Research Article

Asymptotic Stabilizability of a Class of Stochastic Nonlinear Hybrid Systems

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The problem of the asymptotic stabilizability in probability of a class of stochastic nonlinear control hybrid systems (with a linear dependence of the control) with state dependent, Markovian, and any switching rule is considered in the paper. To solve the issue, the Lyapunov technique, including a common, single, and multiple Lyapunov function, the hybrid control theory, and some results for stochastic nonhybrid systems are used. Sufficient conditions for the asymptotic stabilizability in probability for a considered class of hybrid systems are formulated. Also the stabilizing control in a feedback form is considered. Furthermore, in the case of hybrid systems with the state dependent switching rule, a method for a construction of stabilizing switching rules is proposed. Obtained results are illustrated by examples and numerical simulations.

1. Introduction

In recent years there has been a growing interest in hybrid systems. At the same time one of the basic problems in the control theory is the problem of the stability and the stabilizability of dynamic systems. For this reason, in the paper we deal with the stabilizability problem of nonlinear stochastic hybrid systems.

It is known that, if a common Lyapunov function exists, then the hybrid system is stable for any switching [1, 2]. In the absence of the common Lyapunov function, stability properties of the hybrid system in general depend on the switching signal, and in this case the hybrid system is not stable for any switching rules, but only for the so-called stabilizing switching rules [1]. In this case more general single Lyapunov, multiple Lyapunov, and single Lyapunov-like functions have been introduced [1–5].

Stability problem for stochastic hybrid systems with any switching is considered in [2], for stochastic hybrid systems with state dependent switching in [6–8], and for stochastic hybrid systems with Markovian switching in [8–10]. Stabilization problem for hybrid systems is considered for Markovian switching rule in [11–14] and for state dependent switching rule in [4]. For more details concerning the stability

and stabilization problem of hybrid systems, the author refers the reader to [15] and its references.

In the present paper ideas of a feedback control, proposed by Florchinger for nonhybrid stochastic nonlinear control systems [16], are used and combined with the concept of the hybrid control theory [1, 3, 6, 10] to derive results for the asymptotic stabilizability of stochastic nonlinear control hybrid systems (with a linear dependence of the control) described by Itô stochastic differential equations with any, state dependent, and Markovian switching rule. It is assumed that the trivial solution of unforced hybrid system is stable in probability (wherein some of subsystems of unforced hybrid systems can be unstable). By applying the obtained control the trivial solution of hybrid system becomes asymptotically stable in probability. To find sufficient conditions for the asymptotic stabilizability in probability of considered class of hybrid systems and to determine the control in the feedback form the Lyapunov function technique including common, single, and multiple Lyapunov function is applied. In the case of state dependent switching rules author proposes also a design method for stabilizing switching rules, which are based on the knowledge of regions of decreasing a Lyapunov function along solution.

This paper is divided into three sections and is organized as follows. In the first section we recall some basic definitions related to the issue of stability of stochastic systems and some stability results of hybrid systems. In the second section we present main results regarding the asymptotic stabilizability of stochastic nonlinear hybrid systems with any, state dependent, and Markovian switching rule. Also we give two examples to illustrate obtained stabilizability criteria. In the third section we summarize the obtained results.

2. Mathematical Preliminaries

Throughout this paper we use the following notation. We mark $\mathbb{T} = [t_0, \infty)$, $t_0 \geq 0$. Let $\Xi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying usual conditions. Let $(\mathbf{w}(t))_{t \geq 0} = [w_1(t), \dots, w_M(t)]_{t \geq 0}^T$ be the m -dimensional standard Wiener process defined on the probability space Ξ . Let $\mathcal{S} = \{1, \dots, N\}$ be the set of states and let $\sigma : \mathbb{T} \rightarrow \mathcal{S}$ be the switching rule. We assume that processes $w_k(t)$ and $\sigma(t)$ are both $\{\mathcal{F}_t\}_{t \geq 0}$ adapted.

Let us consider a stochastic nonlinear hybrid system described by vector Itô differential equations as follows:

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \sigma(t)) dt + \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t), \sigma(t)) dw_k(t), \quad (1)$$

$$(\mathbf{x}(t_0), \sigma(t_0)) = (\mathbf{x}_0, \sigma_0),$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $(\mathbf{x}_0, \sigma_0) \in \mathbb{R}^n \times \mathcal{S}$ is an initial condition, and $t \in \mathbb{T}$. Functions $\mathbf{f} : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{f} = [f^1, f^2, \dots, f^n]^T$, and $\mathbf{g}_k : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{g}_k = [g_k^1, g_k^2, \dots, g_k^n]^T$, $k = 1, 2, \dots, M$, are continuous and locally Lipschitz and satisfy the following conditions:

$$\forall l \in \mathcal{S} \quad \mathbf{f}(\mathbf{0}, l) = \mathbf{g}_1(\mathbf{0}, l) = \mathbf{g}_2(\mathbf{0}, l) = \dots = \mathbf{g}_M(\mathbf{0}, l) = \mathbf{0}. \quad (2)$$

We consider three types of the switching rules σ : Markovian, any, and state dependent switching rule.

