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Research Article

On the Stochastic Stability and Boundedness of Solutions for Stochastic Delay Differential Equation of the Second Order

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We present two qualitative results concerning the solutions of the following equation: $\ddot{x}(t) + g(\dot{x}(t)) + bx(t-h) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), x(t-h))$; the first result covers the stochastic asymptotic stability of the zero solution for the above equation in case $p \equiv 0$, while the second one discusses the uniform stochastic boundedness of all solutions in case $p \not\equiv 0$. Sufficient conditions for the stability and boundedness of solutions for the considered equation are obtained by constructing a Lyapunov functional. Two examples are also discussed to illustrate the efficiency of the obtained results.

1. Introduction

During the last twenty years, the theory of stochastic differential equations has successfully attracted considerable attentions of scholars; for example, see [1–13]. Since then, the number of contributions to statistics, numerics, and control theory of stochastic differential equations has been rapidly increasing, since stochastic modelling plays an important role in formulation and analysis in modelling physical, technical, biological and economical dynamical systems in which significant uncertainty is present.

Stochastic delay differential equation, also known as stochastic functional differential equation, is a natural generalization of stochastic ordinary differential equation by allowing the coefficients to depend on the past values. The Razumikhin argument, generalized Itô formula, and Euler-Maruyama formula play an important role in studying stochastic differential equations. Unfortunately, it is generally not possible to give explicit expressions for the solutions to stochastic differential equations. Therefore, most of the papers are interested in being able to characterize at least qualitatively the behaviour of the solutions. Thus, Lyapunov theory is a powerful tool for qualitative analysis of stochastic differential equations, since the advantage of this method can judge the behaviour of systems without any prior knowledge of the explicit solutions, while the greatest disadvantage of the Lyapunov approach is that no universal method has been given, which enables us to find a Lyapunov function or determine that no such function exists.

It is worth mentioning that there are few results on the stability and boundedness of solutions for first-order stochastic delay differential equations; for example, see [8, 14–18].

In 2004, Kolmanovskii and Shaikhet [8] investigated the conditions of asymptotic mean-square stability for first-order stochastic delay differential equation of neutral type:

$$\dot{x}(t) + ax(t) + bx(t-h) + c\dot{x}(t-h) + \sigma x(t-\tau)\dot{\xi}(t) = 0, \quad |c| < 1, \tag{1}$$

where a and b are two positive constants and $\xi(t)$ is a standard Wiener process.

Later, in 2006, Rodkina and Basin [17] obtained global asymptotic stability conditions for nonlinear stochastic systems with state delay as follows:

$$dx(t) = -aN(x(t))\,dt - bN(x(t-h))\,dt + \sigma(t, x_t)\,dw_t,$$
$$x(s) = \phi(s), \quad s \in [-h, 0]. \tag{2}$$

The Lyapunov Krasovskii and degenerate functionals techniques are used. In addition, nontrivial examples of nonlinear systems satisfying the obtained stability conditions are given.

On the other hand, the corresponding problem for the stability and boundedness of solutions of second-order stochastic delay differential equations was studied far less often. So our main aim in this paper is to establish new results on the stability and boundedness for solutions of second-order stochastic delay differential equation of the type

$$\begin{aligned}\ddot{x}(t) + g(\dot{x}(t)) + bx(t-h) + \sigma x(t)\dot{\omega}(t) \\ = p(t, x(t), \dot{x}(t), x(t-h)),\end{aligned} \tag{3}$$

where b and σ are two positive constants, and h is a positive constant delay; g and p are two continuous functions with $g(0) = 0$; $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_m(t)) \in \mathbb{R}^m$ is an m-dimensional standard Brownian motion defined on the probability space (also called Wiener process), a stochastic process representing the noise [19].

2. Preliminaries and Stability Result

Consider the following nonautonomous n-dimensional stochastic delay differential equation (SDDE):

$$\begin{aligned}dx(t) = f(t, x(t), x(t-\tau))\,dt \\ + g(t, x(t), x(t-\tau))\,dB(t), \quad t \geq 0,\end{aligned} \tag{4}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given nonlinear continuous functions, $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ is an m-dimensional standard Brownian motion, and $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$ is a solution of the stochastic delay differential equation (4) with initial data of $\{x(\theta) : -\tau \leq \theta \leq 0\}$, $x_0 \in \mathscr{C}([-\tau, 0]; \mathbb{R}^n)$. We assume that f and g satisfy the following conditions.

