle \neq |\text{out}\rangle$, because the state $|\text{in}\rangle$ is unstable against decay into many particle $|\text{out}\rangle$ states due to interaction processes allowed by lack of Poincaré invariance. A nontrivial Bogolubov transformation between past and future positive frequency modes implies that particles are created from the vacuum and this is one of the reasons for $|\text{in}\rangle \neq |\text{out}\rangle$. So, it is another line of reasoning which supports the side suggested in this section approach. Namely, the actual advantage of our approach compared to the more conservative direct constructions, using the vierbeins, metric, and various connections, is that it allows various generalizations and alternatives of the standard gravity. In this framework, in particular, we develop an alternative *distortion gauge induced* fiber-bundle formulation of gravitation—*General Gauge Principle* (GGP) [101]. In this, we restrict ourself to consider only the simplest spacetime deformation map, $\breve{\Omega} : M_4 \to V_4$ ($\Omega^\mu{}_\nu \equiv \delta_\nu^\mu$). Whereas given the principal fiber bundle $\breve{P}(V_4, G_V; \breve{s})$ with the structure group G_V, the local coordinates $\breve{p} \in \breve{P}$ are $\breve{p} = (\breve{x}, U_V)$, where $\breve{x} \in V_4$ and $U_V \in G_V$, the total bundle space $\breve{P}$ is a smooth manifold and the surjection $\breve{s}$ is a smooth map $\breve{s} : \breve{P} \to V_4$. The collection of matter fields of arbitrary spins $\breve{\Phi}(\breve{x})$ takes values in standard fiber over $\breve{x} : \breve{s}^{-1}(\breve{\mathcal{U}}_i) = \breve{\mathcal{U}}_i \times \breve{F}_{\breve{x}}$ where $\{\breve{\mathcal{U}}_i\}$ is a set of open coverings of V_4. The action of the structure group G_V on $\breve{P}$ defines an isomorphism of the Lie algebra $\breve{\mathfrak{g}}$ of G_V onto the Lie algebra of vertical vector fields on $\breve{P}$ tangent to the fiber at each $\breve{p} \in \breve{P}$ called fundamental. Note that an invariance of the Lagrangian $L_{\breve{\Phi}}$ under spacetime diffeomorphisms in V_4 implies an invariance of $L_{\breve{\Phi}}$ under the G_V group and vice versa if, and only if, the generalized local gauge transformations of the fields $\breve{\Phi}(\breve{x})$ and their covariant derivative $\breve{\nabla}_\mu\breve{\Phi}(\breve{x})$ are introduced by finite local U_V gauge transformations:

$$\begin{aligned} \breve{\Phi}'(\breve{x}) &= U_V(\breve{x})\breve{\Phi}(\breve{x}),\\ \left[\breve{\gamma}^\mu(\breve{x})\breve{\nabla}_\mu\breve{\Phi}(\breve{x})\right]' &= U_V(\breve{x})\left[\breve{\gamma}^\mu(\breve{x})\breve{\nabla}_\mu\breve{\Phi}(\breve{x})\right]. \end{aligned} \tag{52}$$

Here $\breve{\nabla}_\mu$ denotes the covariant derivative agreed with the metric $\breve{g}^{\mu\nu} = (1/2)(\breve{\gamma}^\mu\breve{\gamma}^\nu + \breve{\gamma}^\nu\breve{\gamma}^\mu)$. For example, $\widetilde{\gamma}^\mu(\widetilde{x}) = \widetilde{e}_a{}^\mu(\widetilde{x})\gamma^a$ for the spinor field ($j = 1/2$), where γ^a are the Dirac matrices, $\breve{\nabla}_\mu = \breve{\partial}_\mu + \breve{\Gamma}_\mu$, $\breve{\Gamma}_\mu(\breve{x}) = (1/2)J^{ab}\breve{e}_a{}^\nu(\breve{x})\breve{\partial}_\mu\breve{e}_{b\nu}(\breve{x})$, and $J^{ab} = -(1/4)[\gamma^a, \gamma^b]$. To involve a drastic revision of the role of gauge fields in the physical concept of the spacetime deformation/distortion, we generalize the standard gauge scheme by exploring a new special type of *distortion* gauge field, (a) $(\equiv a_m(x^l), x^l \in M_4)$, which is assumed to act on the external spacetime groups. That is, we also consider the principle fiber bundle, $P(M_4, U^{\text{loc}}; s)$, with the base space M_4, the structure group U^{loc}, and the surjection s. The matter fields $\Phi(x)$ take values in the standard fiber which is the Hilbert vector space where a linear representation $U(x)$ of group U^{loc} is given. This space can be regarded as the Lie algebra of the group U^{loc} upon which the Lie algebra acts according to the law of the adjoint representation: $a \leftrightarrow ada\,\Phi \to [a, \Phi]$. The GGP framework accounts for the *gravitation gauge group* G_V generated by the hidden local internal symmetry U^{loc}. We pursue a principle goal to build up the *world-deformation* tensor, $\breve{\Omega}(F) = \breve{D}(a)\breve{f}(a)$, where F is the differential form of gauge field $F = (1/2)F_{nm}\vartheta^n \wedge \vartheta^m$. We connect the structure group G_V, further, to the nonlinear realization of the Lie group G_D of *distortion* of the M_4. The nonlinear realization technique or the method of phenomenological Lagrangians [104–110] provides a way to determine the transformation properties of fields defined on the quotient space. In accord, we treat the distortion group G_D and its stationary subgroup $H = SO(3)$, respectively, as the dynamical group and its algebraic subgroup. The fundamental field is distortion gauge field (a) and, thus, all the fundamental gravitational structures in fact, the metric as much as the coframes and connections, acquire a *distortion-gauge induced* theoretical interpretation. We derive the Maurer-Cartan structure equations, where the distortion fields (a) are treated as the Goldstone fields. This framework explores the most important processes of rearrangement of vacuum state and a spontaneous breaking of gravitation gauge symmetry at huge energies.

2.2. TSSD: Revisited. For the benefit of the reader, a brief outline of the key ideas behind the TSSD [82], as a guiding principle, is given in this section to make the rest of the paper understandable. Before we report on the physical foundation of post-Riemannian geometry, we may remark on the form of generic spacetime deformation, $\pi(x)$. When torsion is nonvanishing, the affine connection is no longer coincident with the Levi-Civita connection, and the geometry is no longer Riemannian, but one has a RC spacetime, U_4, with a nonsymmetric, but metric-compatible, connection. Teleparallel gravity, in turn, represented a new way of including torsion into general relativity, an alternative to the scheme provided by the usual Einstein-Cartan-Sciama-Kibble approach. However, the gravitational coupling of the fundamental fields in teleparallel gravity is a very controversial subject [15, 111–114]. The basic difficulty lies in the definition of the spin connection and consequently in the correct form of the gravitational coupling prescription. For a specific choice of the free parameters, teleparallel gravity becomes completely equivalent to GR. In this case it is usually referred to as the teleparallel equivalent of GR. From this point of view, curvature and torsion are simply alternative ways of describing the gravitational field and are consequently related to the same degrees of freedom of gravity. Teleparallel gravity attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. The fundamental difference between these two theories above was that, whereas in the former a torsion is a propagating field having as a source—the energy-momentum tensor, in the latter a torsion is not a propagating field having as a source—the total spin, a point which can be considered a drawback of this model. In fact, the two physical interpretations of torsion described above are clearly conflictive. This problem can be solved just only by experiment. Therefore, we have to separate, from the very outset, these two completely different cases. This reasoning supports our choice of a double deformation map. Namely, following [82], we assume that the spacetime deformation $\pi(x)$ comprises the two ingredient deformations $(\dot{\pi}(\dot{x}), \sigma(x))$. Hence, local tetrad deformations $(\check{e}, \check{\vartheta}) \rightarrow (e, \vartheta)$ (13) are performed according to the following heuristic map, in *two steps* (two-step deformation map):



Thereby the first deformation proper matrix, $\dot{\pi}(\dot{x}) := (\dot{\pi}_b{}^a)(\dot{x})$ $(|\dot{\pi}_b{}^a| = 1)$, satisfies the following peculiar condition:

$$\dot{\pi}_c{}^a\left(\dot{x}\right)\dot{\partial}_\mu \dot{\pi}^{-1\,c}{}_b\left(\dot{x}\right) = \check{\omega}^a{}_{b\mu}(\check{x}), \tag{53}$$

where $\check{\omega}^a{}_{b\mu}(\check{x})$ is a $SO(3,1)$ valued Lorentz (i.e., traceless with respect to its indices a and b) spin connection defined on the semi-Riemannian space. Under a local spacetime deformation $\dot{\pi}(\dot{x})$, the tetrad changes according to

$$\begin{gathered}\dot{e}_c = \dot{\pi}_c{}^a \check{e}_a, \qquad \dot{\vartheta}^c = \dot{\pi}^c{}_b \check{\vartheta}^b, \\ \dot{e}_a \otimes \dot{\vartheta}^a = \dot{\Omega}^a{}_b \check{e}_a \otimes \check{\vartheta}^b.\end{gathered} \tag{54}$$

A particular solution to (53) is then (see Appendix A)

$$\dot{\pi}\left(\dot{x}\right) = \dot{\pi}(0)\exp\left[-\int_0^{\dot{x}} \check{\omega}_\mu(\check{x})\, d\dot{x}'^\mu\right]. \tag{55}$$

However, a general solution can be obtained by replacing $\dot{\pi}(0) \rightarrow \pi_B(\dot{x}) \equiv \dot{\pi}(0)B(\dot{x})$ in expression (55), where $B(\dot{x})$ is any proper matrix with determinant 1 $(|B(\dot{x})| = 1)$. The solution (55) resembles the exponential of a bivector. The bivectors are isomorphic to skew-symmetric matrices which generate orthogonal matrices with determinant 1 through the exponential map. Therefore, bivectors are used to generate rotations in any dimension through the exponential map and are a useful tool for classifying such rotations. Recall that, in the case of spacetime rotations, the geometric algebra is $Cl_{3,1}(R)$, and the subspace of bivectors is $\wedge 2R_{3,1}$. Accordingly, the exponential map (55) generates set of all arbitrary rotations (54) of the orthonormal frame $\check{e}_a(\check{x})$ in tangent space, which form the Lorentz group. On the other hand, the universality of gravitation allows the Levi-Civita connection to be interpreted as part of the spacetime definition. The form of the Riemannian connection (22), which is a function of tetrad fields and their derivatives, shows that the relative orientation of the orthonormal frame $\check{e}_a(\check{x} + d\check{x})$ with respect to $\check{e}_a(\check{x})$ (parallel transported to $(\check{x} + d\check{x})$) is completely fixed by the metric. Since a change in this orientation is described by Lorentz transformations, it does not induce any gravitational effects; therefore, from the point of view of the principle of equivalence, there is no reason to prevent independent (due to arbitrary deformations (54)) Lorentz rotations of local frames in the space under consideration. If we want to use this freedom, the spin connection should contain a part which is independent of the metric, which will realize an independent Lorentz rotation of frames under parallel transport. In this way, we are led to a description of gravity which is not in semi-Riemannian space. If all inertial frames at a given point are treated on an equal footing, the spacetime has to have torsion, which is the antisymmetric

part of the affine connection. By virtue of (53) or (55), a general deformed spin connection vanishes:

$$\begin{aligned}\dot{\omega}^{a}{}_{b\mu} &= \dot{\pi}_{c}{}^{a}\,\breve{\omega}^{c}{}_{d\mu}\,\dot{\pi}^{d}{}_{b} + \dot{\pi}_{c}{}^{a}\,\dot{\partial}_{\mu}\dot{\pi}^{c}{}_{b} \\ &= \dot{e}^{a}{}_{\sigma}\,\dot{\Gamma}^{\sigma}{}_{\rho\mu}\,\dot{e}_{b}{}^{\rho} + \dot{e}^{a}{}_{\rho}\,\dot{\partial}_{\mu}\dot{e}_{b}{}^{\rho} \equiv 0.\end{aligned} \tag{56}$$