Markovian switching σ is given by a right-continuous Markov chain r defined on the probability space Ξ and taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with a generator $\Gamma = [\gamma_{ij}]_{N \times N}$, that is,

$$\mathbb{P}\{r(t + \delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases} \quad (3)$$

where $\delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. We assume that the Markov chain is irreducible, that is, $\text{rank}(\Gamma) = N - 1$, and has a unique stationary distribution $\mathcal{P} = [\pi_1, \pi_2, \dots, \pi_N]^T \in \mathbb{R}^N$ which can be determined by solving

$$\mathcal{P}\Gamma = \mathbf{0} \quad (4)$$

$$\text{subject to } \sum_{i=1}^N \pi_i = 1, \quad \pi_i > 0 \quad \forall i \in \mathcal{S}.$$

In the case of Markovian switching rule we mean the switching rule with fixed and given matrix Γ . To ensure the existence of the solution of the hybrid system (1) in the case of any switching rule we assume in addition that it is any Markovian switching rule that is with any matrix Γ . In the further part of this paper we use the formulation “any switching rule,” when we mean that “any Markovian switching rule.” In the case of the state dependent switching rule we will assume that the switching times are stopping times.

We use the following definitions.

Definition 1. The null solution $\mathbf{x} \equiv \mathbf{0}$ of the stochastic differential equation (1) is

$$(1) \text{ stable in probability if } \forall \epsilon > 0 \quad \forall r > 0 \quad \exists \delta(\epsilon) > 0 \quad |\mathbf{x}_0| < \delta \implies \forall t \geq t_0 \quad P(|\mathbf{x}(t, \mathbf{x}_0, t_0)| > r) < \epsilon, \quad (5)$$

(2) asymptotically stable in probability if it is stable in probability and

$$\forall (\mathbf{x}_0, t_0) \in \mathbb{R}^n \times [0, \infty) \quad P\left(\lim_{t \rightarrow \infty} |\mathbf{x}(t, \mathbf{x}_0, t_0)| = 0\right) = 1. \quad (6)$$

Definition 2. We say that a continuous function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function if

- (i) $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$;
- (ii) $\mathcal{V}(\mathbf{0}) = 0$ and $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \quad \mathcal{V}(\mathbf{x}) > 0$;
- (iii) $\mathcal{V}$ is proper, that is, $\lim_{|\mathbf{x}| \rightarrow \infty} \mathcal{V}(\mathbf{x}) = \infty$.

For any twice differentiable function $\phi(\mathbf{x})$ (i.e., $\phi(\mathbf{x}) \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$) the l th subsystem of the hybrid system (1) has a generator $\mathcal{L}_l$ acting on $\phi(\mathbf{x})$ (the Itô operator for the l th subsystem of system (1)) defined for every subsystem by

$$\mathcal{L}_l \phi(\mathbf{x}) = \sum_{\mu=1}^n f^\mu(\mathbf{x}, l) \frac{\partial \phi(\mathbf{x})}{\partial x_\mu} + \frac{1}{2} \sum_{r,s=1}^n \sum_{k=1}^M g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l) \frac{\partial^2 \phi(\mathbf{x})}{\partial x_r \partial x_s}, \quad (7)$$

$$l \in \mathcal{S}.$$

The hybrid system $(\mathbf{x}(t), \sigma(t))$ for $\sigma(t) = r(t)$ has a generator $\mathcal{L}_l$, $l \in \mathcal{S}$, acting on any twice differentiable function $\phi_l \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$ in the following way:

$$\mathcal{L}_l \phi_l(\mathbf{x}) = \sum_{\mu=1}^n f^\mu(\mathbf{x}, l) \frac{\partial \phi_l(\mathbf{x})}{\partial x_\mu} + \frac{1}{2} \sum_{r,s=1}^n \sum_{k=1}^M g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l) \frac{\partial^2 \phi_l(\mathbf{x})}{\partial x_r \partial x_s} + \sum_{j=1}^N \gamma_{lj} \phi_j(\mathbf{x}). \quad (8)$$

Sufficient conditions for the asymptotic stability in probability of the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system (1) with Markovian switching rule r in terms of the Lyapunov function are given by the following theorem.

Theorem 3 (see [10]). *Let $D \subset \mathbb{R}^n$ be an open neighborhood of $\mathbf{0}$. Suppose that for each $l \in \mathcal{S}$ there exists a Lyapunov function $\mathcal{V}_l : D \rightarrow \mathbb{R}$ such that*

$$\forall l \in \mathcal{S} \quad \forall \mathbf{x} \in D \setminus \{\mathbf{0}\} \quad \widetilde{\mathcal{L}}_l \mathcal{V}_l(\mathbf{x}) \leq 0 (< 0). \quad (9)$$

Then the null solution $\mathbf{x} \equiv \mathbf{0}$ is (asymptotically) stable in probability.

Definition 4. A Lyapunov function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\forall l \in \mathcal{S} \quad \forall \mathbf{x} \neq \mathbf{0} \quad \mathcal{L}_l \mathcal{V}(\mathbf{x}) < 0 \quad (10)$$

is called a common Lyapunov function for the hybrid system (1).

Using Theorem 3 it is easy to conclude the following.

Fact 1. Note that it is a known fact that if there exists a common Lyapunov function for the hybrid system (1), then the null solution of (1) is asymptotically stable for any Markovian switching rule.

Definition 5. A Lyapunov function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\forall \mathbf{x} \neq \mathbf{0} \quad \mathcal{L}_{\sigma_{st}} \mathcal{V}(\mathbf{x}) < 0 \quad (11)$$

for a switching rule σ_{st} is called a single Lyapunov function for the hybrid system (1).

