Research Article

The Dynamics of a Cubic Nonlinear System with No Equilibrium Point

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We study the dynamics of a three-dimensional nonlinear system with cubic nonlinearity and no equilibrium points with the use of Poincaré maps, Lyapunov Exponents, and bifurcations diagrams. The system has rich dynamics: chaotic behavior, regular orbits, and 3-tori periodicity. Finally, the proposed system is also reported to verify electronic circuit modeling feasibility.

1. Introduction

A lot of work has been done in the field of dynamical system and many systems (Lorenz, Chua, Duffing, Van der pol, Sprott, and many others) have been exhaustively studied. The dynamics of such systems are well known and their properties are used in mechanical and electrical applications and experiments [1–4].

In the last two decades a new field of dynamical systems has been “discovered” and attracts the attention of scientists: dynamical systems with no equilibrium points or with conjugate equilibrium points.

Equilibrium points are important because their stability determines the dynamics of the system [5–7]. In particular, a stable equilibrium point is a point for which the trajectories around it remain close for small perturbations. On the other hand, an unstable equilibrium point is a point for which the trajectories around it escape even for small perturbations and remove the system from its initial state.

Equilibrium points are connected with criteria and theorems that determine the existence of chaotic behavior of a system (Melnikov function, Shilnikov chaos, etc.) [8]. The loss of equilibrium points means that the conventional Shilnikov criteria cannot be applied to prove the chaos in the flow.

A dynamical system with no equilibrium points is categorized as chaotic system with hidden attraction because the loss of equilibrium points means that its basin of attraction does not intersect with small neighborhoods of any equilibrium points.

Sprott (1994) was the first to introduce a simple flow with no equilibrium points [2]. Since then many researcher have introduced many systems with no equilibrium points or with conjugate equilibrium points [9–15].

In this work we study a modified version of the initial Sprott model with a cubic nonlinearity and a constant parameter A .

We made a numerical study of the system and used tools such as Poincaré maps, Lyapunov Characteristic Exponents, bifurcations diagrams [16, 17].

The system has rich dynamics. In general it has a chaotic behavior but for certain initial values and different values for the parameter A the system may have regular orbits (quasiperiodic or periodic). It is important to note here that for some values of initial conditions we detected transient hyperchaotic behavior of the system.

In Section 2 we analyze the system and present the behavior of the system for different values of the constant parameter A and different values of the initial conditions. In Section 3 we present an electronic circuit that implements

FIGURE 1: Bifurcation diagrams for different initial conditions taken for $y = 0$ and $y' > 0$.

the above nonlinear system and finally conclude in Section 4 of the paper.

2. Analysis

We study a nonlinear system with cubic nonlinearity:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x^3 - zy, \\ \dot{z} &= y^2 - A,\end{aligned}\tag{1}$$

where $A > 0$ is the parameter of the system. As it is obvious, since $A > 0$, the system has no equilibrium point.

We used many tools to analyze numerically the above system: Bifurcation diagrams, Poincaré maps (for $y = 0$), and Lyapunov Characteristic Exponents. We simulated the system for many thousands of different initial conditions and different values of the parameter A .

The numerical work was done with the help of Mathematica and the programming Languages C and True Basic by using the classical fourth-order Runge-Kutta method.

As we see from the bifurcations diagrams (Figure 1) the system is in general chaos. This is confirmed by the calculations of the Lyapunov Characteristic Exponents (LCEs)

(Figure 2) where we see that for many different initial conditions and different values of the parameter A there are one positive LCE, one negative LCE, and one LCE that equals zero. This confirms the chaotic behavior of the system for these initial conditions.

Furthermore, from the study of the LCEs we detected that for many initial conditions and values of A the system has a transient hyperchaotic behavior: that is, the system has two positive LCEs (Figure 3).

In what follows we will present three examples of the dynamics of the system, for different values of the parameter A .

First, for the Case Where $A = 5.16$. We see from the Poincaré section (Figure 4) that the system has both chaotic behavior and regular orbits (quasiperiodic and periodic). Also a paradigm of a 3-tori quasiperiodicity is detected.

For example, for the initial conditions ($x_0 = -0.8, y_0 = 0.0, \text{ and } z_0 = 1.0$), the system has one positive LCE, one negative LCE, and one LCE that equals zero (Figure 5). This confirms the chaotic behavior. For these initial conditions the trajectories in various planes are given by Figure 6.

For the initial conditions ($x_0 = -0.8, y_0 = 0.0, \text{ and } z_0 = 1.0$) the system has three LCEs that equal zero (Figure 7). That is, the motion is regular and this can be seen in Figure 8.

FIGURE 2: Lyapunov Characteristic Exponents for different initial conditions.

FIGURE 3: Lyapunov Characteristic Exponents for different initial conditions where transient hyperchaotic behavior is detected.

FIGURE 4: Poincaré section for $A = 5.16$.

FIGURE 5: Lyapunov Characteristic Exponents for $A = 5.16$ and initial conditions: $x_0 = -0.8$, $y_0 = 0.0$, and $z_0 = 1.0$.

FIGURE 6: Trajectories for $A = 5.16$ and initial conditions: $x_0 = -0.8$, $y_0 = 0.0$, and $z_0 = 1.0$.

FIGURE 7: Lyapunov Characteristic Exponents for $A = 5.16$ and initial conditions: $x_0 = -1.6525$, $y_0 = 0.0$, and $z_0 = 0.0$.

FIGURE 8: Trajectories for $A = 5.16$ and initial conditions: $x_0 = -1.6525$; $y_0 = 0.0$; and $z_0 = 0.0$.

FIGURE 9: Poincaré section for $A = 0.6$. FIGURE 10: Lyapunov Characteristic Exponents for $A = 0.6$ and initial conditions: $x_0 = -1.2$, $y_0 = 0.0$, and $z_0 = 0.0$.

