Research Article

On ϕ -Absorbing Primary Elements in Lattice Modules

Sachin Ballal and Vilas Kharat

Department of Mathematics, Savitribai Phule Pune University, Pune 411 007, India

Correspondence should be addressed to Sachin Ballal; ballalshyam@gmail.com

Received 18 December 2014; Accepted 31 March 2015

Academic Editor: Andrei V. Kelarev

Copyright © 2015 S. Ballal and V. Kharat. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let L be a C -lattice and let M be a lattice module over L . Let $\phi : M \rightarrow M$ be a function. A proper element $P \in M$ is said to be ϕ -absorbing primary if, for $x_1, x_2, \dots, x_n \in L$ and $N \in M$, $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \phi(P)$ together imply $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$, for some $i \in \{1, 2, \dots, n\}$. We study some basic properties of ϕ -absorbing primary elements. Also, various generalizations of prime and primary elements in multiplicative lattices and lattice modules as ϕ -absorbing elements and ϕ -absorbing primary elements are unified.

1. Introduction

A lattice L is said to be *complete*, if, for any subset S of L , we have $\vee S, \wedge S \in L$. Since every complete lattice is bounded, 1_L (or 1) denotes the greatest element and 0_L (or 0) denotes the smallest element of L . A complete lattice L is said to be a *multiplicative lattice*, if there is a defined binary operation “ $\cdot$ ” called multiplication on L satisfying the following conditions:

- (1) $a \cdot b = b \cdot a$, for all $a, b \in L$,
- (2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in L$,
- (3) $a \cdot (\vee_{\alpha} b_{\alpha}) = \vee_{\alpha} (a \cdot b_{\alpha})$, for all $a, b_{\alpha} \in L$,
- (4) $a \cdot 1 = a$, for all $a \in L$.

Henceforth, $a \cdot b$ will be simply denoted by ab .

An element $p \neq 1$ of a multiplicative lattice L is said to be *prime* if $ab \leq p$ implies either $a \leq p$ or $b \leq p$, for $a, b \in L$. Radical of an element $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\sqrt{a} = \vee \{x \in L \mid x^n \leq a, \text{ for some } n \in \mathbb{Z}^+\}$.

An element c of a complete lattice L is said to be *compact* if $c \leq \vee_{\alpha} a_{\alpha}$ implies $c \leq \vee_{i=1}^n a_{\alpha_i}$, where $n \in \mathbb{Z}^+$. The set of all compact elements of a lattice L is denoted by L_* . By a C -lattice we mean a multiplicative lattice L with a multiplicatively closed set S of compact elements which generates L under join.

A complete lattice M is said to be a *lattice module* over a multiplicative lattice L , if there is a multiplication between

elements of M and L , denoted by aN for $a \in L$ and $N \in M$, which satisfies the following properties:

- (1) $(ab)N = a(bN)$,
- (2) $(\vee_{\alpha} a_{\alpha})(\vee_{\beta} N_{\beta}) = \vee_{\alpha, \beta} (a_{\alpha} N_{\beta})$,
- (3) $1_L N = N$,
- (4) $0_L N = 0_M$, for all $a, b, a_{\alpha} \in L$ and for all $N, N_{\beta} \in M$,

where 1_M denotes the greatest element of M and 0_M denotes the smallest element of M .

For $N \in M$ and $a \in L$, denote $(N : a) = \vee \{X \in M : aX \leq N\}$. For $a, b \in L$, $(a : b) = \vee \{x \in L : bx \leq a\}$ and for $A, B \in M$, $(A : B) = \vee \{x \in L : xB \leq A\}$. For $N \in M$, $\sqrt[n]{N} = \vee \{x \in L : x^n 1_M \leq N\}$ for some positive integer n and it is also denoted by $\sqrt{(N : 1_M)}$. For $N \in M$, we define $\sqrt[n]{N} = \sqrt{(N : 1_M)} 1_M$. An element $N \in M$ is said to be *weak join principal* if it satisfies the following identity $a \vee (0_M : N) = (aN : N)$ for all $a \in L$.

A lattice module M over a multiplicative lattice L is called a *multiplication lattice module* if for $N \in M$ there exists an element $a \in L$ such that $N = a 1_M$.

An element $N \neq 1_M$ in M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a 1_M \leq N$, that is, $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$.

An element $N \in M$ is called *compact* if $N \leq \vee_{\alpha} A_{\alpha}$ implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some $\alpha_1, \alpha_2, \dots, \alpha_n$. If each element of M is a join of principal (compact) elements of

M , then M is called *principally generated lattice* (compactly generated lattice).

Çallıalp et al. [1] studied the concepts of weakly prime and almost prime elements in multiplicative lattices as extensions of, respectively, weakly prime and almost prime ideals in commutative rings. An element $p \neq 1$ in L is said to be *weakly prime* if $0 \neq ab \leq p$ implies either $a \leq p$ or $b \leq p$ and *almost prime* if $ab \leq p$ and $ab \not\leq p^2$ implies either $a \leq p$ or $b \leq p$ for $a, b \in L$. In [2], the authors generalized these concepts, respectively, to weakly primary and almost primary elements in multiplicative lattices. A proper element $p \in L$ is said to be *weakly primary* if $0 \neq ab \leq p$ implies either $a \leq p$ or $b \leq \sqrt[p]{p}$ and *almost primary* if $ab \leq p$ and $ab \not\leq p^2$ implies either $a \leq p$ or $b \leq \sqrt[p]{p}$ for $a, b \in L$.