Fact 2 (see [6]). Note that it is a known fact that if there exists a single Lyapunov function for the hybrid system (1) for a switching rule σ_{st} , then the null solution of (1) is asymptotically stable for the switching rule σ_{st} . Such switching rules are called stabilizing switching rules.

Definition 6. A family of Lyapunov functions $\mathcal{V}_l : \mathbb{R}^n \rightarrow \mathbb{R}$, $l \in \mathcal{S}$ satisfying

$$\forall l \in \mathcal{S} \quad \forall \mathbf{x} \neq \mathbf{0} \quad \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) < 0 \quad (12)$$

is called a multiple Lyapunov function for the hybrid system (1).

Since we deal only with the stability problem of the null solution in the further part of this paper we use the formulation “system is stable,” when we mean that “the null solution of system is stable.”

3. Stabilizability of Hybrid Systems

The aim of this paper is to establish sufficient conditions for the asymptotic stabilizability in probability for the stochastic

nonlinear control hybrid system given by vector Itô differential equations as follows:

$$\begin{aligned} d\mathbf{x}(t) &= \left[\mathbf{f}_0(\mathbf{x}(t), \sigma(t)) + \sum_{i=1}^m u_i(\mathbf{x}(t), \sigma(t)) \mathbf{f}_i(\mathbf{x}(t), \sigma(t)) \right] dt \\ &\quad + \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t), \sigma(t)) dw_k(t), \\ (\mathbf{x}(t_0), \sigma(t_0)) &= (\mathbf{x}_0, \sigma_0), \end{aligned} \quad (13)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the control vector, $(\mathbf{x}_0, \sigma_0) \in \mathbb{R}^n \times \mathcal{S}$ is an initial condition, and $t \in \mathbb{T}$. Functions $\mathbf{f}_i : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{f}_i = [f_i^1, \dots, f_i^n]^T$, and $\mathbf{g}_k : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{g}_k = [g_k^1, \dots, g_k^n]^T$, $i = 0, 1, \dots, m$, $k = 1, 2, \dots, M$, are continuous and Lipschitz and satisfy the following conditions:

$$\begin{aligned} \forall l \in \mathcal{S} \quad \mathbf{f}_0(\mathbf{0}, l) &= \mathbf{f}_1(\mathbf{0}, l) = \dots = \mathbf{f}_m(\mathbf{0}, l) \\ &= \mathbf{g}_1(\mathbf{0}, l) = \mathbf{g}_2(\mathbf{0}, l) = \dots = \mathbf{g}_M(\mathbf{0}, l) = \mathbf{0}. \end{aligned} \quad (14)$$

We use the following definition of the stabilizability.

Definition 7. The hybrid system (13) is said to be asymptotically stabilizable in probability, if there exists a switching signal $\sigma : \mathbb{T} \rightarrow \mathcal{S}$ and the associated feedback control law $\mathbf{u} : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$, such that the hybrid system (13) is asymptotically stable in probability.

Let us consider nonhybrid stochastic control nonlinear systems given by [16]:

$$\begin{aligned} d\mathbf{x}(t) &= \left[\mathbf{f}_0(\mathbf{x}(t)) + \sum_{i=1}^m u_i(\mathbf{x}(t)) \mathbf{f}_i(\mathbf{x}(t)) \right] dt \\ &\quad + \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t)) dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \end{aligned} \quad (15)$$

We introduce the following notation. The second order differential operator $\mathcal{L}$ (infinitesimal generator for the unforced stochastic differential system deduced from (15)) acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\mathcal{L}\phi(\mathbf{x}) = \sum_{\mu=1}^n f_0^\mu(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial x_\mu} + \frac{1}{2} \sum_{r,s=1}^n \sum_{k=1}^M g_k^r(\mathbf{x}) g_k^s(\mathbf{x}) \frac{\partial^2 \phi(\mathbf{x})}{\partial x_r \partial x_s} \quad (16)$$

and associated with this operator set $\mathbf{W}$

$$\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}\phi(\mathbf{x}) = 0\}. \quad (17)$$

The differential operator $\mathcal{L}^i$ acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\mathcal{L}^i \phi(\mathbf{x}) = \sum_{\mu=1}^n f_i^\mu(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial x_\mu}, \quad i = 1, 2, \dots, m \quad (18)$$

and associated with this operator set $\mathbf{W}^u$

$$\mathbf{W}^u = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}^i \phi(\mathbf{x}) = \mathbf{0} \ \forall i = 1, \dots, m\}. \quad (19)$$

Then the following corollary can be formulated from theorem given by Florchinger in [16].

Corollary 8. Assume that there exists a Lyapunov function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $\forall \mathbf{x} \in \mathbb{R}^n \ \mathcal{L}\mathcal{V}(\mathbf{x}) \leq 0$,
- (ii) $\mathbf{W} \cap \mathbf{W}^u = \{\mathbf{0}\}$.

Then, the control law $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as follows:

$$u_i(\mathbf{x}) = -\mathcal{L}^i \mathcal{V}(\mathbf{x}), \quad i = 1, 2, \dots, m, \quad (20)$$

stabilizes asymptotically in probability the stochastic control system (15).