In fact, a general linear connection, $\dot{\Gamma}^{\mu}{}_{\rho\sigma}$, is related to the corresponding spin connection, $\dot{\omega}^{a}{}_{b\mu}$, through the inverse

$$\dot{\Gamma}^{\mu}{}_{\rho\sigma} = \dot{e}_{a}{}^{\mu}\,\dot{\partial}_{\sigma}\dot{e}^{a}{}_{\rho} + \dot{e}_{a}{}^{\mu}\,\dot{\omega}^{a}{}_{b\sigma}\,\dot{e}^{b}{}_{\rho} = \dot{e}_{a}{}^{\mu}\,\dot{\partial}_{\sigma}\dot{e}^{a}{}_{\rho}, \tag{57}$$

which is the the Weitzenböck connection revealing the Weitzenböck spacetime W_4 of the teleparallel gravity. Thus, $\dot{\pi}(\dot{x})$ can be referred to as the Weitzenböck deformation matrix. The Weitzenböck connection is a connection presenting a nonvanishing torsion, but a vanishing curvature. This recovers a particular case of the teleparallel gravity theory with the dynamical torsion. All magnitudes related to the teleparallel gravity will be denoted with an over "•". Equations (56) and (57) are simply different ways of expressing the property that the total—that is, acting on both indices—derivative of the tetrad vanishes identically. According to the TSSD-map, the next first deformation matrices $\sigma(x) := (\sigma_{b}{}^{a})(x)$ contribute to corresponding ingredient parts according to (A.5)–(A.8). In particular, the tetrad changes according to

$$\begin{aligned} e_{c} &= \sigma_{c}{}^{d}\,\dot{e}_{d}, \qquad \vartheta^{c} = \sigma^{c}{}_{e}\,\dot{\vartheta}^{e}, \\ e_{c}\otimes\vartheta^{c} &= \overline{\chi}_{d}{}^{c}\,\dot{e}_{c}\otimes\dot{\vartheta}^{d} = \Omega_{b}{}^{a}\,\breve{e}_{a}\otimes\breve{\vartheta}^{b},\end{aligned} \tag{58}$$

and a general spin connection transforms according to

$$\omega'^{a}{}_{b\mu} = \sigma_{c}{}^{a}\,\dot{\omega}^{c}{}_{d\mu}\,\sigma^{d}{}_{b} + \sigma_{c}{}^{a}\,\partial_{\mu}\sigma^{c}{}_{b}. \tag{59}$$

So,

$$\overset{(\sigma)}{\omega}{}^{a}{}_{b\mu} := \omega'^{a}{}_{b\mu} = \sigma_{c}{}^{a}\,\partial_{\mu}\sigma^{c}{}_{b} \tag{60}$$

is referred to as a $GL(4,R)$ valued *deformation-related frame connection*, which represents the *deformed properties of the frame* only. Then, it follows that the affine connection, Γ, related to (28) and (58) tetrad deformations, transforms through

$$\begin{aligned}\Gamma^{\mu}{}_{\rho\sigma} &= e_{a}{}^{\mu}\,\partial_{\sigma}e^{a}{}_{\rho} + e_{a}{}^{\mu}\,\overset{(\pi)}{\omega}{}^{a}{}_{b\sigma}e^{b}{}_{\rho} \\ &= \sigma_{a}{}^{\mu}\,\partial_{\sigma}\sigma^{a}{}_{\rho} + \sigma_{a}{}^{\mu}\,\overset{(\sigma)}{\omega}{}^{a}{}_{b\sigma}\,\sigma^{b}{}_{\rho},\end{aligned} \tag{61}$$

where, according to (A.7), we have $\sigma_{a}{}^{\mu}\,\sigma_{\mu}{}^{b} = \delta^{b}_{a}$, also the procedure can be inverted $\sigma_{a}{}^{\mu}\,\sigma_{\nu}{}^{a} = \delta^{\mu}_{\nu}$, and

$$\overset{(\pi)}{\omega}{}^{a}{}_{b\mu} := \omega^{a}{}_{b\mu} = \pi^{a}{}_{c}\,\breve{\omega}^{c}{}_{d\mu}\pi^{db} + \pi^{a}{}_{c}\,\partial_{\mu}\pi^{cb} \tag{62}$$

is a more general $GL(4,R)$ valued spin connection. For our convenience, hereafter the notation, $\{\overset{(A)}{e}_{a}, \overset{(A)}{\vartheta}{}^{b}\}$ $(A = \pi,\sigma)$, will be used for general linear frames

$$\begin{aligned}\left\{\overset{(A)}{e}_{a}, \overset{(A)}{\vartheta}{}^{b}\right\} &= \left\{\left(\overset{(\pi)}{e}_{a}, \overset{(\sigma)}{e}_{a}\right), \left(\overset{(\pi)}{\vartheta}{}^{b}, \overset{(\sigma)}{\vartheta}{}^{b}\right)\right\} \\ &\equiv \left\{\left(e_{a}, \dot{e}_{a}\right), \left(\vartheta^{b}, \dot{\vartheta}^{b}\right)\right\},\end{aligned} \tag{63}$$

where $\overset{(A)}{e}_{a} \rfloor \overset{(A)}{\vartheta}{}^{b} = \delta^{b}_{a}$, or in components, $\overset{(A)}{e}_{a}{}^{\mu}\,\overset{(A)}{e}{}^{b}{}_{\mu} = \delta^{b}_{a}$, and also the procedure can be inverted $\overset{(A)}{e}{}^{a}{}_{\rho}\,\overset{(A)}{e}_{a}{}^{\sigma} = \delta^{\sigma}_{\rho}$, provided

$$\overset{(A)}{e}_{a}{}^{\mu} = \left(\overset{(\pi)}{e}_{a}{}^{\mu}, \overset{(\sigma)}{e}_{a}{}^{\mu}\right) \equiv \left(e_{a}{}^{\mu}, \sigma_{a}{}^{\mu}\right). \tag{64}$$

The affine connection (61) can then be rewritten in the abbreviated form

$$\Gamma^{\mu}{}_{\rho\sigma} = \overset{(A)}{e}_{a}{}^{\mu}\,\partial_{\sigma}\,\overset{(A)}{e}{}^{a}{}_{\rho} + \overset{(A)}{e}_{a}{}^{\mu}\,\overset{(A)}{\omega}{}^{a}{}_{b\sigma}\,\overset{(A)}{e}{}^{b}{}_{\rho}. \tag{65}$$

Since the first deformation matrices $\pi(x)$ and $\sigma(x)$ are arbitrary functions, the transformed general spin connections $\overset{(\pi)}{\omega}(x)$ and $\overset{(\sigma)}{\omega}(x)$, as well as the affine connection (65), are independent of tetrad fields and their derivatives. In what follows, therefore, we will separate the notions of space and connections, the metric-affine formulation of gravity. A metric-affine space (M_4, g, Γ) is defined to have a metric and a linear connection that need not be dependent on each other. The lifting of the constraints of metric-compatibility and symmetry yields the new geometrical property of the spacetime, which are the *nonmetricity* 1-form $\overset{(A)}{N}_{ab}$ and the affine *torsion* 2-form $\overset{(A)}{T}{}^{a}$ representing a translational misfit (for a comprehensive discussion see [35–37, 94]). These, together with the *curvature* 2-form $\overset{(A)}{R}_{a}{}^{b}$, symbolically can be presented as [8, 9]

$$\left(\overset{(A)}{N}_{ab}, \overset{(A)}{T}{}^{a}, \overset{(A)}{R}_{a}{}^{b}\right) \sim \overset{(A)}{\mathscr{D}}\left(\overset{(A)}{g}_{ab}, \overset{(A)}{\vartheta}{}^{a}, \overset{(A)}{\Gamma}_{a}{}^{b}\right), \tag{66}$$

where $\overset{(A)}{\mathscr{D}}$ is the *covariant exterior derivative*. It is well known that, due to the affine character of the connection space [115], one can always add a tensor to a given connection without spoiling the covariance of derivative (A.17). Let us define then a translation in the connection space. Suppose a point in this space will be a Lorentz connection, $\overset{(\pi)}{\omega}(x) := \overset{(\pi)}{\omega}{}^{bc}{}_{\mu}(x)J_{bc}dx^{\mu}$, presenting simultaneously curvature and torsion written in the language of differential forms as

$$\begin{aligned}\overset{(\pi)}{R} &= d\overset{(\pi)}{\omega} + \overset{(\pi)}{\omega}\overset{(\pi)}{\omega} \equiv \mathscr{D}_{\overset{(\pi)}{\omega}}\overset{(\pi)}{\omega}, \\ \overset{(\pi)}{T} &= de + \overset{(\pi)}{\omega}e \equiv \mathscr{D}_{\overset{(\pi)}{\omega}}e,\end{aligned} \tag{67}$$

where $\mathscr{D}_{\overset{(\pi)}{\omega}}$ denotes the covariant differential in the connection $\overset{(\pi)}{\omega}$. Now, given two connections $\overset{(\sigma)}{\omega}(x)$ and $\overset{(\pi)}{\omega}(x)$,

the difference $k = \overset{(\sigma)}{\omega} - \overset{(\pi)}{\omega}$ is also a 1-form assuming values in the Lorentz Lie algebra, but transforming covariantly, whereas its covariant derivative is $\mathscr{D}_{\overset{(\pi)}{\omega}} k = dk + \{\overset{(\pi)}{\omega}, k\}$. The effect of adding a covector k to a given connection $\overset{(\pi)}{\omega}$, therefore, is to change its curvature and torsion 2-forms:

$$\overset{(\sigma)}{R} = \overset{(\pi)}{R} + \mathscr{D}_{\overset{(\pi)}{\omega}} k + kk, \qquad \overset{(\sigma)}{T} = \overset{(\pi)}{T} + ke. \tag{68}$$