Standing Hypothesis (H1). Both f and g satisfy the local Lipschitz condition and for all given $b > 0$, $p \geq 2$, $f(t, 0, 0) \in \mathscr{L}^1([0, b]; \mathbb{R}^n)$, and $g(t, 0, 0) \in \mathscr{L}^p([0, b]; \mathbb{R}^{n\times m})$ (see [20]). Then, there must be a stopping time $\beta = \beta(\omega) > 0$ such that (4) with $x_0 \in \mathscr{C}^p_{\mathscr{F}_0}$ has a unique maximal local solution for $t \in [t_0, \beta)$.

In this section, for the stability result, we impose *Standing Hypothesis (H2)*:

$$f(t, 0, 0) = 0, \quad g(t, 0, 0) = 0, \quad \forall t \geq 0. \tag{5}$$

Hence, the stochastic delay differential equation (4) admits the zero solution $x(t; 0) \equiv 0$, for any given initial value $x_0 \in \mathscr{C}([-\tau, 0]; \mathbb{R}^n)$.

Definition 1. The zero solution of the stochastic differential equation is said to be stochastically stable or stable in probability, if for every pair of $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r) > 0$ such that

$$P\{|x(t; x_0)| < r \ \forall t \geq 0\} \geq 1 - \epsilon, \quad \text{whenever } |x_0| < \delta. \tag{6}$$

Otherwise, it is said to be stochastically unstable.

Definition 2. The zero solution of the stochastic differential equation is said to be stochastically asymptotically stable, if it is stochastically stable, and moreover for every $\epsilon \in (0, 1)$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$P\left\{\lim_{t\to 0} x(t; x_0) = 0\right\} \geq 1 - \epsilon, \quad \text{whenever } |x_0| < \delta_0. \tag{7}$$

Let $\mathscr{K}$ denote the family of all continuous nondecreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$, if $r > 0$. In addition, $\mathscr{K}_\infty$ denotes the family of all functions $\rho \in \mathscr{K}$ with $\lim_{r\to\infty}\rho(r) = \infty$. Let $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of nonnegative functions $V(t, x)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are once continuously differentiable in t and twice continuously differentiable in x. For each $V \in \mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, define an operator $\mathscr{L}V : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}\mathscr{L}V(t, x, y) = V_t(t, x) + V_x(t, x) f(t, x, y) \\ + \frac{1}{2}\operatorname{trace}\left[g^T(t, x, y) V_{xx}(t, x) g(t, x, y)\right],\end{aligned} \tag{8}$$

where

$$\begin{aligned}V_t(t, x) &= \frac{\partial V(t, x)}{\partial t}, \\ V_x(t, x) &= \left(\frac{\partial V(t, x)}{\partial x_1}, \ldots, \frac{\partial V(t, x)}{\partial x_n}\right), \\ V_{xx}(t, x) &= \left(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\right)_{n\times n}.\end{aligned} \tag{9}$$

It should be emphasized that the operator $\mathscr{L}V$ is defined on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, although the function V is defined only on $\mathbb{R}^+ \times \mathbb{R}^n$.

To prove our main stability result, we shall introduce the following theorems.

Theorem 3 (see [21, 22]). *Assume that there exist $V \in \mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\mu \in \mathscr{K}_\infty$ such that*

$$\begin{aligned}\mu(|x|) \leq V(t, x), \qquad \mathscr{L}V(t, x, y) \leq 0, \\ \forall (t; x; y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n.\end{aligned} \tag{10}$$

Then, (4) with $x_0 \in \mathscr{C}^p_{\mathscr{F}_0}$ has a unique global solution for $t > 0$ denoted by $x(t; t_0)$ (this is a special case of Corollary 12 in [22] with $G \equiv b \equiv \kappa \equiv 0$).

Furthermore, $V(0) = 0$; then, the zero solution of the stochastic delay differential equation is stochastically stable (see [21]).

Theorem 4 (see [21]). *Assume that there exist $V \in \mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\mu_1, \mu_2, \mu_3 \in \mathscr{K}$ such that*

$$\begin{aligned}\mu_1(|x|) \leq V(t, x) \leq \mu_2(|x|), \\ \mathscr{L}V(t, x, y) \leq -\mu_3(|x|),\end{aligned} \tag{11}$$

for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$. Then, the zero solution of the stochastic delay differential equation is stochastically asymptotically stable.

Now we present the main stability result of (3) with $p \equiv 0$.

Theorem 5. *Further to the basic assumptions imposed on a, b, σ, h, g, and p, suppose that the following conditions are satisfied.*

(i) $g(y)/y \geq a > 0$ *for* $y \neq 0$ *and* $|g'(y)| \leq l$, $l > 0$, *for all* y.