3.1. Hybrid Systems with Markovian Switching Rule. Using Theorem of Khasminskii et al. [10] we can also formulate sufficient conditions for the asymptotic stabilizability in probability for a stochastic nonlinear control hybrid system with Markovian switching rule described by vector Itô differential equations as follows:

$$\begin{aligned} d\mathbf{x}(t) &= \left[\mathbf{f}_0(\mathbf{x}(t), \sigma(t)) + \sum_{i=1}^m u_i(\mathbf{x}(t), \sigma(t)) \mathbf{f}_i(\mathbf{x}(t), \sigma(t)) \right] dt \\ &+ \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t), \sigma(t)) dw_k(t), \\ (\mathbf{x}(t_0), \sigma(t_0)) &= (\mathbf{x}_0, \sigma_0), \end{aligned} \quad (21)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the control vector, $(\mathbf{x}_0, \sigma_0) \in \mathbb{R}^n \times \mathcal{S}$ is an initial condition, and $t \in \mathbb{T}$. Switching rule $\sigma(t)$ is given by Markovian switching rule $r(t)$ described by (3). Functions $\mathbf{f}_i : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{f}_i = [f_i^1, \dots, f_i^n]^T$, and $\mathbf{g}_k : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{g}_k = [g_k^1, \dots, g_k^n]^T$, $i = 0, 1, \dots, m$, $k = 1, 2, \dots, M$, are continuous and Lipschitz and satisfy conditions (14).

Following the idea of Florchinger [16] we introduce the following notation. The second order differential operator $\mathcal{L}_l$ (infinitesimal generator for the l th subsystem of the unforced stochastic differential hybrid system with Markovian switching rule deduced from (21)) acting on any function $\phi_l \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\begin{aligned} \mathcal{L}_l \phi_l(\mathbf{x}) &= \sum_{\mu=1}^n f_0^\mu(\mathbf{x}, l) \frac{\partial \phi_l(\mathbf{x})}{\partial x_\mu} \\ &+ \frac{1}{2} \sum_{r,s=1}^n \sum_{k=1}^M g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l) \frac{\partial^2 \phi_l(\mathbf{x})}{\partial x_r \partial x_s} \\ &+ \sum_{j=1}^N \gamma_{lj} \phi_j(\mathbf{x}), \quad l \in \mathcal{S} \end{aligned} \quad (22)$$

and associated with this operator set $\widetilde{\mathbf{W}}_l$

$$\widetilde{\mathbf{W}}_l = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}_l \phi_l(\mathbf{x}) = 0\}, \quad l \in \mathcal{S}. \quad (23)$$

The differential operator $\mathcal{L}_l^i$ acting on any function $\phi_l \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\mathcal{L}_l^i \phi_l(\mathbf{x}) = \sum_{\mu=1}^n f_i^\mu(\mathbf{x}, l) \frac{\partial \phi_l(\mathbf{x})}{\partial x_\mu}, \quad i = 1, 2, \dots, m, \quad l \in \mathcal{S} \quad (24)$$

and associated with this operator set $\widetilde{\mathbf{W}}_l^u$

$$\widetilde{\mathbf{W}}_l^u = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}_l^i \phi_l(\mathbf{x}) = 0 \ \forall i = 1, \dots, m\}, \quad l \in \mathcal{S}, \quad (25)$$

where the superindex u indicates the influence of the control coefficients $\mathbf{f}_i$.

The following theorem gives sufficient conditions for the asymptotic stabilizability in probability of the stochastic hybrid system (21).

Theorem 9. Assume that there exist Lyapunov functions $\mathcal{V}_l : \mathbb{R}^n \rightarrow \mathbb{R}$, $l \in \mathcal{S}$, such that

- (i) $\forall l \in \mathcal{S} \ \forall \mathbf{x} \in \mathbb{R}^n \ \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) \leq 0$,
- (ii) $\forall l \in \mathcal{S} \ \widetilde{\mathbf{W}}_l \cap \widetilde{\mathbf{W}}_l^u = \mathbf{0}$.

Then, the control law $\mathbf{u} : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$ defined as follows:

$$u_i(\mathbf{x}, l) = -\mathcal{L}_l^i \mathcal{V}_l(\mathbf{x}), \quad i = 1, 2, \dots, m, \quad l \in \mathcal{S}, \quad (26)$$

stabilizes asymptotically in probability the stochastic control hybrid system (21).

Proof. Using Assumption (i) we can conclude that the infinitesimal generator $\mathcal{L}_l$ for the l th subsystem of the hybrid system (21) can be estimated as follows:

$$\begin{aligned} \forall l \in \mathcal{S} \quad \forall \mathbf{x} \in \mathbb{R}^n \\ \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) = \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) - \sum_{i=1}^m (\mathcal{L}_l^i \mathcal{V}_l(\mathbf{x}))^2 \leq 0, \end{aligned} \quad (27)$$

$$\mathcal{L}_l \mathcal{V}_l(\mathbf{x}) = 0 \iff \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) = \sum_{i=1}^m (\mathcal{L}_l^i \mathcal{V}_l(\mathbf{x}))^2 = 0.$$

Hence we can conclude that

$$\begin{aligned} \forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \widetilde{\mathbf{W}}_l \cap \widetilde{\mathbf{W}}_l^u, \\ \forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^n \setminus (\widetilde{\mathbf{W}}_l \cap \widetilde{\mathbf{W}}_l^u) \end{aligned} \quad (28)$$

and since condition (ii) is satisfied we obtain

$$\forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}_l(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \quad (29)$$

Now the thesis follows from Theorem 3. $\square$

The family of Lyapunov functions satisfying (29) is called the multiple Lyapunov function for the hybrid system (21).