The concept of prime elements in multiplicative lattices is further generalised to 2-absorbing and weakly 2-absorbing elements in multiplicative lattices by Jayaram et al. [3]. An element $q < 1$ in L is said to be *2-absorbing* if $abc \leq q$ implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$ and is said to be *weakly 2-absorbing element* if $0 \neq abc \leq q$ implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$, for $a, b, c \in L$. Joshi and Ballal [4] defined the concept of n -prime elements in multiplicative lattices as a generalization of prime elements. An element $p < 1$ of a multiplicative lattice L is said to be an n -prime if we can express it as meet of at most n primes, where n is a positive integer.

In [5], the authors defined the concepts of n -absorbing, weakly n -absorbing, and n -almost n -absorbing elements in multiplicative lattices as generalizations of, respectively, 2-absorbing, weakly 2-absorbing, and almost prime elements, where $n \geq 2$. An element $p < 1$ of a multiplicative lattice L is called *n -absorbing* if $x_1 x_2 x_3 \cdots x_{n+1} \leq p$ implies $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \leq p$ and called *weakly n -absorbing* if, for $x_1, x_2, x_3, \dots, x_{n+1} \in L$, $0 \neq x_1 x_2 x_3 \cdots x_{n+1} \leq p$ implies $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \leq p$, $i \in \{1, 2, \dots, n\}$, and $x_1, x_2, x_3, \dots, x_{n+1} \in L$. An element $p < 1$ of a multiplicative lattice L is called *n -almost n -absorbing* if $x_1 x_2 x_3 \cdots x_{n+1} \leq p$ and $x_1 x_2 x_3 \cdots x_{n+1} \not\leq p^n$ together imply $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \leq p$ for $i \in \{1, 2, \dots, n\}$, $x_1, x_2, x_3, \dots, x_{n+1} \in L$.

Manjarekar and Bingi [6] unified the theory of generalizations of prime and primary elements in multiplicative lattice as ϕ -prime and ϕ -primary elements. Let $\phi : L \rightarrow L$ be a function. A proper element $p \in L$ is said to be ϕ -prime if $ab \leq p$ and $ab \not\leq \phi(p)$ implies either $a \leq p$ or $b \leq p$ and it is said to be ϕ -primary if $ab \leq p$ and $ab \not\leq \phi(p)$ implies either $a \leq p$ or $b \leq \sqrt[p]{\phi(p)}$, for $a, b \in L$.

As a generalization of primary ideals in commutative rings, Badawi et al. [7] introduced the concept of 2-absorbing primary ideals. In this paper, we extend the various generalizations of prime ideals and primary ideals in commutative rings to lattice modules. Also, we unify various generalizations of prime and primary elements in multiplicative lattices and lattice modules, respectively, as ϕ -absorbing elements and ϕ -absorbing primary elements.

For basic concepts and terminologies of lattice modules, one may refer to [8–11] and for multiplicative lattices, one may refer to [12–14].

2. ϕ -Absorbing Primary Elements

We introduce the concepts of ϕ -absorbing elements and ϕ -absorbing primary elements in lattice modules which generalizes, respectively, the concepts of prime and primary elements in multiplicative lattices and lattice modules (see [1–3, 5, 6]).

Essentially, we have the following definitions, where M is a lattice module over a multiplicative lattice L and $n \geq 1$.

Definition 1. A proper element $P \in M$ is said to be *n -absorbing* if $x_1 x_2 \cdots x_n N \leq P$ implies $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq P$ for $i \in \{1, 2, \dots, n\}$, where $x_1, x_2, \dots, x_n \in L$ and $N \in M$.

Definition 2. Let $\phi : M \rightarrow M$ be a function. A proper element $P \in M$ is said to be *ϕ -absorbing* if $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \phi(P)$ together imply $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq P$ for $i \in \{1, 2, \dots, n\}$, where $x_1, x_2, \dots, x_n \in L$ and $N \in M$.

Definition 3. A proper element $P \in M$ is said to be *n -absorbing primary* if $x_1 x_2 \cdots x_n N \leq P$ implies $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for $i \in \{1, 2, \dots, n\}$, where $x_1, x_2, \dots, x_n \in L$ and $N \in M$.

Definition 4. Let $\phi : M \rightarrow M$ be a function. A proper element $P \in M$ is said to be *ϕ -absorbing primary* if $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \phi(P)$ together imply $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for $i \in \{1, 2, \dots, n\}$, where $x_1, x_2, \dots, x_n \in L$ and $N \in M$.

Let M be a lattice module over a multiplicative lattice L . For a map $\phi_\alpha : M \rightarrow M$, we have the following.

- (1) $\phi_0 : \phi(P) = 0$ defines weakly n -absorbing primary elements of M .
- (2) $\phi_2 : \phi(P) = (P : 1_M)P$ defines almost n -absorbing primary elements of M .
- (3) $\phi_{m+1} (n \geq 1) : \phi(P) = (P : 1_M)^m P$ defines m -almost n -absorbing primary elements of M .
- (4) $\phi_\omega : \phi(P) = \bigwedge_{m=1}^\infty (P : 1_M)^m P$ defines ω -absorbing primary elements of M .

Remark 5. It follows immediately from the definition that any n -absorbing primary element of M is a ϕ -absorbing primary. However, the converse does not necessarily holds. In fact, we have the following theorem in which the converse is true under certain condition.