For the Case Where $A = 0.6$. From the Poincaré sections (Figure 9) we see that the regions of regular orbits are much smaller than the regions of the previous example.

For the initial conditions ($x_0 = -1.2$, $y_0 = 0.0$, and $z_0 = 0.0$) the system has three LCEs that are equal to zero (Figure 10) and this means that the system has regular orbits. This is also confirmed by the trajectories of the system (Figure 11).

Also for the initial conditions ($x_0 = 2.0$, $y_0 = 0.0$, and $z_0 = 2.0$) the system has one positive LCE, one negative LCE, and one LCE that equal zero (Figure 12) and the system has a chaotic behavior (Figure 13).

For the Case Where $A = 12.7$. A more complicated structure of the Poincaré section can be seen for the parameter $A = 12.7$ (Figure 14). Beyond the regions of chaotic behavior and regular orbits a paradigm of a 3-tori periodicity also appears.

For the initial conditions ($x_0 = -1.4$, $y_0 = 0.0$, and $z_0 = 0.0$) the 3-tori periodicity paradigm appears. For this case, as it is shown from the figures of the LCEs and the trajectories (Figures 15 and 16) and as it is expected, the system has regular orbits.

3. Circuit Realization of the System

Circuitual design of chaotic systems plays a crucial role in the field of nonlinear science not only for providing a

simple experimental confirmation of phenomena related to nonlinear dynamics but also due to its applications in many engineering approaches, such as secure communication, signal processing, random bit generator, or path planning for autonomous mobile robot [18–23]. In addition, circuitual implementation of chaotic systems also provides an effective approach for investigating new dynamics of such theoretical models [24, 25]. For example, time-series of chaotic signals or chaotic attractors can be observed at the oscilloscope easily or experimental bifurcation diagram can be obtained by varying the value of variable resistors [26, 27].

In this work, an electronic circuit (Figure 17) is introduced for implementing system (1), which is designed by using the operational amplifiers approach [27]. It has three integrators (U_1 , U_5 , and U_7) and an inverting amplifier (U_2), which are implemented with the TL084, as well as four signals multipliers (U_3 , U_4 , U_6 , and U_8) by using the AD633. By applying Kirchhoff's circuit laws, the corresponding circuitual equations of designed master circuit can be written as

$$\begin{aligned}\dot{x} &= \frac{1}{R_1 C_1} y, \\ \dot{y} &= -\frac{1}{R_4 C_2 100V} x^3 - \frac{1}{R_5 C_2 10V} zy, \\ \dot{z} &= \frac{1}{R_6 C_3 10V} y^2 - \frac{1}{R_7 C_3} V_A,\end{aligned}\quad (2)$$

FIGURE 11: Trajectories for $A = 0.6$ and initial conditions: $x_0 = -1.2$, $y_0 = 0.0$, and $z_0 = 0.0$.

FIGURE 12: Lyapunov Characteristic Exponents for $A = 0.6$ and initial conditions: $x_0 = 2.0$, $y_0 = 0.0$, and $z_0 = 2.0$.

FIGURE 13: Trajectories for $A = 0.6$ and initial conditions: $x_0 = 2.0$, $y_0 = 0.0$, and $z_0 = 2.0$.

FIGURE 14: Poincaré section for $A = 12.7$.

FIGURE 15: Lyapunov Characteristic Exponents for $A = 12.7$ and initial conditions: $x_0 = -1.4$, $y_0 = 0.0$, and $z_0 = 0.0$.

FIGURE 16: Trajectories for $A = 12.7$ and initial conditions: $x_0 = -1.4$, $y_0 = 0.0$, and $z_0 = 0.0$.

FIGURE 17: The schematic of the circuit that emulates the proposed dynamical system (1).

FIGURE 18: Simulation phase portraits produced by the OrCAD and the respective ones produced by the system's arithmetic integration, for $A = 1$.

where x , y , and z are the voltages in the outputs of the operational amplifiers U_2 , U_5 , and U_7 . Normalizing the differential equations of system (2) by using $\tau = t/RC$ we can see that this system is equivalent to the proposed dynamical system (1). The circuit components have been selected as $R_1 = R_2 = R_3 = R_7 = 10 \text{ k}\Omega$, $R_4 = 0.1 \text{ k}\Omega$, $R_5 = R_6 = 1 \text{ k}\Omega$, $C_1 = C_2 = C_3 = 10 \text{ nF}$ and V_A adjust the value of the parameter A , while the power supplies of all active devices are $\pm 15V_{DC}$.

The designed circuit is implemented in the electronic simulation package Cadence OrCAD and the obtained results are displayed in Figures 18 and 19. In more details, these figures depict the simulation phase portraits produced by the OrCAD and the respective ones produced by the system's arithmetic integration, for $A = 1$ and $A = 5$. The comparison of the chaotic attractors proves that the theoretical attractors are similar with the circuital ones. So, the designed circuit emulates very well the proposed system's dynamic behavior.

FIGURE 19: Simulation phase portraits produced by the OrCAD and the respective ones produced by the system's arithmetic integration, for $A = 5$.

4. Conclusions

We study a nonlinear system with cubic nonlinearity and no equilibrium point through numerical simulations and confirm that the system has rich dynamics.

Specifically, the system has, in general, chaotic behavior. A transient hyperchaotic (two positive LCEs) behavior is also detected.

Also, for different initial conditions and different values of the parameter A the system may have regular orbits (periodic and quasiperiodic). Furthermore, examples of a 3-tori periodicity may appear for different values of the constant parameter A and different initial conditions.

Finally, the designed nonlinear electronic circuit emulates very well the proposed system's dynamic behavior.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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