3.2. Hybrid Systems with Any Switching Rule. Let us consider a stochastic nonlinear control hybrid system with any switching rule described by the following vector Itô differential equations:

$$\begin{aligned} d\mathbf{x}(t) &= \left[\mathbf{f}_0(\mathbf{x}(t), \sigma(t)) + \sum_{i=1}^m u_i(\mathbf{x}(t), \sigma(t)) \mathbf{f}_i(\mathbf{x}(t), \sigma(t)) \right] dt \\ &\quad + \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t), \sigma(t)) dw_k(t), \\ (\mathbf{x}(t_0), \sigma(t_0)) &= (\mathbf{x}_0, \sigma_0), \end{aligned} \quad (30)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the control vector, $(\mathbf{x}_0, \sigma_0) \in \mathbb{R}^n \times \mathcal{S}$ is an initial condition, and $t \in \mathbb{T}$. Switching rule σ is any Markovian switching rule. Functions $\mathbf{f}_i : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{f}_i = [f_i^1, \dots, f_i^n]^T$, and $\mathbf{g}_k : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{g}_k = [g_k^1, \dots, g_k^n]^T$, $i = 0, 1, \dots, m$, $k = 1, 2, \dots, M$, are continuous and Lipschitz and satisfy conditions (14).

Following the idea of Florchinger [16] we introduce the notation. The second order differential operator $\mathcal{L}_l$ (infinitesimal generator for the l th subsystem of the unforced stochastic differential hybrid system with any switching rule deduced from (30)) acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\begin{aligned} \mathcal{L}_l \phi(\mathbf{x}) &= \sum_{\mu=1}^n f_0^\mu(\mathbf{x}, l) \frac{\partial \phi(\mathbf{x})}{\partial x_\mu} \\ &\quad + \frac{1}{2} \sum_{r,s=1}^n \sum_{k=1}^M g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l) \frac{\partial^2 \phi(\mathbf{x})}{\partial x_r \partial x_s}, \quad l \in \mathcal{S} \end{aligned} \quad (31)$$

and associated with this operator set $\mathbf{W}_l$

$$\mathbf{W}_l = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}_l \phi(\mathbf{x}) = 0\}, \quad l \in \mathcal{S}. \quad (32)$$

The differential operator $\mathcal{L}_l^i$ acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$

$$\mathcal{L}_l^i \phi(\mathbf{x}) = \sum_{\mu=1}^n f_i^\mu(\mathbf{x}, l) \frac{\partial \phi(\mathbf{x})}{\partial x_\mu}, \quad i = 1, 2, \dots, m, \quad l \in \mathcal{S} \quad (33)$$

and associated with this operator set $\mathbf{W}_l^u$

$$\mathbf{W}_l^u = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}_l^i \phi(\mathbf{x}) = 0 \quad \forall i = 1, \dots, m\}, \quad l \in \mathcal{S}. \quad (34)$$

Using this notation we formulate the next theorem that gives sufficient conditions for the asymptotic stabilizability in probability of the stochastic hybrid system (30).

Theorem 10. Assume that there exists a Lyapunov function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $\forall l \in \mathcal{S} \quad \forall \mathbf{x} \in \mathbb{R}^n \quad \mathcal{L}_l \mathcal{V}(\mathbf{x}) \leq 0$
- (ii) $\forall l \in \mathcal{S} \quad \mathbf{W}_l \cap \mathbf{W}_l^u = \{\mathbf{0}\}.$

Then, the control law $\mathbf{u} : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$ defined as follows:

$$u_i(\mathbf{x}, l) = -\mathcal{L}_l^i \mathcal{V}(\mathbf{x}), \quad i = 1, 2, \dots, m, \quad l \in \mathcal{S}, \quad (35)$$

stabilizes asymptotically in probability the stochastic control hybrid system (30).

Proof. Using Assumption (i) we can conclude that the infinitesimal generator $\mathcal{L}_l$ for the l th subsystem of the hybrid system (30) can be estimated as follows:

$$\begin{aligned} \forall l \in \mathcal{S} \quad \forall \mathbf{x} \in \mathbb{R}^n \\ \mathcal{L}_l \mathcal{V}(\mathbf{x}) = \mathcal{L}_l \mathcal{V}(\mathbf{x}) - \sum_{i=1}^m (\mathcal{L}_l^i \mathcal{V}(\mathbf{x}))^2 \leq 0, \end{aligned} \quad (36)$$

$$\mathcal{L}_l \mathcal{V}(\mathbf{x}) = 0 \iff \mathcal{L}_l \mathcal{V}(\mathbf{x}) = \sum_{i=1}^m (\mathcal{L}_l^i \mathcal{V}(\mathbf{x}))^2 = 0.$$

Hence we can conclude that

$$\begin{aligned} \forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \mathbf{W}_l \cap \mathbf{W}_l^u, \\ \forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^n \setminus (\mathbf{W}_l \cap \mathbf{W}_l^u) \end{aligned} \quad (37)$$

and since condition (ii) is satisfied we obtain

$$\forall l \in \mathcal{S} \quad \mathcal{L}_l \mathcal{V}(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \quad (38)$$

By means of equation (38) one can conclude that function $\mathcal{V}(\mathbf{x})$ is the common Lyapunov function for the hybrid system (30) and hence from Fact 1 it follows that the hybrid system (30) is asymptotically stable in probability under the control law given by (35). $\square$

Remark 11. Note that from condition (38) and assumption (iii) of Definition 2 it follows that every subsystem of hybrid system (30) with control $\mathbf{u}$ is globally asymptotically stable in probability (see [17]) and hence the set of initial conditions from which stability of hybrid system (30) can be achieved (the so-called null controllable region) is the whole $\mathbb{R}^n$.