Theorem 6. Let M be a lattice module over a C -lattice L . Then every ϕ -absorbing primary element $P \in M$ with $(P : 1_M)^n P \not\leq \phi(P)$ is n -absorbing primary.

Proof. Suppose that $(P : 1_M)^n P \not\leq \phi(P)$ and $x_1 x_2 \cdots x_n N \leq P$, for $x_1, x_2, \dots, x_n \in L$, $N \in M$. If $x_1 x_2 \cdots x_n N \not\leq \phi(P)$, then as P is ϕ -absorbing primary, we have $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$, which concludes that P is n -absorbing primary.

Now, suppose that $x_1 x_2 \cdots x_n N \leq \phi(P)$. We assume that $x_1 x_2 \cdots x_{n-k} (P : 1_M)^k N \leq \phi(P)$, for all $k \in \{1, 2, \dots, n-1\}$. If $x_1 x_2 \cdots x_{n-k} (P : 1_M)^k N \not\leq \phi(P)$, then there exists $a_1, a_2, \dots, a_k \leq (P : 1_M)$ such that $x_1 x_2 \cdots x_{n-k} a_1 a_2 \cdots a_k N \not\leq \phi(P)$. Hence, $x_1 x_2 \cdots x_{n-k} (x_{n-k+1} \vee a_1)(x_{n-k+2} \vee a_2) \cdots (x_n \vee a_k) N \leq P$ and $x_1 x_2 \cdots x_{n-k} (x_{n-k+1} \vee a_1)(x_{n-k+2} \vee a_2) \cdots (x_n \vee a_k) N \not\leq \phi(P)$.

Since p is ϕ -absorbing primary, $x_1 x_2 \cdots x_{n-k} (x_{n-k+1} \vee a_1) \cdots (x_n \vee a_k) \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n-k} (x_{n-k+1} \vee a_1)(x_{n-k+2} \vee a_2) \cdots (x_n \vee a_k) N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n-1\}$ and so $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n-k} x_{n-k+1} x_{n-k+2} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n-1\}$ or $x_1 x_2 \cdots x_n \leq (P : 1_M)$.

Similarly, we assume that, for $\{i_1, i_2, \dots, i_{n-k}\} \subseteq \{1, 2, \dots, n\}$, $1 \leq k \leq n$, $x_{i_1} x_{i_2} \cdots x_{i_{n-k}} (P : 1_M)^k N \leq \phi(P)$. Also, we assume that $x_{i_1} x_{i_2} \cdots x_{i_{n-k}} (P : 1_M)^k P \leq \phi(P)$, $1 \leq k \leq n$, because if $x_{i_1} x_{i_2} \cdots x_{i_{n-k}} (P : 1_M)^k P \not\leq \phi(P)$, then $x_{i_1} x_{i_2} \cdots x_{i_{n-k}} a_1 a_2 \cdots a_k P \not\leq \phi(P)$, where $a_1 a_2 a_3 \cdots a_k \leq (P : 1_M)$ and so $x_1 x_2 \cdots x_{n-k} (x_{n-k+1} \vee a_1)(x_{n-k+2} \vee a_2) \cdots (x_n \vee a_k) (N \vee P) \leq \sqrt[n]{P}$ and $x_1 x_2 \cdots x_{n-k} (x_{n-k+1} \vee a_1)(x_{n-k+2} \vee a_2) \cdots (x_n \vee a_k) (N \vee P) \not\leq \phi(P)$.

Now, since P is ϕ -absorbing primary, $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$, for some $i \in \{1, 2, \dots, n-1\}$.

Also, $(P : 1_M)^n P \not\leq \phi(P)$ implies that there exist $a_1, a_2, \dots, a_n \leq (P : 1_M)$ with $a_1 a_2 \cdots a_n P \not\leq \phi(P)$. We have $(x_1 \vee a_1)(x_2 \vee a_2) \cdots (x_n \vee a_n) (N \vee P) \leq P$ and $(x_1 \vee a_1)(x_2 \vee a_2) \cdots (x_n \vee a_n) (N \vee P) \not\leq \phi(P)$. Now, P is ϕ -absorbing primary; therefore, $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n-k} x_{n-k+1} x_{n-k+2} \cdots x_n N \leq P$, for some $i \in \{1, 2, \dots, n-1\}$ or $x_1 x_2 \cdots x_n \leq (P : 1_M)$ and consequently P is ϕ -absorbing primary. $\square$

From the above theorem, it follows that if P is a ϕ -primary element of M that is not n -absorbing primary, then $(P : 1_M)^n P \leq \phi(P)$.

Corollary 7. Let M be a lattice module over a C -lattice L . If $P \in M$ is weakly n -absorbing primary that is not n -absorbing primary, then $(P : 1_M)^n P = 0$.

Theorem 8. Let M be a multiplication lattice module over C -lattice L and let $P \in M$. Then the following holds.

- (1) Let $\psi_1, \psi_2 : M \rightarrow M$ be two functions with $\psi_1 \leq \psi_2$, that is, $\psi_1(N) \leq \psi_2(N)$ for each $N \in M$. Then P is ψ_2 -absorbing primary if it is ψ_1 -absorbing primary.
- (2) Consider the following statements.