Since $k^a{}_{bc}$ is a Lorentz-valued covector, it is necessarily antisymmetric in the first two indices. Presenting $k^a{}_{bc} = (1/2)k^a{}_{(bc)} + (1/2)k^a{}_{[bc]}$, we may define $k^a{}_{[bc]} \equiv t^a{}_{bc}$, such that

$$k^a{}_{bc} = \frac{1}{2}\left(t_a{}^c{}_b + t_b{}^c{}_a - t^a{}_{bc}\right). \tag{69}$$

Turning to the connection appearing in the covariant derivative (A.17): $\overset{(\pi)}{\Omega}{}^a{}_{bc} \equiv \overset{(\pi)}{\omega}{}^a{}_{bc} - \overset{(\pi)}{K}{}^a{}_{bc}$, a translation in the connection space with parameter $k^a{}_{bc}$ corresponds to

$$\overset{(\sigma)}{\Omega}{}^a{}_{bc} = \overset{(\pi)}{\Omega}{}^a{}_{bc} + k^a{}_{bc} \equiv \overset{(\pi)}{\omega}{}^a{}_{bc} - \overset{(\pi)}{K}{}^a{}_{bc} + k^a{}_{bc}. \tag{70}$$

Since $k^a{}_{bc}$ has always the form of a contortion tensor (69), the above connection is equivalent to $\overset{(\sigma)}{\Omega}{}^a{}_{bc} = \overset{(\pi)}{\omega}{}^a{}_{bc} - \overset{(\sigma)}{K}{}^a{}_{bc}$, with $\overset{(\sigma)}{K}{}^a{}_{bc} = \overset{(\pi)}{K}{}^a{}_{bc} - k^a{}_{bc}$ being another contortion tensor: $\overset{(\sigma)}{K}{}^\lambda{}_{\mu\nu} = \overset{(\pi)}{K}{}^\lambda{}_{\mu\nu} - k^\lambda{}_{\mu\nu}$. There are actually infinitely many choices for $t^a{}_{bc}$, each one defining a different translation in the connection space and consequently yielding a connection $\overset{(\pi)}{\omega}{}^a{}_{bc}$ with different curvature and torsion.

3. Outline of the Key Points of TSSD Gauge MAG Theory

At low energies the spacetime group associated with the matter fields is the Poincaré group. An extension of the Poincaré gauge theory of gravity constructed in the RC geometry, to the most general spacetime symmetry gauge theory, the MAG theory, has been developed by Hehl and collaborators [17, 18]. The MAG theory has the most general type of covariant derivative: in addition to curvature and torsion, the MAG also has nonmetricity, that is, a nonmetric compatible connection. Hence parallel transport no longer preserves length and angle. Note that there are only indications (but no conclusive evidence) for assuming invariance of physical systems under the action of the entire affine group and that in MAG one is far from actually calculating S-matrix elements. Although the theoretical structure of this theory has been developed, we do not yet have much understanding of what new physics are allowed by the MAG theory. One source of improved understanding is exact solutions; see, for example, [116–118] and references therein. But due to the highly nonlinear nature of theory, exact solutions are not easily found unless they have a great deal of symmetry. However, to carry in full generality through the extension of TSSD-ideas as applied to more general metric-affine gravity, it reasonable as a next step to gauge immediately the 4 + 16 parameter *affine* group $A(4,R) = R^4 \otimes GL(4,R)$, which lacks a metric structure altogether, and to introduce the metric *subsequently*. General affine invariance adds *dilation* and *shear* invariance as physical symmetries to Poincaré invariance, and both of these symmetries are of physical importance. Dilation invariance is a crucial component of particle physics in the high energy regime. Shear invariance was shown to yield representations of hadronic matter; the corresponding shear current can be related to hadronic quadrupole excitations. From this in the framework of the gauge theory of the affine group with a metric supplemented, as a physically meaningful field theory, it is speculated that the invariance under affine transformations played an essential part at an early stage of the universe, such that todays Poincaré invariance might be a remnant of affine invariance after some symmetry breaking mechanism. Thus MAG encompasses the PG as a subcase. The rather comprehensive gauging of the affine group was done in [17, 18], where a rigorous mathematical justification of the relevant spacetime structures and equations with more details is to be found. But a brief outline of the key points of relevance to the context of TSSD can be stated here, in which some conventions and results will be borrowed from the presentation given in [17, 18, 42]. To this aim, one enlarges at any point of the base manifold $x \in \mathscr{M}$ a tangent space $T_x\mathscr{M}_4$ to an *affine tangent space* $A_x\mathscr{M}_4$ by allowing freely translating elements of $T_x\mathscr{M}_4$ to different points $p \in A_x\mathscr{M}_4$. The collection of all affine tangent spaces $A_x\mathscr{M}_4$ forms the *affine bundle* $A\mathscr{M}_4$. An *affine frame* of $\mathscr{M}_4$ at x is a pair $(\overset{(A)}{e}{}_a, p)$ (whereas before $A = \pi, \sigma$) consisting of a linear frame $\overset{(A)}{e}{}_a \in L_x\mathscr{M}_4$ and a point $p \in A_x\mathscr{M}_4$. The origin of $A_x\mathscr{M}_4$ is that point $o_x \in A_x\mathscr{M}$ for which the affine frame $(\overset{(A)}{e}{}_a, o_x) \in A_x\mathscr{M}_4$ reduces to the linear frame $\overset{(A)}{e}{}_a \in L_x\mathscr{M}_4$. The transformation behavior of an affine frame $(\overset{(A)}{e}{}_a, p)$ under an affine transformation (Λ, τ) with $\tau = \tau^a \in T^4 \simeq R^4$ and $\Lambda = \Lambda_a{}^b \in GL(4,R)$ reads

$$\begin{aligned}\left(\overset{(A)}{e}, p\right) \overset{(\Lambda,\tau)}{\longrightarrow} \left(\overset{(A)}{e}{}', p'\right) &= \left(\overset{(A)}{e}\Lambda, p + \tau\right) \\ &= \left(\overset{(A)}{e}{}_b \Lambda_a{}^b, p + \tau^a \overset{(A)}{e}{}_a\right).\end{aligned} \tag{71}$$

The affine group acts transitively on the affine tangent spaces $A\mathscr{M}_4$: any two affine frames of some $A_x\mathscr{M}_4$ can be related by a unique affine transformation. Consequently, the notion of an affine frame should be enlarged to include all $GL(4,R)$-representations needed. The gauging is accomplished by the introduction of the *generalized affine connection* as a prescription $(\overset{(A)}{\Gamma}{}^{(L)}, \overset{(A)}{\Gamma}{}^{(T)})$, which maps infinitesimally neighbouring affine tangent spaces $A_x\mathscr{M}_4$, $A_{\tilde{x}}\mathscr{M}_4$, where $\tilde{x} = x + dx$, by an $A(4,R)$-transformation onto each other. The generalized affine connection consists of a $GL(4,R)$-valued 1-form $\overset{(A)}{\Gamma}{}^{(L)}$ and an R^4-valued one-form $\overset{(A)}{\Gamma}{}^{(T)}$, both of which generate the required $A(4,R)$-transformation. The two affine tangent

spaces get now related by an affine transformation according to the prescription

$$dp = \overset{(A)}{\Gamma}{}^{(T)a}\overset{(A)}{e}{}_{a}, \qquad d\overset{(A)}{e}{}_{a} = \overset{(A)}{\Gamma}{}_{a}^{(L)b}\overset{(A)}{e}{}_{b}. \tag{72}$$

Introducing origins in $A_x\mathscr{M}_4$, that is, by soldering $A_x\mathscr{M}_4$ to $\mathscr{M}_4$, one losts translational invariance in $A_x\mathscr{M}_4$ but gained a local one-to-one correspondence *between translations* in $A_x\mathscr{M}_4$ *and diffeomorphisms* on $\mathscr{M}_4$. However, one can introduce translational invariance by demanding diffeomorphism invariance instead and continue to work with this modified notion of translation invariance. The diffeomorphisms themself, as horizontal transformations in their active interpretation, cannot be gauged according to the usual gauge principle and thus do not furnish their own gauge potential. In the Lagrangian formulation of a general metric-affine theory it was assumed that the matter fields are described in terms of manifields. Actually, the matter fields are supposed to be represented by vector- or spinor-valued p-forms. However, in MAG one goes beyond Poincaré invariance, assuming that matter fields might undergo not only Poincaré transformations but also the more general linear transformations. In this case, the unavailability of local Lorentz frames poses no problem in the context of boson fields. They are naturally constructed so as to be capable of carrying the action of $SL(4,R)$, instead of the Lorentz group, whether in a local frame or holonomically. However, this is not true of the conventional fermion fields one uses to represent matter, and a linear action should nevertheless be realized, through the use of infinite-component linear field representations of the double covering of the linear, affine, and diffeomorphism groups. More general spinor representations than the Poincaré-representations have to be constructed. Such representations of the matter fields corresponding to the affine group must exist. Otherwise it does not make sense to demand $A(4,R)$-invariance of a nonvacuum field theory. The construction of the spinor representations of fermionic matter fields in MAG, the so-called *manifields*, is illustrated in [17, 18]. These representations turn out to be infinite dimensional, due to the noncompactness of the gauge (sub)group $GL(4,R)$. Fermions are then assigned to spinor manifields. The restriction of $GL(4,R)$ to $SO(1,n-1)$ reduces the manifield representations to the familiar spinor representations. The actual gauging of the affine group introduced, in addition to the matter fields f, the gravitational gauge potentials $\overset{(A)}{\Gamma}{}^{(T)}$ and $\overset{(A)}{\Gamma}{}^{(L)}$. Following the common practice, we will use as gauge potential the translation invariant $\overset{(A)}{\vartheta}{}^{a}$ in place of the translational part $\overset{(A)}{\Gamma}{}^{(T)}$ of the affine connection, simply because it has the immediate interpretation as a reference (co)frame. Expanded in a holonomic frame, the components of $\overset{(A)}{\vartheta}{}^{a}$ and $\overset{(A)}{\Gamma}{}^{(T)}$ differ just by a Kronecker symbol, as is clear from the definition of $\overset{(A)}{\vartheta}{}^{a}$. Also the homogeneous transformation behavior of $\overset{(A)}{\vartheta}{}^{a}$ will turn out to be quite convenient. For the action of the $GL(4,R)$-gauge potential, below the shorthand notation $\overset{(A)}{\Gamma}{}_{a}^{\ b}$ is used instead of $(\overset{(A)}{\Gamma}{}_{c}^{(L)d}L^{c}{}_{d})_{a}^{\ b}$. The introduction of a metric into MAG is mandatory since we are interested in a realistic macroscopic gravity theory that contains GR in some limit. So, it was assumed as a *metric* of the general form $\overset{(A)}{g} = \overset{(A)}{g}{}_{ab}\,\overset{(A)}{\vartheta}{}^{a}\otimes\overset{(A)}{\vartheta}{}^{b}$ with coefficients $\overset{(A)}{g}{}_{ab}$ which are independent of the coframe $\overset{(A)}{\vartheta}{}^{a}$. The gauge potentials $(\overset{(A)}{\vartheta}{}^{a}, \overset{(A)}{\Gamma}{}_{a}^{\ b})$ become true dynamical variables if one has to add to the minimally coupled matter Lagrangian $L_m^{(\pi)}$ a gauge Lagrangian V. Here, we restrict ourself considering only first order Lagrangian V, which must be expressed in terms of the gauge potentials and their first derivatives. The total action reads then