Example 12 (stabilizability of hybrid systems with any switching rule). Let us consider a particular case of hybrid system (30)

$$\begin{aligned} d\mathbf{x}(t) &= \begin{bmatrix} -0.5 \\ -0.5 \text{sat}^2(x_1) \end{bmatrix} \mathbf{x}(t) dt + u_1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix} dt \\ &\quad + u_2 \begin{bmatrix} 0 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} x_2 \text{sat}(x_1) \\ x_1 \end{bmatrix} dw(t), \quad \text{for } \sigma = 1, \end{aligned} \quad (39)$$

$$\begin{aligned} d\mathbf{x}(t) &= \begin{bmatrix} -0.02 \text{sat}^2(x_2) \\ -0.02 \end{bmatrix} \mathbf{x}(t) dt + u_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt \\ &\quad + \begin{bmatrix} 0.2x_2 \\ 0.2x_1 \text{sat}(x_2) \end{bmatrix} dw(t), \quad \text{for } \sigma = 2, \end{aligned} \quad (40)$$

$$(\mathbf{x}(0), \sigma(0)) = (\mathbf{x}_0, \sigma_0),$$

FIGURE 1: An exemplary stable path of system (39) without control u .

where $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$, $\mathbf{u} = [u_1, u_2]^T \in \mathbb{R}^2$, $\mathcal{S} = \{1, 2\}$, and function $\text{sat} : \mathbb{R} \rightarrow \mathbb{R}$ is given as follows:

$$\text{sat}(y) = \begin{cases} -1 & y < -1 \\ y & y \in [-1, 1] \\ 1 & y > 1. \end{cases} \quad (41)$$

We look for the feedback $\mathbf{u} : \mathbb{R}^2 \times \mathcal{S} \rightarrow \mathbb{R}^2$ under which the hybrid system (39) is asymptotically stable in probability for any switching. Let us choose the Lyapunov function as follows:

$$\mathcal{V}(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{x}. \quad (42)$$

We obtain

$$\begin{aligned} \mathcal{L}_1 V(\mathbf{x}) &= 0, & \mathbf{W}_1 &= \mathbb{R}^2, & \mathcal{L}_1^1 V(\mathbf{x}) &= x_1^2, \\ \mathcal{L}_1^2 V(\mathbf{x}) &= x_2^2, & \mathbf{W}_1^u &= \{(0, 0)\}, \\ \mathcal{L}_2 V(\mathbf{x}) &= 0, & \mathbf{W}_2 &= \mathbb{R}^2, & \mathcal{L}_2^1 V(\mathbf{x}) &= x_1^2 + x_2^2, \\ \mathcal{L}_2^2 V(\mathbf{x}) &= 0, & \mathbf{W}_2^u &= \{(0, 0)\}. \end{aligned} \quad (43)$$

We note that $\mathbf{W}_l \cap \mathbf{W}_l^u = \{(0, 0)\}$ for $l \in \mathcal{S} = \{1, 2\}$ and hence from Theorem 10 it follows that the hybrid system (39) is asymptotically stable in probability for any switching under the feedback control given as follows:

$$\mathbf{u}(\mathbf{x}, 1) = [-x_1^2, -x_2^2], \quad \mathbf{u}(\mathbf{x}, 2) = [-x_1^2 - x_2^2, 0]. \quad (44)$$

Note that function $\mathcal{V}(\mathbf{x}) = (1/2)\mathbf{x}^T \mathbf{x}$ is a common Lyapunov function for the hybrid system (39).

An exemplary simulation of a trajectory of the hybrid system (39) is shown in Figures 1 and 2.

3.3. Hybrid Systems with State Dependent Switching Rule. In this section we consider a stochastic nonlinear control hybrid

FIGURE 2: An exemplary asymptotically stable path of system (39) with control u given by (44).

system with state dependent switching rule described by vector Itô differential equations as follows:

$$\begin{aligned} d\mathbf{x}(t) &= \left[\mathbf{f}_0(\mathbf{x}(t), \sigma(t)) + \sum_{i=1}^m u_i(\mathbf{x}(t), \sigma(t)) \mathbf{f}_i(\mathbf{x}(t), \sigma(t)) \right] dt \\ &\quad + \sum_{k=1}^M \mathbf{g}_k(\mathbf{x}(t), \sigma(t)) dw_k(t), \\ (\mathbf{x}(t_0), \sigma(t_0)) &= (\mathbf{x}_0, \sigma_0), \end{aligned} \quad (45)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the control vector, $(\mathbf{x}_0, \sigma_0) \in \mathbb{R}^n \times \mathcal{S}$ is an initial condition, and $t \in \mathbb{T}$. Functions $\mathbf{f}_i : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{f}_i = [f_i^1, \dots, f_i^n]^T$, and $\mathbf{g}_k : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$, $\mathbf{g}_k = [g_k^1, \dots, g_k^n]^T$, $i = 0, 1, \dots, m$, $k = 1, 2, \dots, M$, are continuous and Lipschitz and satisfy conditions (14).

Following the methodology introduced in [6] we assume that

$$\sigma(t) = l_j, \quad t \in [\tau_j, \tau_{j+1}), \quad j = 0, \dots, j^*, \quad (46)$$

where $\tau_0 = t_0$, and $l_j \in \mathcal{S}$ with $l_j \neq l_{j+1}$ and j^* is the number of switches. Here $j^* \leq \infty$ and $\tau_{j^*+1} \leq \infty$. The preliminary assumption about switching is that switching instants τ_j are stopping times and the corresponding active system has a unique solution in the interval $[\tau_j, \tau_{j+1})$.