- (a) P is n -absorbing primary.
- (b) P is weakly n -absorbing primary.
- (c) P is ω -absorbing primary.
- (d) P is m -almost n -absorbing primary.
- (e) P is almost n -absorbing primary.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

- (3) P is ω -absorbing primary if and only if it is m -almost n -absorbing primary for $m \geq 1$.

Proof. (1) Suppose that $P \in M$ is ψ_1 -absorbing primary and also suppose that $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \psi_2(P)$ for $x_1, x_2, \dots, x_n \in L$ and $N \in M$. Since $\psi_1(N) \leq \psi_2(N)$ for each $N \in M$, we have $x_1 x_2 \cdots x_n N \not\leq \psi_1(P)$. It follows from the fact $x_1 x_2 \cdots x_n N \leq P$, $x_1 x_2 \cdots x_n N \not\leq \psi_1(P)$, and P is ψ_1 -absorbing primary that $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$ and so, P is ψ_2 -absorbing primary.

(2) By definition, every n -absorbing primary element is weakly n -absorbing primary and therefore (a) $\Rightarrow$ (b) holds.

(b) $\Rightarrow$ (c) Suppose that P is weakly n -absorbing primary and also suppose that $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \bigwedge_{m=1}^{\infty} (P : 1_M)^m P$, for $x_1, x_2, \dots, x_n \in L$ and $N \in M$. Then $x_1 x_2 \cdots x_n N \neq 0_M$. By the assumption, $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$ and so, P is ω -absorbing primary.

(c) $\Rightarrow$ (d) Suppose that P is ω -absorbing primary and also suppose that $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq (P : 1_M)^m P$ for $x_1, x_2, \dots, x_n \in L$, $N \in M$, and $m \geq 1$. Then $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \bigwedge_{m=1}^{\infty} (P : 1_M)^m P$. Since P is ω -absorbing primary, it follows that $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$ and so, P is m -almost n -absorbing primary.

The statement (e) is a particular case of the statement (d) for $m = 1$ and therefore (d) $\Rightarrow$ (e) holds.

(3) Suppose that $P \in M$ is m -almost n -absorbing primary for $m \geq 1$ and also suppose that $x_1 x_2 \cdots x_n N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq \bigwedge_{m=1}^{\infty} (P : 1_M)^m P$ for $x_1, x_2, \dots, x_n \in L$ and $N \in M$. Then $x_1 x_2 \cdots x_m N \leq P$ and $x_1 x_2 \cdots x_n N \not\leq (P : 1_M)^m P$ for some $m \geq 1$. Since P is m -almost n -absorbing primary, we have $x_1 x_2 \cdots x_n \leq (P : 1_M)$ or $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n N \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$. Consequently, P is ω -absorbing primary.

The converse follows from (c) $\Rightarrow$ (d). $\square$

Corollary 9. Let M be a lattice module over a C -lattice L . If $P \in M$ is ϕ -absorbing primary, where $\phi \leq \phi_{n+2}$, then P is ω -absorbing primary.

Proof. Suppose that $P \in M$ is ϕ -absorbing primary. If P is n -absorbing primary, then, by Theorem 8, it is ω -absorbing primary. Now, if P is not n -absorbing primary, then, by Theorem 6, $(P : 1_M)^n P \leq \phi(P) \leq \phi_{n+2}(P) = (P : 1_M)^{n+1} P$. Consequently, $\phi(P) = (P : 1_M)^k P$, for each $k \geq n$. And, by Theorem 8(3), P is ω -absorbing primary. $\square$

Lemma 10. Let L be a multiplicative lattice. If $p \in L$ is ϕ -absorbing primary with $\phi(\sqrt[n]{p}) = \sqrt[n]{\phi(p)}$, then $\sqrt[n]{p}$ is also ϕ -absorbing.

Proof. (I) Suppose that $x_1 x_2 \cdots x_{n+1} \leq \sqrt[n]{p}$ and $x_1 x_2 \cdots x_{n+1} \not\leq \phi(\sqrt[n]{p}) = \sqrt[n]{\phi(p)}$, for $x_1, x_2, \dots, x_{n+1} \in L$. If $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \not\leq \sqrt[n]{p}$ for $i \in \{1, 2, \dots, n\}$, then we have $(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})^s \not\leq p$ for any positive integer s and for $i \in \{1, 2, \dots, n\}$.

(II) Now, since $x_1 x_2 \cdots x_{n+1} \leq \sqrt[n]{p}$, there exists a positive integer k such that $(x_1 x_2 \cdots x_{n+1})^k = x_1^k x_2^k \cdots x_{n+1}^k \leq p$.

(III) As $x_1 x_2 \cdots x_{n+1} \notin \sqrt{\phi(p)}$, we have $x_1^t x_2^t \cdots x_{n+1}^t \notin \phi(p)$ for a positive integer t .

From the assumption and (I), (II), and (III), it follows that $(x_1 x_2 \cdots x_n)^k = x_1^k x_2^k \cdots x_n^k \leq \sqrt{p}$. Consequently, $x_1 x_2 \cdots x_n \leq \sqrt[k]{p}$ and therefore $\sqrt{p}$ is ϕ -absorbing primary. $\square$

It is easy to observe that if $(L_1, \wedge_1, \vee_1, \circ_1)$ and $(L_2, \wedge_2, \vee_2, \circ_2)$ are multiplicative lattices then $L_1 \times L_2$ is also a multiplicative lattice with componentwise meet, join, and multiplication.