$$S = \int\left[V\left(\overset{(A)}{g}{}_{ab}, \overset{(A)}{\vartheta}{}^{a}, \overset{(A)}{N}{}_{ab}, \overset{(A)}{T}{}^{a}, \overset{(A)}{R}{}_{a}^{\ b}\right) + L_m^{(\pi)}\left(\overset{(A)}{g}{}_{ab}, \overset{(A)}{\vartheta}{}^{a}, \Psi, D\Psi\right)\right]. \tag{73}$$

If Ψ, as a p-form, represents a matter field (fundamentally a representation of the $SL(4,R)$ or of some of its subgroups), its first order Lagrangian $L_m^{(\pi)}$ will be embedded in metric-affine spacetime by the minimal coupling procedure, that is, exterior covariant derivatives feature in the kinetic terms of the Lagrangian instead of only exterior ones. Just as ordinary stress is the analogue of the (Hilbert) energy-momentum density, in MAG, one has, in addition, the spin current and the dilation plus shear currents inducing the torsion and nonmetricity fields, respectively. Both spin and dilation plus shear are components of the hypermomentum current, symmetric for dilation plus shear and antisymmetric for spin: *spin current* $\otimes$ *dilation current* $\otimes$ *shear current*. And these currents ought to couple to the corresponding post-Riemannian structures. In accord, the material currents are defined as follows:

$$\overset{(A)}{t}{}^{ab} := 2\frac{\delta L_m^{(\pi)}}{\delta\overset{(A)}{g}{}_{ab}}, \qquad \overset{(A)}{\theta}{}_{a} := \frac{\delta L_m^{(\pi)}}{\delta\overset{(A)}{\vartheta}{}^{a}}, \qquad \overset{(\pi)}{\Delta}{}_{b}^{\ a} := \frac{\delta L_m^{(\pi)}}{\delta\overset{(\pi)}{\Gamma}{}_{a}^{\ b}}, \tag{74}$$

where $\overset{(A)}{t}{}^{ab}$ is the metric (and symmetric) energy-momentum current of matter (Hilbert current), whereas one believes that $\overset{(A)}{\theta}{}_{a}$ should be the dual 3-form relating to the canonical energy-momentum tensor, $\overset{(A)}{\theta}{}_{a}^{\ \mu}$, by

$$\star\,\overset{(A)}{\theta}{}_{a} = \frac{1}{3!}\,\overset{(A)}{\theta}{}_{a}^{\ \mu}\varepsilon_{\mu\nu\alpha\beta}\,\overset{(A)}{\vartheta}{}^{\nu\alpha\beta}, \tag{75}$$

where we used the abbreviated notations for the wedge product monomials, $\overset{(A)}{\vartheta}{}^{\mu\nu\alpha\cdots} = \overset{(A)}{\vartheta}{}^{\mu}\wedge\overset{(A)}{\vartheta}{}^{\nu}\wedge\overset{(A)}{\vartheta}{}^{\alpha}\wedge\cdots$, and $\star$ denotes the Hodge dual (Appendix B). The canonical energy-momentum density is identical with the dynamical tetrad energy-momentum density $e\,\overset{(A)}{\theta}{}_{\mu}^{\ a} = \overset{(A)}{\tau}{}_{\mu}^{\ a}$, where

$e := \det|e^{\mu}_{a}| = \sqrt{-g}$, and the canonical tensor $e\overset{(A)}{\theta}_{\mu\nu} = \overset{(A)}{\tau}_{\mu\nu} = \overset{(A)}{e}_{a\nu}\overset{(A)}{\tau}_{\mu}{}^{a}$ is generally not symmetric. The relation between the tetrad dynamical energy-momentum tensor and the metric dynamical energy-momentum tensor for matter fields is $\overset{(A)}{\theta}_{(\mu\nu)} = t_{\mu\nu}$. The canonical hypermomentum current $\overset{(\pi)}{\Delta}{}^{a}{}_{b}$, which couples to the linear connection, can be decomposed according to

$$\overset{(\pi)}{\Delta}_{ab} = \overset{(\pi)}{S}_{ab} + \frac{1}{4}\overset{(\pi)}{g}_{ab}\Delta + \widehat{\mathbb{X}}_{\alpha\beta}, \tag{76}$$

with $\overset{(\pi)}{S}_{ab} := \overset{(\pi)}{\Delta}_{[ab]}$ as (dynamical) spin current, $\Delta := \overset{(\pi)}{\Delta}{}^{c}{}_{c}$, as dilation current, and $\widehat{\mathbb{X}}_{ab}$ as symmetric and tracefree shear current, which is a bit more remote from direct observation than the other currents. From variation of a total action (73), we find the matter and the gauge field equations as follows:

$$\begin{aligned}
&D\left(\frac{\partial L_m^{(\pi)}}{\partial(D\Psi)}\right) - (-1)^p\frac{\partial L_m^{(\pi)}}{\partial\Psi} = 0, \quad \text{(matter)}\\
&D\left(2\frac{\partial V}{\partial\overset{(A)}{N}_{ab}}\right) + 2\frac{\partial V}{\partial\overset{(A)}{g}_{ab}} = -\overset{(A)}{t}{}^{ab}, \quad \text{(0th)}\\
&D\left(\frac{\partial V}{\partial\overset{(A)}{T}{}^{a}}\right) + \frac{\partial V}{\partial\overset{(A)}{\vartheta}{}^{a}} = -\overset{(A)}{\theta}_{a}, \quad \text{(1st)}\\
&\frac{\partial\overset{(A)}{\omega}_{a'}{}^{b'}}{\partial\overset{(\pi)}{\omega}_{a}{}^{b}}\wedge\left[D\left(\frac{\partial V}{\partial\overset{(A)}{R}_{a'}{}^{b'}}\right) + \overset{(A)}{\vartheta}{}^{a'}\wedge\frac{\partial V}{\partial\overset{(A)}{T}{}^{b'}}\right.\\
&\qquad\left. + 2\overset{(A)}{g}_{b'c}\frac{\partial V}{\partial\overset{(A)}{N}_{a'c}}\right] = -\overset{(\pi)}{\Delta}{}^{a}{}_{b}, \quad \text{(2nd)}
\end{aligned} \tag{77}$$

where we take into account that the variation of $L_m^{(\pi)}$ with respect to the affine connection in MAG is equivalent to the variation with respect to the torsion (or contortion) or spin connection: $(\partial\overset{(A)}{\omega}_{a'}{}^{b'}/\partial\overset{(\pi)}{\omega}_{a}{}^{b} = \partial\overset{(A)}{\Gamma}_{a'}{}^{b'}/\partial\overset{(\pi)}{\Gamma}_{a}{}^{b})$. Note that, using the Noether identities, the zeroth field equation can be shown to be redundant, provided the matter equations hold. In first, the canonical energy-momentum of the translational gauge potential $\overset{(A)}{\vartheta}{}^{a}$ can be written in standard form

$$\begin{aligned}
\frac{\partial V}{\partial\overset{(A)}{\vartheta}{}^{a}} &= \overset{(A)}{e}_{a}\rfloor V - \left(\overset{(A)}{e}_{a}\rfloor\overset{(A)}{T}{}^{b}\right)\wedge\frac{\partial V}{\partial\overset{(A)}{T}{}^{b}}\\
&\quad - \left(\overset{(A)}{e}_{a}\rfloor\overset{(A)}{R}_{b}{}^{c}\right)\wedge\frac{\partial V}{\partial\overset{(A)}{R}_{b}{}^{c}} - \left(\overset{(A)}{e}_{a}\rfloor\overset{(A)}{N}_{bc}\right)\frac{\partial V}{\partial\overset{(A)}{N}_{bc}}.
\end{aligned} \tag{78}$$

Thus, the equations of the standard MAG theory written in the framework of the first order Lagrangian, which is expressed in terms of the gauge potentials and their first derivatives, can be recovered for $A = \pi$:

$$\begin{aligned}
&D\left(\frac{\partial L_m^{(\pi)}}{\partial(D\Psi)}\right) - (-1)^p\frac{\partial L_m^{(\pi)}}{\partial\Psi} = 0, \quad \text{(matter)}\\
&D\left(2\frac{\partial V}{\partial\overset{(\pi)}{N}_{ab}}\right) + 2\frac{\partial V}{\partial\overset{(\pi)}{g}_{ab}} = -\overset{(\pi)}{t}{}^{ab}, \quad \text{(0th)}\\
&D\left(\frac{\partial V}{\partial\overset{(\pi)}{T}{}^{a}}\right) + \frac{\partial V}{\partial\overset{(\pi)}{\vartheta}{}^{a}} = -\overset{(\pi)}{\theta}_{a}, \quad \text{(1st)}\\
&D\left(\frac{\partial V}{\partial\overset{(\pi)}{R}_{a'}{}^{b'}}\right) + \overset{(\pi)}{\vartheta}{}^{a'}\wedge\frac{\partial V}{\partial\overset{(\pi)}{T}{}^{b'}} + 2\overset{(\pi)}{g}_{b'c}\frac{\partial V}{\partial\overset{(\pi)}{N}_{a'c}}\\
&\quad = -\overset{(\pi)}{\Delta}{}^{a}{}_{b}, \quad \text{(2nd)}.
\end{aligned} \tag{79}$$