A set of regions $\{\Omega_l, l \in \mathcal{S}\}$ is called an active-region set of (45) if the l th subsystem is active on Ω_l and $\mathbb{R}^n \setminus \{\mathbf{0}\} \subset \bigcup_{l \in \mathcal{S}} \Omega_l$. Let $\text{int}(\Omega_l)$ denote the interior of Ω_l and we say $\{\text{int}(\Omega_l), l \in \mathcal{S}\}$ is an interior of $\{\Omega_l, l \in \mathcal{S}\}$. To ensure the existence of the solution of hybrid system (45) we assume also that [6].

Property A. The interior of $\{\Omega_l, l \in \mathcal{S}\}$ is still an active-region set of system (45), that is,

$$\mathbb{R}^n \setminus \{\mathbf{0}\} \subset \bigcup_{l \in \mathcal{S}} \text{int}(\Omega_l). \quad (47)$$

Let us denote the domain of the active-region set of system (45) by

$$\text{Dom} = \bigcup_{l \in \mathcal{S}} (\{l\} \times \text{int}(\Omega_l)). \quad (48)$$

We choose times of switches as follows [18]:

$$\tau_j = \inf \{t > t_0 : (\sigma(t), x(t)) \notin \text{Dom}\}, \quad j = 1, 2, \dots \quad (49)$$

This is well-defined stopping time for the diffusion process described by (45) [18]. We denote such special class of state dependent switching rules satisfying (46)–(49) by Σ . Note that a switching rule $\sigma \in \Sigma$ is a stochastic switching rule because of its dependence on stochastic process $\mathbf{x}(t)$. Our aim is to find a special partition Ω_l such that every switching rule $\sigma \in \Sigma$ is a stabilizing switching rule for considered class of stochastic hybrid systems.

Following the idea of Florchinger [16] we introduce the notation. The second order differential operator $\mathcal{L}_l$, $l \in \mathcal{S}$, (infinitesimal generator for the l th subsystem of the unforced stochastic differential hybrid system with state dependent switching rule deduced from (45)) acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$ is defined by (31). Sets $\mathbf{W}_l$ are given as follows:

$$\mathbf{W}_l = \{\mathbf{x} \in \Omega_l : \mathcal{L}_l \phi(\mathbf{x}) = 0\}, \quad l \in \mathcal{S}. \quad (50)$$

The differential operator $\mathcal{L}_l^i$, $l \in \mathcal{S}$, $i = 1, 2, \dots, m$, acting on any function $\phi \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$, is defined by (33). Set $\mathbf{W}_l^u$ is given as follows:

$$\mathbf{W}_l^u = \{\mathbf{x} \in \Omega_l : \mathcal{L}_l^i \phi(\mathbf{x}) = 0 \quad \forall i = 1, \dots, m\}, \quad l \in \mathcal{S}. \quad (51)$$

The following theorem gives sufficient conditions for the asymptotic stabilizability of the stochastic hybrid system (45).

Theorem 13. Assume that there exists a Lyapunov function $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $\bigcup_{l \in \mathcal{S}} \Omega_l = \mathbb{R}^n$, where $\Omega_l = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{L}_l \mathcal{V}(\mathbf{x}) \leq 0\}$,
- (ii) $\forall l \in \mathcal{S} \quad \mathbf{W}_l \cap \mathbf{W}_l^u = \{\mathbf{0}\}$.

Then, the control law $\mathbf{u} : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$ defined as follows:

$$u_i(\mathbf{x}, l) = -\mathcal{L}_l^i \mathcal{V}(\mathbf{x}), \quad i = 1, 2, \dots, m, \quad l \in \mathcal{S}, \quad (52)$$

stabilizes asymptotically in probability the stochastic control hybrid system (45).

Proof. Using Assumption (i) similarly as in the proof of Theorem 10 we can conclude that the infinitesimal generator $\mathcal{L}_l$ for the l th subsystem of the hybrid system (45) can be estimated as follows:

$$\mathcal{L}_l \mathcal{V}(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \Omega_l \setminus (\mathbf{W}_l \cap \mathbf{W}_l^u), \quad l \in \mathcal{S}. \quad (53)$$

And hence

$$\mathcal{L}_{\sigma_{st}(\mathbf{x})} \mathcal{V}(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \neq \mathbf{0}, \quad \sigma_{st} \in \Sigma. \quad (54)$$

By means of (54) we can conclude that function $\mathcal{V}(\mathbf{x})$ is the single Lyapunov function for the hybrid system (45) and hence from Fact 2 it follows that the hybrid system (45) is asymptotically stable in probability under the control law (52). $\square$

We note that in the assumptions of Theorems 9–13 condition (i) (different in each theorem) guarantees that the corresponding unforced hybrid system is stable in probability, but since sets $\widetilde{\mathbf{W}}_l$, $\mathbf{W}_l$ can contain some vectors $\mathbf{x} \neq \mathbf{0}$ the corresponding unforced system does not need to be asymptotically stable in probability.

Remark 14. Note that in general, the presence of input constraints inherently limits the set of initial conditions from which stability can be achieved (the so-called null controllable region) [19]. For considered hybrid systems with any switching the null controllable region is the whole $\mathbb{R}^n$. In the case of the hybrid system with Markovian and state-dependent switching rule the important problem of the characterization of the null-controllable region is not discussed in this paper.