Also, if M_1 and M_2 are lattice modules over multiplicative lattices L_1 and L_2 , respectively, then $M_1 \times M_2$ is a lattice module over $L_1 \times L_2$ with componentwise meet, join, and multiplication given by $(a, b)(N_1, N_2) = (aN_1, bN_2)$, where $(a, b) \in L_1 \times L_2$ and $(N_1, N_2) \in M_1 \times M_2$.

Lemma 11. Let $M = M_1 \times M_2$ and $L = L_1 \times L_2$, where M_i is a lattice module over C -lattice L_i , for $i = 1, 2$. Then $((P_1 : 1_{M_1}), (P_2 : 1_{M_2})) = ((P_1, P_2) : (1_{M_1}, 1_{M_2}))$, for $P_1 \in M_1$ and $P_2 \in M_2$.

Proof. Let $(x, y) \in L_*$. Now, $(x, y) \leq ((P_1 : 1_{M_1}), (P_2 : 1_{M_2}))$:

$$\begin{aligned} &\Leftrightarrow x \leq (P_1 : 1_{M_1}) \text{ and } y \leq (P_2 : 1_{M_2}), \\ &\Leftrightarrow x1_{M_1} \leq P_1 \text{ and } y1_{M_2} \leq P_2, \\ &\Leftrightarrow (x, y)(1_{M_1}, 1_{M_2}) = (x1_{M_1}, y1_{M_2}) \leq (P_1, P_2), \\ &\Leftrightarrow (x, y) \leq (P_1, P_2) : (1_{M_1}, 1_{M_2}). \end{aligned}$$

$\square$

Lemma 12. Let $M = M_1 \times M_2$ and $L = L_1 \times L_2$, where M_i is a lattice module over C -lattice L_i , for $i = 1, 2$. Then $\sqrt[n]{(P_1, P_2)} = (\sqrt[n]{P_1}, \sqrt[n]{P_2})$, for $P_1 \in M_1$ and $P_2 \in M_2$.

Proof. Let $(N_1, N_2) \in M_*$ with $(N_1, N_2) \leq \sqrt[n]{(P_1, P_2)} = \sqrt{(P_1, P_2) : (1_{M_1}, 1_{M_2})}$. Then $(N_1, N_2) \leq (a, b)(1_{M_1}, 1_{M_2})$, where $(a, b) \leq \sqrt{(P_1, P_2) : (1_{M_1}, 1_{M_2})}$.

Now, $(a, b) \leq \sqrt{(P_1, P_2) : (1_{M_1}, 1_{M_2})} \Rightarrow (a, b)^n \leq (P_1, P_2) : (1_{M_1}, 1_{M_2})$ for a positive integer n :

$$\begin{aligned} &\Rightarrow a^n \leq (P_1 : 1_{M_1}) \text{ and } b^n \leq (P_2 : 1_{M_2}) \text{ (by Lemma 11),} \\ &\Rightarrow (a, b) \leq \sqrt[n]{(P_1 : 1_{M_1}), \sqrt[n]{(P_2 : 1_{M_2})}}, \\ &\Leftrightarrow (a, b)(1_{M_1}, 1_{M_2}) \leq (\sqrt[n]{(P_1 : 1_{M_1})}, \sqrt[n]{(P_2 : 1_{M_2})}) (1_{M_1}, 1_{M_2}) \\ &= (\sqrt[n]{(P_1 : 1_{M_1})}1_{M_1}, \sqrt[n]{(P_2 : 1_{M_2})}1_{M_2}), \\ &\Leftrightarrow (a, b)(1_{M_1}, 1_{M_2}) \leq (\sqrt[n]{P_1}, \sqrt[n]{P_2}). \end{aligned}$$

Consequently,

$$\sqrt[n]{(P_1, P_2)} \leq (\sqrt[n]{P_1}, \sqrt[n]{P_2}). \quad (I)$$

Next, let $(S_1, S_2) \in M_*$ be such that $(S_1, S_2) \leq (\sqrt[n]{P_1}, \sqrt[n]{P_2})$:

$$\Rightarrow S_1 \leq \sqrt[n]{P_1} \text{ and } S_2 \leq \sqrt[n]{P_2};$$

$$\Rightarrow S_1 \leq \sqrt{(P_1 : 1_{M_1})}1_{M_1} \text{ and } S_2 \leq \sqrt{(P_2 : 1_{M_2})}1_{M_2};$$

$$\Rightarrow S_1 \leq x1_{M_1} \text{ and } S_2 \leq y1_{M_2} \text{ for some } x, y \in L \text{ with } x \leq \sqrt{(P_1 : 1_{M_1})} \text{ and } y \leq \sqrt{(P_2 : 1_{M_2})};$$

$$\Rightarrow x^t \leq (P_1 : 1_{M_1}) \text{ and } y^r \leq (P_2 : 1_{M_2}) \text{ for some positive integers } t \text{ and } r.$$

Choose $k = t + r$. Then $x^k \leq (P_1 : 1_{M_1})$ and $y^k \leq (P_2 : 1_{M_2})$:

$$\Rightarrow (x, y)^k = (x^k, y^k) \leq ((P_1 : 1_{M_1}), (P_2 : 1_{M_2}));$$

$$\Rightarrow (x, y) \leq \sqrt[k]{((P_1 : 1_{M_1}), (P_2 : 1_{M_2}))};$$

$$\Rightarrow (x, y)(1_{M_1}, 1_{M_2}) \leq \sqrt[k]{((P_1 : 1_{M_1}), (P_2 : 1_{M_2}))} (1_{M_1}, 1_{M_2});$$

$$\Rightarrow (S_1, S_2) \leq (x, y)(1_{M_1}, 1_{M_2}) \leq \sqrt[k]{((P_1 : 1_{M_1}), (P_2 : 1_{M_2}))} (1_{M_1}, 1_{M_2}) = \sqrt[k]{(P_1, P_2)}.$$