Consequently, the (2nd) equation defines a nondynamical torsion, such that torsion at a given point in spacetime does not vanish only if there is matter at this point, represented in the Lagrangian density by a function which depends on torsion. Unlike the metric, which is related to matter through a differential field equation, torsion does not propagate. However, equations (79) can be equivalently replaced by the set of *modified MAG equations* for $A = \sigma$:

$$\begin{aligned}
&D\left(\frac{\partial L_m^{(\pi)}}{\partial(D\Psi)}\right) - (-1)^p\frac{\partial L_m^{(\pi)}}{\partial\Psi} = 0, \quad \text{(matter)}\\
&D\left(2\frac{\partial V}{\partial\overset{(\sigma)}{N}_{ab}}\right) + 2\frac{\partial V}{\partial\overset{(\sigma)}{g}_{ab}} = -\overset{(\sigma)}{t}{}^{ab}, \quad \text{(0th)}\\
&D\left(\frac{\partial V}{\partial\overset{(\sigma)}{T}{}^{a}}\right) + \frac{\partial V}{\partial\overset{(\sigma)}{\vartheta}{}^{a}} = -\overset{(\sigma)}{\theta}_{a}, \quad \text{(1st)}\\
&\Theta^{ab'}_{a'b}(\pi(x),\sigma(x))\wedge\left[D\left(\frac{\partial V}{\partial\overset{(\sigma)}{R}_{a'}{}^{b'}}\right) + \overset{(\sigma)}{\vartheta}{}^{a'}\wedge\frac{\partial V}{\partial\overset{(\sigma)}{T}{}^{b'}}\right.\\
&\qquad\left. + 2\overset{(\sigma)}{g}_{b'c}\frac{\partial V}{\partial\overset{(\sigma)}{N}_{a'c}}\right] = -\overset{(\pi)}{\Delta}{}^{a}{}_{b}, \quad \text{(2nd)}
\end{aligned} \tag{80}$$

where

$$\frac{\partial\overset{(\sigma)}{\omega}_{a'}{}^{b'}}{\partial\overset{(\pi)}{\omega}_{a}{}^{b}} := \Theta^{ab'}_{a'b}(\pi(x),\sigma(x)), \tag{81}$$

in which the torsion is *dynamical* if only $\Theta^{ab'}_{a'b}(\pi(x),\sigma(x)) \neq \delta^{a'}_{a}\delta^{b}_{b'}$. Actually, in testing the general MAG equations (77), we desire, in some limit, to recover the field equation

for different (sub)cases; then we have to put on *Lagrange multipliers*, whereas, in the case of TSSD-PG, one has to kill nonmetricity; TSSD-EC is the TSSD-PG with the curvature scalar as gravitational Lagrangian; in the case of TSSD-teleparallel gravity (TSSD-GR$_{||}$) in a Weitzenböck spacetime, one has to remove nonmetricity and curvature; and, finally, in the case of GR in a Riemannian space, curvature scalar as Lagrangian, one has to remove nonmetricity and torsion. To recover, for example, GR, let us in (68) choose $t^a{}_{bc}$ as the torsion of the connection $\overset{(\pi)}{\omega}{}^a{}_{bc}$; that is, $t^a{}_{bc} = \overset{(\pi)}{T}{}^a{}_{bc}$. In this case, $k^a{}_{bc} = \overset{(\pi)}{K}{}^a{}_{bc}$, and we are left with the torsionless, a $SO(3,1)$ valued Lorentz spin connection of general relativity: $\overset{(\sigma)}{\Omega}{}^a{}_{bc} = \breve{\omega}^a{}_{bc}$. Then, one can choose for the missing piece of the gauge field Lagrangian V_{GR} the corresponding 4-form of the curvature scalar [8, 9] and take Lagrange multipliers for extinguishing nonmetricity and torsion:

$$V_{\mathrm{GR}} = -\frac{1}{4æ}\breve{R}^{ab}\wedge\breve{\eta}_{ab} + \frac{1}{2}\overset{(A)}{N}_{ab}\wedge{}^{(1)}\lambda^{ab} + \overset{(A)}{T}{}^a\wedge{}^{(2)}\lambda_a, \quad (82)$$

where æ is the coupling constant relating to the Newton gravitational constant $æ = 8\pi G/c^4$, $\breve{R}_a{}^b$ is the curvature tensor of the independent field variable $\breve{\Gamma}_a{}^b$, and $\breve{\eta}_{ab}$ is the eta basis consisting in the Hodge dual of exterior products of tetrads by means of the Levi-Civita object (Appendix B).

The RC manifold, U_4, is a particular case of general metric-affine manifold M_4, restricted by the metricity condition, when a nonsymmetric linear connection is said to be metric compatible. Taking the antisymmetrized derivative of the metric condition gives an identity between the curvature of the spin connection and the curvature of the Christoffel connection

$$\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right)\overset{(A)}{e}{}_{\rho b} - R^{\sigma}{}_{\rho\mu\nu}(\Gamma)\overset{(A)}{e}{}_{\sigma}{}^{a} = 0, \quad (83)$$

where we take into account (65), so

$$\begin{aligned}\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right) &= \partial_\mu\overset{(A)}{\omega}{}_{\nu}{}^{ab} - \partial_\nu\overset{(A)}{\omega}{}_{\mu}{}^{ab} \\ &\quad + \overset{(A)}{\omega}{}_{\mu}{}^{ac}\overset{(A)}{\omega}{}_{\nu c}{}^{b} - \overset{(A)}{\omega}{}_{\nu}{}^{ac}\overset{(A)}{\omega}{}_{\mu c}{}^{b}, \end{aligned} \quad (84)$$

$$R^{\sigma}{}_{\rho\mu\nu}(\Gamma) = \partial_\mu\Gamma^{\sigma}_{\nu\rho} - \partial_\nu\Gamma^{\sigma}_{\mu\rho} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda}.$$

Hence, the relations between the scalar curvatures for the U_4 manifold read

$$\begin{aligned}\overset{(A)}{R}\left(\overset{(A)}{\omega}\right) &\equiv \overset{(A)}{e}{}_{a}{}^{\mu}\overset{(A)}{e}{}_{b}{}^{\nu}\overset{(A)}{R}{}_{\mu\nu}{}^{ab}\left(\overset{(A)}{\omega}\right) \\ &= R(g,\Gamma) \equiv g^{\rho\gamma}R^{\mu}{}_{\rho\mu\nu}(\Gamma).\end{aligned} \quad (85)$$

This means that the Lorentz and diffeomorphism invariant scalar curvature, R, becomes either a function of $\overset{(A)}{e}{}_{\mu}^{a}$ only or a function of $g_{\mu\nu}$ only. Certainly, it can be seen by noting that the Lorentz gauge transformations can be used to fix the six antisymmetric components of $\overset{(A)}{e}{}_{\mu}^{a}$ to vanish. Then in both cases diffeomorphism invariance fixes four more components out of the six $g_{\mu\nu}$, with the four components $g_{0\mu}$ being nondynamical, obviously, leaving only two dynamical degrees of freedom. This shows the equivalence of the vierbein and metric formulations holds. To recover the TSSD-U_4 theory, one can choose for the missing piece the EC Lagrangian V_{EC} (here without cosmological constant):

$$V_{\mathrm{EC}} = -\frac{1}{4æ}\overset{(A)}{R}{}^{ab}\wedge\overset{(A)}{\eta}_{ab} + \frac{1}{2}\overset{(A)}{N}_{ab}\wedge\lambda^{ab}, \quad (86)$$

provided $(1/2)\overset{(A)}{R}{}^{ab}\wedge\overset{(A)}{\eta}_{ab}$ is equal to $\overset{(A)}{R}\overset{(A)}{\eta}$. The variation of the total action (73), given by the sum of the gravitational field action $S_{\mathrm{EC}} = \int V_{\mathrm{EC}}$ with the Lagrangian (86) and the macroscopic matter sources $S_m^{(\pi)}$, with respect to the $\overset{(A)}{\vartheta}{}^a$, 1-form $\overset{(A)}{\omega}{}^{ab} = \overset{(A)}{\omega}{}_{\mu}^{ab}\wedge dx^\mu$ and Ψ, gives

$$\begin{aligned}&(1)\ \frac{1}{2}\overset{(A)}{\eta}_{abc}\wedge\overset{(A)}{R}{}^{bc} = -æ\overset{(A)}{\theta}{}_a, \\ &(2)\ \Theta^{a'b'}_{ab}\wedge D\overset{(A)}{\eta}_{ab} = æ\overset{(\pi)}{S}_{ab}, \\ &(3)\ \frac{\delta L_m^{(\pi)}}{\delta\Psi} = 0, \qquad \frac{\delta L_m^{(\pi)}}{\delta\overline{\Psi}} = 0,\end{aligned} \quad (87)$$

where $\partial\overset{(A)}{\omega}{}^{a'b'}/\partial\overset{(\pi)}{\omega}{}^{ab} := \Theta^{ab}_{a'b'}(\pi(x),\sigma(x))$. The metricity condition $\overset{(A)}{N}_{ab} = \overset{(A)}{\mathscr{D}}\overset{(A)}{g}_{ab} = 0$ yields $\overset{(\pi)}{S}_{ab} + \overset{(\pi)}{S}_{ba} = 0$, and hence $D\overset{(A)}{\eta}_{ab} = \overset{(A)}{\eta}_{abc}\wedge\overset{(A)}{T}{}^c$. Consequently, the field equations (87) become

$$\begin{aligned}&(1)\ \frac{1}{2}\breve{R}^{(A)}_{ca}\wedge\overset{(A)}{\vartheta}{}^c = æ\overset{(A)}{\theta}{}_a, \\ &(2)\ \Theta^{a'b'}_{ab}\wedge\star\overset{(A)}{\mathscr{T}}_{a'b'} = -\frac{1}{2}æ\star\overset{(\pi)}{\Sigma}_{ab}, \\ &(3)\ \frac{\delta L_m^{(\pi)}}{\delta\Psi} = 0, \qquad \frac{\delta L_m^{(\pi)}}{\delta\overline{\Psi}} = 0,\end{aligned} \quad (88)$$

where $\star\overset{(\pi)}{\Sigma}_{ab} = -\star\overset{(\pi)}{\Sigma}_{ba}$ is the dual 3-form corresponding to the canonical spin tensor, which is identical with the dynamical spin tensor $\overset{(\pi)}{S}_{abc}$; namely,

$$\star\overset{(\pi)}{\Sigma}_{ab} = \overset{(\pi)}{S}{}^{\mu}{}_{ab}\varepsilon_{\mu\nu\alpha\beta}\overset{(\pi)}{\vartheta}{}^{\nu\alpha\beta} \quad (89)$$

provided

$$\begin{aligned}\star\overset{(A)}{\mathscr{T}}_{ab} &:= \frac{1}{2}\star\left(\overset{(A)}{Q}_a\wedge\overset{(A)}{e}_b\right) = \overset{(A)}{Q}{}^c\wedge\overset{(A)}{\vartheta}{}^d\varepsilon_{cdab} \\ &= \frac{1}{2}\overset{(A)}{Q}{}^c{}_{\mu\nu}\wedge\overset{(A)}{e}{}^d{}_{\alpha}\varepsilon_{abcd}\overset{(A)}{\vartheta}{}^{\mu\nu\alpha},\end{aligned} \quad (90)$$

and that

$$\overset{(A)}{Q}{}^a = \overset{(A)}{D}\overset{(A)}{\vartheta}{}^a = d\overset{(A)}{\vartheta}{}^a + \overset{(A)}{\omega}{}^a{}_b\wedge\overset{(A)}{\vartheta}{}^b. \quad (91)$$