Example 15 (stabilizability of hybrid systems with the state dependent switching rule). Let us consider a particular case of the hybrid system (45) which is a modification of the example considered by Florchinger [16]

$$d\mathbf{x}(t) = \begin{bmatrix} -0.8 & 1 \\ 0.0 & -0.8 \end{bmatrix} \mathbf{x}(t) dt + u \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} dt \quad (55)$$

$$+ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) dw(t), \quad \text{for } \sigma = 1,$$

$$d\mathbf{x}(t) = \begin{bmatrix} -0.5 & 0.0 \\ -0.8 & -0.5 \end{bmatrix} \mathbf{x}(t) dt + u \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt \quad (56)$$

$$+ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) dw(t), \quad \text{for } \sigma = 2,$$

$$(\mathbf{x}(0), \sigma(0)) = (\mathbf{x}_0, \sigma_0),$$

where $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$, $\mathbf{u} = u \in \mathbb{R}$, $\mathcal{S} = \{1, 2\}$.

We look for the stabilizing switching rule $\sigma_{st} \in \Sigma$ and the feedback control $\mathbf{u} : \mathbb{R}^2 \times \mathcal{S} \rightarrow \mathbb{R}$ under which the hybrid system (55) is asymptotically stable in probability. Let us choose a Lyapunov function of the form

$$\mathcal{V}(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{x}. \quad (57)$$

We obtain

$$\mathcal{L}_1 \mathcal{V}(\mathbf{x}) = x_1 x_2 - 0.3x_1^2 - 0.3x_2^2,$$

$$\Omega_1 = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1 x_2 - 0.3x_1^2 - 0.3x_2^2 = -0.3(x_2 - 3x_1) \left(x_2 - \frac{x_1}{3} \right) \leq 0 \right\},$$

$$\mathcal{L}_1^1 \mathcal{V}(\mathbf{x}) = x_1^2 - x_2^2,$$

$$\mathcal{L}_2 \mathcal{V}(\mathbf{x}) = -0.8x_1 x_2, \quad \Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1 x_2 \geq 0\},$$

$$\mathcal{L}_2^1 \mathcal{V}(\mathbf{x}) = x_1^2 + x_2^2. \quad (58)$$

FIGURE 3: Regions Ω_1 and Ω_2 with nonempty intersection and sets $W_l, W_l^u, l = 1, 2$, defined by (58)-(59).

From (58) it follows that the sets $W_l, W_l^u, l = 1, 2$, defined by (50) and (51), respectively, have the form

$$\begin{aligned} W_1 &= \{ \mathbf{x} = (x_1, x_2) \in \Omega_1 : x_2 = 3x_1 \vee x_2 = \frac{x_1}{3} \}, \\ W_2 &= \{ \mathbf{x} = (x_1, x_2) \in \Omega_2 : x_1 = 0 \vee x_2 = 0 \}, \\ W_1^u &= \{ \mathbf{x} = (x_1, x_2) \in \Omega_1 : x_1 = x_2 \vee x_1 = -x_2 \}, \\ W_2^u &= \{(0, 0)\}. \end{aligned} \quad (59)$$

The illustration of regions Ω_l and sets $W_l, W_l^u, l = 1, 2$, defined by (58)-(59), is given in Figure 3.

We note that

$$W_1 \cap W_1^u = \{(0, 0)\}, \quad W_2 \cap W_2^u = \{(0, 0)\} \quad (60)$$

and hence from Theorem 13 it follows that the hybrid system (55) is asymptotically stable in probability under feedback given as follows:

$$\mathbf{u}(\mathbf{x}, 1) = -x_1^2 + x_2^2, \quad \mathbf{u}(\mathbf{x}, 2) = -x_1^2 - x_2^2, \quad (61)$$

and the stabilizing switching rule of the form

$$\sigma_{st}(t) = \begin{cases} 1 & \text{if } \mathbf{x}(t) \in \Omega_1 \\ 2 & \text{if } \mathbf{x}(t) \in \Omega_2. \end{cases} \quad (62)$$

We can conclude also that function $\mathcal{V}(\mathbf{x}) = (1/2)\mathbf{x}^T \mathbf{x}$ is the single Lyapunov function for the hybrid system (55).

An exemplary simulation of a trajectory of the hybrid system (55) is shown in Figures 4 and 5.

4. Conclusions

In this paper stochastic nonlinear control hybrid systems, consisting of unstable and stable subsystems described by Itô stochastic differential equations, have been analyzed in terms of the stabilizability. The asymptotic stabilizability in probability problem, for considered class of hybrid systems with any, state dependent, and Markovian switching rules, has

FIGURE 4: An exemplary stable path of system (55) without control u .

FIGURE 5: An exemplary asymptotically stable path of system (55) with control u defined by (61)-(62).

been discussed. It has been assumed that the trivial solution of unforced hybrid system is stable in probability while some subsystems of unforced hybrid systems still can be unstable. By applying the obtained control the trivial solution of hybrid system becomes asymptotically stable in probability.

To find sufficient stabilizability conditions and to obtain the control law in a feedback form, the Lyapunov function technique (including a common, single, and a multiple Lyapunov function), the hybrid control theory, and some results of Florchinger [16] and Khasminskii et al. [10] have been used. Furthermore a method for a construction of stabilizing switching rules in the case of hybrid systems with state dependent switching rule has been given. The obtained results have been illustrated by examples.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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