Consequently,

$$(\sqrt[n]{P_1}, \sqrt[n]{P_2}) \leq \sqrt[n]{(P_1, P_2)}. \quad (II)$$

From (I) and (II), result follows. $\square$

Theorem 13. Let $M = M_1 \times M_2$ and $L = L_1 \times L_2$, where M_i is a lattice module over C -lattice L_i , for $i = 1, 2$, and let $\phi : M \rightarrow M$ be a function. If $P_1 \in M_1$ is a weakly n -absorbing primary such that $(0, 1_{M_2}) \leq \phi(P_1, 1_{M_2})$, then $(P_1, 1_{M_2}) \in M_1 \times M_2$ is ϕ -absorbing primary.

Proof. Let $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in L$ and $(N_1, N_2) \in M$ be such that $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(N_1, N_2) = (a_1 a_2 \cdots a_n N_1, b_1 b_2 \cdots b_n N_2) \leq (P_1, 1_{M_2})$ and $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(N_1, N_2) = (a_1 a_2 \cdots a_n N_1, b_1 b_2 \cdots b_n N_2) \notin \phi(P_1, 1_{M_2})$.

Since $(0, 1_{M_2}) \leq \phi(P_1, 1_{M_2})$, we have $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)(N_1, N_2) = (a_1 a_2 \cdots a_n N_1, b_1 b_2 \cdots b_n N_2) \leq (0, 1_{M_2})$ and so $0 \neq a_1 a_2 \cdots a_n N_1 \leq P_1$. Since $P_1 \in M_1$ is weakly n -absorbing primary, we have $a_1 a_2 \cdots a_n \leq (P_1 : 1_{M_1})$ or $a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n N_1 \leq \sqrt[n]{P_1}$, for some $i \in \{1, 2, \dots, n\}$. This implies that $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n) = (a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n) \leq ((P_1 : 1_{M_1}), (1_{M_2} : 1_{M_2})) = ((P_1, 1_{M_2}) : (1_{M_1}, 1_{M_2}))$ or $(a_1, b_1)(a_2, b_2) \cdots (a_{i-1}, b_{i-1})(a_{i+1}, b_{i+1}) \cdots (a_n, b_n)(N_1, N_2) \leq (\sqrt[n]{P_1}, 1_{M_2}) = \sqrt[n]{(P_1, 1_{M_2})}$ for some $i \in \{1, 2, \dots, n\}$, by Lemmas 11 and 12. Therefore, $(P_1, 1_{M_2})$ is ϕ -absorbing primary element of $M_1 \times M_2$. $\square$

Theorem 14. Let M be a lattice module over a C -lattice L and $P \in M$. Then P is ϕ -absorbing primary if and only if $(P : x_1 x_2 \cdots x_{n-1} N) = (P : x_1 x_2 \cdots x_{n-1} 1_M)$ or $(P : x_1 x_2 \cdots x_{n-1} N) = (\phi(P) : x_1 x_2 \cdots x_{n-1} N)$ or $(P : x_1 x_2 \cdots x_{n-1} N) = (\sqrt{P} : x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n-1} N)$ for some $i \in \{1, 2, \dots, n-1\}$, for $x_1, x_2, \dots, x_{n-1} \in L$ and $N \in M$ with $x_1 x_2 \cdots x_{n-1} N \notin \sqrt{P}$.

Proof. Suppose that $P \in M$ is ϕ -absorbing primary and $x_1x_2 \cdots x_{n-1}N \not\leq \sqrt[n]{P}$, for $x_1, x_2, \dots, x_{n-1} \in L$ and $N \in M$. Let $r \in L_*$ be such that $r \leq (P : x_1x_2 \cdots x_{n-1}N)$ which essentially implies that $x_1x_2 \cdots x_{n-1}rN \leq P$.

We have the following two cases.

Case 1. If $rx_1x_2 \cdots x_{n-1}N \not\leq \phi(P)$, then as P is ϕ -absorbing primary, we have $x_1x_2 \cdots x_{n-1}r \leq (P : 1_M)$ or $x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_{n-1}rN \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n-1\}$. Therefore, $r \leq (P : x_1x_2 \cdots x_{n-1}1_M)$ or $r \leq (\sqrt[n]{P} : x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_{n-1}N)$ for some $i \in \{1, 2, \dots, n-1\}$.

Case 2. If $rx_1x_2 \cdots x_{n-1}N \leq \phi(P)$, then $r \leq (\phi(P) : x_1x_2 \cdots x_{n-1}N)$. So $(P : x_1x_2 \cdots x_{n-1}N) \leq (\phi(P) : x_1x_2 \cdots x_{n-1}N)$.

Now, from Cases 1 and 2, it follows that $(P : x_1x_2 \cdots x_{n-1}N) = (P : x_1x_2 \cdots x_{n-1}1_M)$ or $(P : x_1x_2 \cdots x_{n-1}N) = (\phi(P) : x_1x_2 \cdots x_{n-1}N)$ or $(P : x_1x_2 \cdots x_{n-1}N) = (\sqrt[n]{P} : x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_{n-1}N)$ for some $i \in \{1, 2, \dots, n-1\}$.