According to (85), the relations between the Ricci scalars read

$$\breve{R} \equiv \breve{R}^{(\sigma)}_{cd} \wedge \dot{\vartheta}^c \wedge \dot{\vartheta}^d = \breve{R}^{(\pi)}_{cd} \wedge \vartheta^c \wedge \vartheta^d. \tag{92}$$

To obtain some feeling for the tensor language then we may recast the first equation in (88) into the form

$$-\frac{e}{4}\breve{R}^{(A)ab}_{\mu\nu} \overset{(A)}{e}{}^{\mu\nu\beta}_{abc} = æ \overset{(A)}{\theta}{}^a_c e \overset{(A)}{e}{}^\beta_a. \tag{93}$$

Making use of the relation,

$$-\breve{R}^{(A)ab}_{\mu\nu} \overset{(A)}{e}{}^{\mu\nu\beta}_{abc} = -2\breve{R}^{(A)} \overset{(A)}{e}{}^\beta_c + 4\breve{R}^{(A)\beta}_c, \tag{94}$$

gives

$$\breve{G}^{(A)\beta}_c \equiv \breve{R}^{(A)\beta}_c - \frac{1}{2} \overset{(A)}{e}{}^\beta_c \breve{R} = æ \overset{(A)}{\theta}{}^\beta_\alpha. \tag{95}$$

We may evaluate the second equation in (88) as

$$\begin{aligned} \frac{\partial \overset{(A)}{\omega}{}^{cd}_\beta}{\partial \overset{(\pi)}{\omega}{}^{c'd'}_{\beta'}} \frac{e}{2} \overset{(A)}{Q}{}^a_{\mu\nu} \overset{(A)}{e}{}^{\mu\nu\beta}_{acd} &= -\frac{e}{2} æ \overset{(\pi)}{S}{}^a_{c'd'} \overset{(A)}{e}{}^{\beta'}_a \\ &= -\frac{e}{2} æ \overset{(\pi)}{S}{}^{\beta'}_{c'd'}. \end{aligned} \tag{96}$$

Taking into account the relation

$$\frac{1}{2} \overset{(A)}{Q}{}^c_{\mu\nu} \overset{(A)}{e}{}^{\mu\nu\beta}_{cab} = \overset{(A)}{e}{}^\mu_a \overset{(A)}{e}{}^\nu_b \overset{(A)}{T}{}^\beta_{\mu\nu} \tag{97}$$

we then obtain

$$\frac{\partial \overset{(A)}{\omega}{}^{cd}_\beta}{\partial \overset{(\pi)}{\omega}{}^{c'd'}_{\beta'}} \overset{(A)}{\mathcal{T}}{}^\beta_{cd} = -\frac{1}{2} æ \overset{(\pi)}{S}{}^{\beta'}_{c'd'}. \tag{98}$$

It is remarkable that the torsion (98), in a natural way, can be made a short-range propagating torsion. Actually, from the above equations, we see that it is the spin $\overset{(\pi)}{S}$ and spacetime deformations $\pi(x)$ and $\sigma(x)$ that define the torsion $\overset{(A)}{Q}$:

$$\overset{(A)}{Q}{}^\nu_{\mu\rho} = \overset{(A)}{T}{}^\nu_{\mu\rho} + \frac{1}{2}\delta^\nu_\mu \overset{(A)}{T}{}^\lambda_{\rho\lambda} - \frac{1}{2}\delta^\nu_\rho \overset{(A)}{T}{}^\lambda_{\mu\lambda}, \tag{99}$$

which through the field equation, in turn, defines Einstein's field tensor $\breve{G}$. A generic spacetime deformation, $\pi(x)$, consists of two ingredient deformations $(\dot{\pi}(\dot{x}), \sigma(x))$ of the orthonormal frame. While the deformation matrix $\dot{\pi}(\dot{x})$ implies a peculiar condition (53), the choice of the $\sigma(x)$ is not fixed yet. This allows us to impose a physical constraint upon the spacetime deformation $\sigma(x)$:

$$\Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\pi(x), \sigma(x)) = \left(\Box + M_T^2\right) \overset{(\sigma)}{T}{}^\nu_{\mu\rho} \left(\overset{(\sigma)}{T}{}^{-1}\right)^{\mu'\rho'}_{\nu'}, \tag{100}$$

where $\Box$ is a generalization of the d'Alembertian operator for covariant derivatives defined on the RC manifold, U_4. Then, the *modified EC equations* (95) and (98) reduce to

$$\breve{G}_{\mu\nu} = æ \overset{(\sigma)}{\theta}_{\mu\nu}, \qquad \left(\Box + M_T^2\right) \overset{(\sigma)}{T}{}^\nu_{\mu\rho} = -\frac{1}{2} æ \overset{(\pi)}{S}{}^\nu_{\mu\rho}, \tag{101}$$

which describe the short-range propagating torsion and spin-spin interaction. At large distances $r > \lambda_T \equiv \hbar/M_T c$ (Compton length), torsion vanishes $\overset{(\sigma)}{T}(r) = 0$. Note that the short-range propagating torsion is of fundamental importance from a view point of microphysics.

We may impose some other physical constraints upon the spacetime deformation $\sigma(x)$, which will be useful for the theory of electromagnetism and charged particles:

$$\Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\pi(x), \sigma(x)) \equiv \Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}\left(\overset{(\sigma)}{T}\right) = 2\varphi_{,\sigma}\varepsilon^{\sigma\nu}_{\mu\rho}\left(\overset{(\sigma)}{T}{}^{-1}\right)^{\mu'\rho'}_{\nu'}, \tag{102}$$

with φ as a scalar or pseudoscalar function of relevant variables. Then

$$\overset{(\pi)}{T}{}^\nu_{\mu\rho} = \Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\pi(x), \sigma(x)) \overset{(\sigma)}{T}{}^{\nu'}_{\mu'\rho'} = 2\varphi_{,\sigma}\varepsilon^{\sigma\nu}_{\mu\rho}, \tag{103}$$

which recovers the term in the Lagrangian of pseudoscalar-photon interaction theory [95–100], such that the nonmetric part of the Lagrangian can be put in the well known form of the $\chi - g$ framework:

$$\begin{aligned} L_I^{(\pi)NM} &= 2(-g)^{1/2} A_\nu A_{\mu,\rho} \overset{(\pi)}{\mathcal{T}}{}^{\nu\mu\rho} \\ &= 4(-g)^{1/2} \varphi_{,\sigma}\varepsilon^{\sigma\nu\mu\rho} A_\nu A_{\mu,\rho}, \quad (\text{mod div}) \end{aligned} \tag{104}$$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ have the usual meaning for electromagnetism. This is equivalent, up to integration by parts in the action integral (modulo a divergence), to the Lagrangian

$$L_I^{(\pi)NM} = (-g)^{1/2} \varphi\varepsilon^{\sigma\nu\mu\rho} F_{\sigma\nu} F_{\mu\rho}. \tag{105}$$

According to (105), the gravitational constitutive tensor $\chi^{\sigma\nu\mu\rho} = \chi^{\mu\rho\sigma\nu} = -\chi^{\mu\rho\nu\sigma}$ [119] of the gravitational fields (e.g., metric $g_{\mu\nu}$, (pseudo) scalar field φ, etc.) reads

$$\chi^{\sigma\nu\mu\rho} = (-g)^{1/2}\left[\frac{1}{2}g^{\sigma\mu}g^{\nu\rho} - \frac{1}{2}g^{\sigma\rho}g^{\mu\nu} + \varphi\varepsilon^{\sigma\nu\mu\rho}\right]. \tag{106}$$

Specifying the physical meaning of the constrains (99), (101), and (102), will be an interesting topic for another investigation elsewhere. Here we finally concentrate on the other (sub)case when $t^a{}_{bc} = \overset{(\pi)}{T}{}^a_{bc} - \overset{(\pi)}{C}{}^a_{bc}$, and the connection $\overset{(\pi)}{\omega}{}^a_{bc}$ vanishes, which characterizes teleparallel gravity. In this case, the resulting connection has the form $\overset{(\sigma)}{\omega}{}^a_{bc} = -\dot{K}^a{}_{bc}$, where $\dot{K}^a{}_{bc}$ is the contortion tensor written in terms of the Weitzenböck torsion $\dot{T}^a{}_{bc} = -\dot{C}^a{}_{bc}$. The particle equation of motion then becomes the force equation of teleparallel

gravity. Hence, the Weitzenböck covariant derivative of the tetrad field vanishes identically: $\dot{\mathscr{D}}_\nu \dot{e}^a{}_\mu \equiv \dot{\partial}_\nu \dot{e}^a{}_\mu - \dot{\Gamma}^\rho{}_{\mu\nu} \dot{e}^a{}_\rho = 0$. This is the so-called distant, or absolute parallelism condition. As a consequence of this condition, the corresponding Weitzenböck spin connection also vanishes identically: $\dot{\omega}^c{}_{a\nu} = \breve{\omega}^c{}_{a\nu} + \dot{K}^c{}_{a\nu} \equiv 0$. Of course, these relations above are true only in one specific class of frames. In fact, since $\dot{\omega}^c{}_{a\nu}$ is the Weitzenböck spin connection, if it vanishes in a given frame, it will be different from zero in any other frame related to the first by a local Lorentz transformation. The teleparallel gravity becomes consistent and fully equivalent with GR, even in the presence of spinor fields if we write the minimal coupling prescription as $\dot{\partial}_a \to \dot{\mathscr{D}}_a = \dot{e}_a{}^\mu \dot{\mathscr{D}}_\mu$ with $\dot{\mathscr{D}}_\mu$ the teleparallel Fock-Ivanenko derivative written in the form $\dot{\mathscr{D}}_\mu := \dot{\partial}_\mu - (i/2)\dot{\Omega}^a{}_{b\mu} J_a{}^b$, where the teleparallel spin connection, $\dot{\Omega}^a{}_{b\mu}$, reads $\dot{\Omega}^a{}_{b\mu} := 0 - \dot{K}^a{}_{b\mu}$. Field equations can be derived from the least action, $\delta \dot{S} = 0$, with the Lagrangian of teleparallel gravity, where we take Lagrange multipliers for extinguishing nonmetricity and curvature. This holds when the deformation $\sigma(x)$ vanishes identically ($\sigma(x) \equiv 0$):