Conversely, suppose that $x_1x_2 \cdots x_nN \leq P$ and $x_1x_2 \cdots x_nN \not\leq \phi(P)$, for $x_1, x_2, \dots, x_n \in L$ and $N \in M$. If $x_1x_2 \cdots x_{n-1}N \leq \sqrt[n]{P}$, then the result is obvious. So, suppose that $x_1x_2 \cdots x_{n-1}N \not\leq \sqrt[n]{P}$.

Now, $(P : x_1x_2 \cdots x_{n-1}N) = (P : x_1x_2 \cdots x_{n-1}1_M)$ or $(P : x_1x_2 \cdots x_{n-1}N) = (\phi(P) : x_1x_2 \cdots x_{n-1}N)$ or $(P : x_1x_2 \cdots x_{n-1}N) = (\sqrt[n]{P} : x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_{n-1}N)$ for some $i \in \{1, 2, \dots, n-1\}$. Since $x_1x_2 \cdots x_nN \leq P$, we have $x_n \leq (P : x_1x_2 \cdots x_{n-1}N)$. But $x_n \not\leq (\phi(P) : x_1x_2 \cdots x_{n-1}N)$ and so $x_1x_2 \cdots x_{n-1}x_n \leq (P : 1_M)$ or $x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_nN \leq \sqrt[n]{P}$ for some $i \in \{1, 2, \dots, n\}$. Consequently, P is ϕ -absorbing primary. $\square$

Theorem 15. Let $M = M_1 \times M_2 \times \cdots \times M_n$ and $L = L_1 \times L_2 \times \cdots \times L_n$, where each M_i is a compactly generated lattice module over a C -lattice L_i , for $i \in \{1, 2, \dots, n\}$. Let $\psi : M \rightarrow M$ such that $\psi(P) = (\psi_1(P_1), \psi_2(P_2), \dots, \psi_n(P_n))$, where $P_i \in M_i$, $\psi_i : M_i \rightarrow M_i$, $i \in \{1, 2, \dots, n\}$, and $P = (P_1, P_2, \dots, P_n)$ is ψ -absorbing primary. Then P_i is a ψ_i -absorbing primary element of M_i , for each i with $P_i \neq 1_{M_i}$.

Proof. Let $P_i \neq 1_{M_i}$, $N_i \in M_i$, and $x_1, x_2, \dots, x_n \in L_i$ be such that $x_1x_2 \cdots x_nN_i \leq P_i$ and $x_1x_2 \cdots x_nN_i \not\leq \psi_i(P_i)$. Thus $(1, \dots, 1, x_1, 1, \dots, 1)(1, \dots, 1, x_2, 1, \dots, 1) \cdots (1, \dots, 1, x_n, 1, \dots, 1)(0, \dots, 0, N_i, 0, \dots, 0) = (0, \dots, 0, x_1x_2 \cdots x_nN_i, 0, \dots, 0) \leq P$, and $(0, \dots, 0, x_1x_2 \cdots x_nN_i, 0, \dots, 0) \not\leq \psi(P)$. As P is ψ -absorbing primary, $(1, 1, \dots, x_1x_2 \cdots x_n, 0, \dots, 0) \leq (P : 1_M)$ or $(1, 1, \dots, x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_nN_i, 0, \dots, 0) \leq \sqrt[n]{P}$. Now, by Lemmas 11 and 12, we have $x_1x_2 \cdots x_n \leq (P_i : 1_{M_i})$ or $x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_nN_i \leq \sqrt[n]{P_i}$ for $i \in \{1, 2, \dots, n\}$ and consequently, P_i is ψ_i -absorbing primary element of M_i , for each i . $\square$

Corollary 16. Let $M = M_1 \times M_2 \times \cdots \times M_n$ and $L = L_1 \times L_2 \times \cdots \times L_n$, where each M_i is a compactly generated lattice module over a C -lattice L_i , for $i \in \{1, 2, \dots, n\}$. If $P = (P_1, P_2, \dots, P_n) \in M$ is ϕ_m -absorbing primary, where $P_i \in M_i$, then $P_i \in M_i$ is ϕ_m -absorbing primary with $P_i \neq 1_{M_i}$ ($n, m \geq 2$).

Proof. We have $\phi_m(P) = (P : 1_M)^{m-1}P = ((P_1 : 1_{M_1})^{m-1}P_1, (P_2 : 1_{M_2})^{m-1}P_2, \dots, (P_n : 1_{M_n})^{m-1}P_n) = (\phi_m(P_1), \phi_m(P_2), \dots, \phi_m(P_n))$ and the result follows from Theorem 15. $\square$

Theorem 17. Let M be a lattice module over a C -lattice L . Suppose that $a1_M \in M$ is a weak join principal element with $a1_M \neq 1_M$ and $(0_M : a1_M) \leq \sqrt{a}$. Then $a1_M$ is m -almost n -absorbing primary, $m \geq 1$, if and only if it is n -absorbing primary.

Proof. Suppose that $a1_M \in M$ is m -almost n -absorbing primary and $x_1x_2 \cdots x_nN \leq a1_M$, for $x_1, x_2, \dots, x_n \in L$ and $N \in M$. If $x_1x_2 \cdots x_nN \not\leq (a1_M : 1_M)^m a1_M$, then $x_1x_2 \cdots x_n \leq (a1_M : 1_M)$ or $x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_nN \leq \sqrt[n]{a1_M}$ for some $i \in \{1, 2, \dots, n\}$ and therefore $a1_M$ is n -absorbing primary.

Now, suppose that $x_1x_2 \cdots x_nN \leq (a1_M : 1_M)^m a1_M$. Since $x_1x_2 \cdots x_nN \leq a1_M$, we have $(x_1 \vee a)x_2 \cdots x_nN \leq a1_M$.

If $(x_1 \vee a)x_2 \cdots x_nN \not\leq (a1_M : 1_M)^m a1_M$, then it follows from the fact $(x_1 \vee a)x_2 \cdots x_nN \leq a1_M$, $(x_1 \vee a)x_2 \cdots x_nN \not\leq (a1_M : 1_M)^m a1_M$, and $a1_M$ is m -almost n -absorbing primary that $x_1x_2 \cdots x_n \leq (a1_M : 1_M)$ or $x_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_nN \leq \sqrt[n]{a1_M}$ for some $i \in \{1, 2, \dots, n\}$ and we are done.

So assume that $(x_1 \vee a)x_2 \cdots x_nN \leq (a1_M : 1_M)^m a1_M$. Then as $x_1x_2 \cdots x_nN \leq (a1_M : 1_M)^n a1_M$, we have $ax_2 \cdots x_nN \leq (a1_M : 1_M)^n a1_M$. Next, $ax_2 \cdots x_nN \leq (a1_M : 1_M)^n a1_M$ and $a1_M$ is weak join principal; together they imply that $x_2 \cdots x_n \leq ((a1_M : 1_M)^n a1_M : a1_M) = (a1_M : 1_M)^n \vee (0_M : a1_M) \leq a \vee (0_M : a1_M) \leq \sqrt{a}$. Consequently, $x_2 \cdots x_nN \leq \sqrt{a}N \leq \sqrt{a}a1_M = \sqrt[n]{a1_M}$ which implies that $a1_M$ is n -absorbing primary.

The converse follows from Theorem 8(2). $\square$

Note. The results pertaining to ϕ -absorbing elements are essentially the corollaries to the respective results of ϕ -absorbing primary elements, as such the results of ϕ -absorbing elements are the immediate consequences of results of ϕ -absorbing primary elements in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research work is an outcome of the project supported by Board of College and University Development, Savitribai Phule Pune University, Pune.

References

- [1] F. Çallıalp, C. Jayaram, and Ü. Tekir, "Weakly prime elements in multiplicative lattices," *Communications in Algebra*, vol. 40, no. 8, pp. 2825–2840, 2012.
- [2] S. B. Ballal, M. T. Gophane, and V. S. Kharat, "On weakly primary elements in multiplicative lattices," *Southeast Asian Bulletin of Mathematics*. In press.

- [3] C. Jayaram, Ü. Tekir, and E. Yetkin, “2-absorbing and weakly 2-absorbing elements in multiplicative lattices,” *Communications in Algebra*, vol. 42, no. 6, pp. 2338–2353, 2014.
- [4] V. V. Joshi and S. B. Ballal, “A note on n -Baer multiplicative lattices,” *Southeast Asian Bulletin of Mathematics*, vol. 39, pp. 67–76, 2015.
- [5] S. B. Ballal and V. S. Kharat, “On generalization of prime, weakly prime and almost prime elements in multiplicative lattices,” *International Journal of Algebra*, vol. 8, no. 9, pp. 439–449, 2014.
- [6] C. S. Manjarekar and A. V. Bingi, “ $\emptyset$ -prime and $\emptyset$ -primary elements in multiplicative lattices,” *Algebra*, vol. 2014, Article ID 890312, 7 pages, 2014.
- [7] A. Badawi, Ü. Tekir, and E. Yetkin, “On 2-absorbing primary ideals in commutative rings,” *Bulletin of the Korean Mathematical Society*, vol. 51, no. 4, pp. 1163–1173, 2014.
- [8] E. A. Al-Khouja, “Maximal elements and prime elements in lattice modules,” *Damascus University for Basic Sciences*, vol. 19, pp. 9–20, 2003.
- [9] F. Callialp and Ü. Tekir, “Multiplication lattice modules,” *Iranian Journal of Science and Technology*, vol. 35, no. 4, pp. 309–313, 2011.
- [10] E. W. Johnson and J. A. Johnson, “Lattice modules over principal element domains,” *Communications in Algebra*, vol. 31, no. 7, pp. 3505–3518, 2003.
- [11] H. M. Nakkar and D. D. Anderson, “Associated and weakly associated prime elements and primary decomposition in lattice modules,” *Algebra Universalis*, vol. 25, no. 2, pp. 196–209, 1988.
- [12] F. Alarcon, D. D. Anderson, and C. Jayaram, “Some results on abstract commutative ideal theory,” *Periodica Mathematica Hungarica*, vol. 30, no. 1, pp. 1–26, 1995.
- [13] D. D. Anderson, “Abstract commutative ideal theory without chain condition,” *Algebra Universalis*, vol. 6, no. 2, pp. 131–145, 1976.
- [14] R. P. Dilworth, “Abstract commutative ideal theory,” *Pacific Journal of Mathematics*, vol. 12, pp. 481–498, 1962.