$$V_{||} = -\frac{1}{4æ} \dot{T}^a \wedge^* \left(-{}^{(1)}\dot{T}_a + 2{}^{(2)}\dot{T}_a + \frac{1}{2}{}^{(3)}\dot{T}_a \right) + \frac{1}{2}\dot{N}_{ab} \wedge^{(1)} \lambda^{ab} + \dot{R}_a{}^b \wedge^{(2)} \lambda^a{}_b, \tag{107}$$

where the three irreducible pieces of the torsion $\overset{(A)}{T}{}^a = {}^{(1)}\overset{(A)}{T}{}^a + {}^{(2)}\overset{(A)}{T}{}^a + {}^{(3)}\overset{(A)}{T}{}^a$ under $SO(1,3)$-decomposition are

$$\begin{aligned} {}^{(1)}\overset{(A)}{T}{}^a &:= \overset{(A)}{T}{}^a - {}^{(2)}\overset{(A)}{T}{}^a - {}^{(3)}\overset{(A)}{T}{}^a, \quad \text{(tentor)}; \\ {}^{(2)}\overset{(A)}{T}{}^a &:= \frac{1}{3}\overset{(A)}{\vartheta}{}^a \wedge \left(\overset{(A)}{e}{}_b \rfloor \overset{(A)}{T}{}^b \right), \quad \text{(trator)}; \\ {}^{(3)}T^a &:= -\frac{1}{3}{}^* \left(\overset{(A)}{\vartheta}{}^a \wedge^* \left(\overset{(A)}{T}{}^b \wedge \overset{(A)}{\vartheta}{}_b \right) \right), \quad \text{(axiator)}. \end{aligned} \tag{108}$$

The projective transformation

$$\overset{(A)}{\Gamma}{}_a{}^b \longrightarrow \overset{(A)}{\Gamma}{}_a{}^b + \delta_a^b P, \tag{109}$$

with arbitrary 1-form field P, leaves the Hilbert-Einstein type Lagrangian invariant, which leads to the connection determined up to a 1-form. Projectively related connections have the same (unparametrized) geodesics. So, only projectively invariant matter Lagrangians would be allowed. This necessitates abandoning this constraint, at the very least replacing semi-Riemannian geometry by Weyl's. Namely, to remove this constraint from the gravitational Lagrangians (82) and (86), following [120, 121], we may lift the Lagrange multiplier λ^{ab} and add a dilaton type massless scalar field to

$$V' = V + \frac{1}{2} d\Phi \wedge \star d\Phi, \tag{110}$$

in the context of the Weyl 1-form $\overset{(A)}{N}$, which is of the type of a gauge potential for dilations anyways:

$$V'_{\mathrm{GR}} = -\frac{1}{4æ}\left(\breve{R}^{ab} \wedge \breve{\eta}_{ab} + \beta \overset{(A)}{N} \wedge\star \overset{(A)}{N} \right) + \overset{(A)}{T}{}^a \wedge^{(2)} \lambda_a, \tag{111}$$

$$V'_{\mathrm{EC}} = -\frac{1}{4æ}\left(\overset{(A)}{R}{}^{ab} \wedge \overset{(A)}{\eta}{}_{ab} + \beta \overset{(A)}{N} \wedge\star \overset{(A)}{N} \right), \tag{112}$$

respectively, provided the trace $\overset{(A)}{\Gamma}{}_c{}^c$ of a connection is closely related to the Weyl 1-form [17, 18], as

$$\begin{aligned} \overset{(A)}{\Gamma}{}_c{}^c &= 2\overset{(A)}{N} + d\ln\sqrt{\left|\det \overset{(A)}{g}{}_{ab}\right|}, \\ d\overset{(A)}{\Gamma}{}_c{}^c &= \overset{(A)}{R}{}_c{}^c = 2d\overset{(A)}{N}. \end{aligned} \tag{113}$$

In case if matter is present and supplies energy-momentum and hypermomentum currents, then the *hypermomentum*, via the second field equation (77), turns out to be proportional to the *post*-Riemannian pieces of the connection.

In case of translational gauge with the most general term $V = d\overset{(A)}{\vartheta}{}^a \wedge \overset{(A)}{H}_a$, quadratic in $\overset{(A)}{\vartheta}{}^a$, the field equation $\delta L_{\mathrm{tot}}/\delta \overset{(A)}{\vartheta}{}^a = 0$ then becomes

$$\frac{\partial \overset{(A)}{\vartheta}{}^a}{\partial \overset{(\pi)}{\vartheta}{}^{a'}} \wedge \left[d\overset{(A)}{H}_{a'} - \overset{(A)}{E}_{a'} \right] = \overset{(A)}{\theta}_a, \tag{114}$$

where $\overset{(A)}{H}_a$ is linear in $d\overset{(A)}{\vartheta}{}^a$, and

$$\begin{aligned} \overset{(A)}{E}_a &= \left(\overset{(A)}{e}{}_a \rfloor d\overset{(A)}{\vartheta}{}^b \right) \wedge \overset{(A)}{H}_b - \frac{1}{2}\overset{(A)}{e}{}_a \rfloor \left(d\overset{(A)}{\vartheta}{}^b \wedge \overset{(A)}{H}_b \right) \\ &= \frac{1}{2}\left[\left(\overset{(A)}{e}{}_a \rfloor d\overset{(A)}{\vartheta}{}^b \right) \wedge \overset{(A)}{H}_b - d\overset{(A)}{\vartheta}{}^b \wedge \left(\overset{(A)}{e}{}_a \rfloor \overset{(A)}{H}_b \right) \right], \end{aligned} \tag{115}$$

the energy-momentum current of the gauge field. This can be considered as a starting point for turning to the Lagrangians with the quadratic ansatz for the kinetic term, that is, quadratic in the field strengths. Note that recently the authors of [122] show that the inverse of the so-called Barbero-Immirzi parameter multiplying the pseudoscalar curvature, because of the topological Nieh-Yan form, can only be appropriately discussed if torsion square pieces are included. Using more conservative direct constructions, they establish the exact relations between both approaches by applying the topological Euler and Pontryagin forms in a Riemann-Cartan space expressed for the first time in terms of irreducible

pieces of the curvature tensor. Only in a Riemann-Cartan spacetime, that is, in a spacetime with torsion, parity violating terms can be brought into the gravitational Lagrangian in a straightforward and natural way. They conclude that Riemann-Cartan spacetime is a natural habitat for chiral fermionic matter fields. However, the standard EC theory is invariant under the spacetime diffeomorphisms and local $SO(3,1)$ transformations acting in the tangent space. In our approach we suggest extending the $SO(3,1)$ to local $GL(4,R)$ group. This group is surely not the invariance of the action of standard theory, and so it takes the latter into diverse other ones, still invariant under spacetime diffeomorphisms.

4. Concluding Remarks

The actual advantage of present approach is twofold. To complete the TSSD theory [82], at first, we build up the *distortion-complex* and show how it restores the *world-deformation tensor*, which still has been put in it by hand. In this approach we suggest to extend the $SO(3,1)$ group of standard EC theory to local $GL(4,R)$ group. The latter is surely not the invariance of the action of standard EC theory, and so it takes the latter into diverse other ones still invariant under spacetime diffeomorphisms. All the fundamental gravitational structures in fact, the metric as much as the coframes and connections, acquire a DC induced theoretical interpretation. Through a nontrivial choice of explicit form of a DC which we have at our disposal, we have a way to construct various generalizations and alternatives of the standard gravity. Secondly, we extend the geometrical ideas of the TSSD theory to study, in the framework of the first order Lagrangian, the physical foundation of the gauge model of more general metric-affine gravity theory with a dynamical torsion. It is remarkable that, in the framework of the first order standard Lagrangian, the equations of the standard MAG theory, in which the equation defining torsion is the algebraic type and in fact no propagation of torsion is allowed, can be equivalently replaced by the *modified equations* in which the torsion, in general, is *dynamical*. In testing the too general MAG equations (77) in some limit, we have to put on Lagrange multipliers to recover the field equation for different (sub)cases: in the case of TSSD-PG, one has to kill nonmetricity; TSSD-EC is the TSSD-PG with the curvature scalar as gravitational Lagrangian, whereas the special physical ansatz for the spacetime deformations yields the short-range propagating spin-spin interaction.

Appendices

A. TSSD: More Details

For a local spacetime deformation $\dot{\pi}(\dot{x})$, we have

$$\begin{gathered}\dot{\Omega}_b{}^a = \dot{\pi}_c{}^a \dot{\pi}_b{}^c = \dot{\Omega}_\mu{}^\nu \breve{e}^a{}_\nu \breve{e}_b{}^\mu, \qquad \dot{\Omega}_\mu{}^\nu = \dot{\pi}_\mu{}^\rho \dot{\pi}_\rho{}^\nu, \\ \dot{e}_\rho = \dot{\pi}_\rho{}^\nu \breve{e}_\nu \equiv \dot{\partial}_\rho = \frac{\partial}{\partial \dot{x}^\rho}, \qquad \dot{\vartheta}^\rho = \dot{\pi}^\rho{}_\mu \breve{\vartheta}^\mu \equiv d\dot{x}^\rho.\end{gathered} \tag{A.1}$$

Since we are interested only in a peculiar condition (53) to be held, then it is completely satisfactory for further consideration to write the first deformation matrix, $\dot{\pi}(\dot{x})$, in the form of a particular solution to (53). To derive the particular solution to (53), we recall that, for an arbitrary matrix M,

$$\operatorname{tr}\left\{M^{-1}\partial_\mu M\right\} = \partial_\mu \ln|M|, \tag{A.2}$$

where $|\cdots|$ denotes the determinant and tr the trace. According to it, in matrix notation $\breve{\omega}_\mu := (\breve{\omega}^a{}_{b\mu})$, (53) becomes

$$\operatorname{tr}\left\{\dot{\pi}\left(\dot{x}\right)\dot{\partial}_\mu \dot{\pi}^{-1}\left(\dot{x}\right)\right\} = -\dot{\partial}_\mu \ln\left|\dot{\pi}\left(\dot{x}\right)\right| = \operatorname{tr}\breve{\omega}_\mu(\breve{x}), \tag{A.3}$$

which gives

$$\left|\dot{\pi}\left(\dot{x}\right)\right| = \left|\dot{\pi}(0)\right| \exp\left\{-\int_0^{\dot{x}} \operatorname{tr}\breve{\omega}_\mu(\breve{x})\, d\dot{x}'^\mu\right\}. \tag{A.4}$$

A particular solution to (53) is then (55). The first deformation matrices $\sigma(x) := (\sigma_b{}^a)(x)$ contribute to corresponding ingredient part, $\chi_b{}^d$, of the general deformation tensor,

$$\begin{gathered}\Omega_b{}^a = \chi_b{}^d \dot{\Omega}_d{}^a = \chi_b{}^d \dot{\Omega}_\rho{}^\nu \breve{e}_\nu{}^a \breve{e}^\rho{}_d, \\ \overline{\chi}_d{}^c = \sigma_e{}^c \sigma_d{}^e, \qquad \overline{\chi}_e{}^d \dot{\pi}_b{}^e = \chi_b{}^e \dot{\pi}_e{}^d,\end{gathered} \tag{A.5}$$

or

$$\Omega_\mu{}^\nu = \chi_\mu{}^\rho \dot{\Omega}_\rho{}^\nu, \qquad \chi_\mu{}^\rho = \chi_b{}^d \breve{e}_d{}^\rho \breve{e}^b{}_\mu. \tag{A.6}$$

The tetrad changes according to

$$\begin{gathered}e_c = \left(\sigma_c{}^d \dot{\pi}_d{}^a\right)\breve{e}_a = \sigma_c{}^d \dot{e}_d, \\ \vartheta^c = \left(\sigma^c{}_e \dot{\pi}^e{}_b\right)\breve{\vartheta}^b = \sigma^c{}_e \dot{\vartheta}^e, \qquad e_\rho = \sigma_\rho{}^\sigma \dot{e}_\sigma, \\ \vartheta^\rho = \sigma_\sigma{}^\rho \dot{\vartheta}^\sigma, \qquad e_\rho = \sigma_\rho{}^c \dot{e}_c, \qquad \vartheta^\rho = \sigma_c{}^\rho \dot{\vartheta}^c, \\ \sigma_\rho{}^c = \sigma_\rho{}^\sigma \dot{e}_\sigma{}^c, \qquad \sigma^\rho{}_c = \sigma^\rho{}_\sigma \dot{e}^\sigma{}_c.\end{gathered} \tag{A.7}$$

The corresponding second deformation matrices read

$$\begin{gathered}\gamma_{cd}(x) = \overline{\chi}_{ee'} \dot{\pi}_c{}^e \dot{\pi}_d{}^{e'}, \\ \dot{\gamma}_{cd}\left(\dot{x}\right) = o_{ab} \dot{\pi}_c{}^a\left(\dot{x}\right) \dot{\pi}_d{}^b\left(\dot{x}\right),\end{gathered} \tag{A.8}$$

where $\overline{\chi}_{ee'} = o_{ab}\sigma_e{}^a \sigma_{e'}{}^b$.

If the nonmetricity tensor $N_{\lambda\mu\nu} = -\mathscr{D}_\lambda g_{\mu\nu} \equiv -g_{\mu\nu;\lambda}$ does not vanish, the general formula for the affine connection written in the spacetime components is [94]

$$\overset{(A)}{\Gamma}{}^\rho{}_{\mu\nu} = \breve{\Gamma}^\rho{}_{\mu\nu} + \overset{(A)}{K}{}^\rho{}_{\mu\nu} - \overset{(A)}{N}{}^\rho{}_{\mu\nu} + \frac{1}{2}\overset{(A)}{N}{}_{(\mu\nu)}^{\ \ \ \rho}. \tag{A.9}$$

Here $\overset{(A)}{K}{}^{\rho}{}_{\mu\nu} := 2\overset{(A)}{Q}{}^{\rho}{}_{(\mu\nu)} + \overset{(A)}{Q}{}^{\rho}{}_{\mu\nu}$ is the non-Riemann part—the affine *contortion tensor*, where the torsion, $\overset{(A)}{Q}{}^{\rho}{}_{\mu\nu} = (1/2)\overset{(A)}{T}{}^{\rho}{}_{\mu\nu} = \overset{(A)}{\Gamma}{}^{\rho}{}_{[\mu\nu]}$ given with respect to a holonomic frame, $d\overset{(A)}{\vartheta}{}^{\rho} = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components. We may introduce the contortion tensors related to the *deformation-related frame connection* (60) and the spin connection (62):

$$\overset{(A)}{K}{}^{c}{}_{a\nu} = \overset{(A)}{\omega}{}^{c}{}_{a\nu} + \overset{(A)}{\Delta}{}^{c}{}_{a\nu}, \tag{A.10}$$

where

$$\overset{(A)}{\Delta}{}_{\mu\rho\nu} = \overset{(A)}{e}{}_{\mu a}\overset{(A)}{e}{}^{a}{}_{[\rho,\nu]} - \overset{(A)}{e}{}_{\rho a}\overset{(A)}{e}{}^{a}{}_{[\mu,\nu]} - \overset{(A)}{e}{}_{\nu a}\overset{(A)}{e}{}^{a}{}_{[\mu,\rho]} \tag{A.11}$$

is referred to as the the Ricci coefficients of rotation. Both the contortion tensor and spin connection are antisymmetric in their first two indices. The Levi-Civita spin connection

$$\breve{\omega}{}^{\mu}{}_{a\rho} = \overset{(\pi)}{e}{}^{\mu}{}_{a:\rho} = \overset{(\pi)}{e}{}^{\mu}{}_{a,\rho} + \breve{\Gamma}{}^{\mu}{}_{\nu\rho}\overset{(\pi)}{e}{}^{\nu}{}_{a} \tag{A.12}$$

is related to the Ricci rotation coefficients, with $\overset{(A)}{K} = 0$; thus,

$$\overset{(A)}{K}{}^{\mu}{}_{a\rho} = \overset{(A)}{\omega}{}^{\mu}{}_{a\rho} - \breve{\omega}{}^{\mu}{}_{a\rho}. \tag{A.13}$$

The relations between the corresponding torsion and contortion tensors read

$$\overset{(A)}{K}{}^{\rho}{}_{\mu\nu} := 2\overset{(A)}{Q}{}^{\rho}{}_{(\mu\nu)} + \overset{(A)}{Q}{}^{\rho}{}_{\mu\nu}, \qquad \overset{(A)}{Q}{}^{\rho}{}_{\mu\nu} = \overset{(A)}{K}{}^{\rho}{}_{[\mu\nu]}, \tag{A.14}$$

where

$$\overset{(A)}{Q}{}^{\rho}{}_{\mu\nu} = \overset{(A)}{\omega}{}^{\rho}{}_{[\mu\nu]} + \overset{(A)}{e}{}^{a}{}_{[\mu,\nu]}\overset{(A)}{e}{}^{\rho}{}_{a}. \tag{A.15}$$

So, the affine connection (65) reads

$$\Gamma^{\mu}{}_{\rho\sigma} = \overset{(\sigma)}{\Gamma}{}^{\mu}{}_{(\rho\sigma)} + \overset{(\sigma)}{Q}{}^{\mu}{}_{\rho\sigma} = \overset{(\pi)}{\Gamma}{}^{\mu}{}_{(\rho\sigma)} + \overset{(\pi)}{Q}{}^{\mu}{}_{\rho\sigma}, \tag{A.16}$$

where $\overset{(A)}{\Gamma}{}^{\mu}{}_{(\rho\sigma)} = \breve{\Gamma}{}^{\mu}{}_{(\rho\sigma)} + 2\overset{(A)}{Q}{}^{\mu}{}_{(\rho\sigma)}$. In the presence of curvature and torsion, the coupling prescription of a general field carrying an arbitrary representation of the Lorentz group will be

$$\partial_{\mu} \longrightarrow \overset{(A)}{\mathscr{D}}_{\mu} = \partial_{\mu} - \frac{i}{2}\left(\overset{(A)}{\omega}{}^{ab}{}_{\mu} - \overset{(A)}{K}{}^{ab}{}_{\mu}\right)J_{ab}. \tag{A.17}$$

B. Some Algebraic Operations

Since the 4D-dimensional teleparallel manifold $\mathscr{M}_4$ is diffeomorphic to R^4, one can choose an orientation on $\mathscr{M}_4$ and restrict the frames to agree with that orientation so that only transformations with values in $GL_4^+(R)$ are allowed. The metric then defines the Hodge dual of differential forms. The following algebraic operations are defined.

The Hodge dual map $\star : \Omega^p \to \Omega^{n-p}$. This is a C^{∞}-linear map $\star : \Omega^p \to \Omega^{n-p}$, which acts on the wedge product monomials of the basis 1-forms as

$$\star\left(\vartheta^{a_1\cdots a_p}\right) = \varepsilon^{a_1\cdots a_n}e_{a_{p+1}\cdots a_n}, \tag{B.1}$$

where e_{a_i} $(i = p+1,\ldots,n)$ are understood as the downindexed 1-forms $e_{a_i} = o_{a_i b}\vartheta^b$ and $\epsilon^{a_1\cdots a_n}$ is the total antisymmetric pseudotensor. A further relation involving Hodge duality reads $^{\star}(\alpha\wedge e_a) = (e_a\rfloor{}^{\star}\alpha)$, while for differential forms α, β of the same degree p, equation $^{\star}(\alpha\wedge\beta) = {}^{\star}(\beta\wedge\alpha)$ holds.

The eta basis is consisting in the Hodge dual of exterior products of tetrads by means of the Levi-Civita object: $\eta^{abcd} := {}^{\star}\vartheta^{abcd}$, which yields

$$\begin{aligned}
\eta^{abc} &:= {}^{\star}\vartheta^{abc} = \eta^{abc}{}_{d}\wedge\vartheta^d,\\
\eta^{ab} &:= {}^{\star}\vartheta^{ab} = \frac{1}{2!}\eta^{ab}{}_{cd}\wedge\vartheta^{cd},\\
\eta^{a} &:= {}^{\star}\vartheta^{a} = \frac{1}{3!}\eta^{a}{}_{bcd}\wedge\vartheta^{bcd},\\
\eta &:= {}^{\star}1 = \frac{1}{4!}\eta_{abcd}\vartheta^{abcd},
\end{aligned} \tag{B.2}$$

such that

$$\begin{aligned}
\vartheta^e\wedge\eta_{abcd} &= -\delta^e_a\eta_{bcd} + \delta^e_d\eta_{abc} - \delta^e_c\eta_{dab} + \delta^e_b\eta_{cda},\\
\vartheta^e\wedge\eta_{abc} &= \delta^e_a\eta_{bc} + \delta^e_c o_{ab} + \delta^e_b\eta_{ca},\\
\vartheta^e\wedge o_{ab} &= -\delta^e_a\eta_b + \delta^e_b\eta_a, \qquad \vartheta^e\wedge\eta_a = \delta^e_a\eta.
\end{aligned} \tag{B.3}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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