(ii) $a > b > 1$.

(iii) $b(a-1) - (l+1)\sigma^2 > 0$.

Then, the zero solution of (3) is stochastically asymptotically stable, provided that

$$h < \min\left\{\frac{b(a-1)-(l+1)\sigma^2}{2ab}, \frac{a(a-b)+a^2-b}{2b(a+2l+2)}\right\}. \quad (12)$$

Proof of Theorem 5. In fact, (3) with $p \equiv 0$ can be transformed into an equivalent system of the following form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(y) - bx + b\int_{t-h}^{t} y(s)\,ds - \sigma x\dot{\omega}(t). \end{aligned} \quad (13)$$

Define the Lyapunov functional $V_1(x_t, y_t)$ as

$$\begin{aligned} V_1(x_t, y_t) = &\int_0^y g(\eta)\,d\eta + bx^2 + \frac{1}{2}(y+ax)^2 \\ &+ \gamma\int_{-h}^{0}\int_{t+s}^{t} y^2(\phi)\,d\phi\,ds, \end{aligned} \quad (14)$$

where $x_t = x(t+s)$, $s \leq 0$, and γ is a positive constant, which will be determined later.

From (14) and (13) and by using Itô formula, we find

$$\begin{aligned} \mathscr{L}V_1(x_t, y_t) = &ay^2 + a^2xy - g^2(y) - bxg(y) \\ &+ bg(y)\int_{t-h}^{t} y(s)\,ds - yg(y) \\ &+ bxy + by\int_{t-h}^{t} y(s)\,ds - axg(y) \\ &- abx^2 + abx\int_{t-h}^{t} y(s)\,ds + \gamma hy^2(t) \\ &- \gamma\int_{t-h}^{t} y^2(s)\,ds + \frac{g'(y)+1}{2}\sigma^2x^2; \end{aligned} \quad (15)$$

since $|g'(y)| \leq l$, $l > 0$ and $g(y)/y \geq a > 0$ by (i), then

$$\begin{aligned} \mathscr{L}V_1(x_t, y_t) \leq &-a^2y^2 - abxy + bly\int_{t-h}^{t} y(s)\,ds \\ &+ bxy + by\int_{t-h}^{t} y(s)\,ds - abx^2 \\ &+ abx\int_{t-h}^{t} y(s)\,ds + \gamma hy^2 \\ &- \gamma\int_{t-h}^{t} y^2(s)\,ds + \frac{l+1}{2}\sigma^2x^2. \end{aligned} \quad (16)$$

Thus, by using the inequality $|uv| \leq (1/2)(u^2+v^2)$, we obtain

$$\begin{aligned} \mathscr{L}V_1(x_t, y_t) \leq &-\left\{\frac{b(a-1)}{2} - \frac{l+1}{2}\sigma^2 - \frac{abh}{2}\right\}x^2 \\ &-\left\{\frac{a(a-b)+a^2-b}{2} - \left(\frac{bl}{2}+\frac{b}{2}+\gamma\right)h\right\}y^2 \\ &+\left(\frac{bl}{2}+\frac{ab}{2}+\frac{b}{2}-\gamma\right)\int_{t-h}^{t} y^2(s)\,ds. \end{aligned} \quad (17)$$

Let us choose $\gamma = b(l+a+1)/2 > 0$. Then, it is easy to see that

$$\begin{aligned} \mathscr{L}V_1(x_t, y_t) \leq &-\left\{\frac{b(a-1)}{2} - \frac{l+1}{2}\sigma^2 - \frac{abh}{2}\right\}x^2 \\ &-\left\{\frac{a(a-b)+a^2-b}{2} - \frac{b(a+2l+2)}{2}h\right\}y^2. \end{aligned} \quad (18)$$

Now, in view of (18), one can conclude for some positive constant α that

$$\mathscr{L}V_1(x_t, y_t) \leq -\alpha\left(x^2+y^2\right), \quad (19)$$

provided that

$$h < \min\left\{\frac{b(a-1)-(l+1)\sigma^2}{2ab}, \frac{a(a-b)+a^2-b}{2b(a+2l+2)}\right\}. \quad (20)$$

Since $\int_{-h}^{0}\int_{t+s}^{t} y^2(\phi)d\phi\,ds$ is nonnegative, then we find

$$V_1(x_t, y_t) \geq \int_0^y g(\eta)\,d\eta + bx^2 + \frac{1}{2}(y+ax)^2, \quad (21)$$

from $g(y)/y \geq a > 0$, therefore, we get

$$V_1(x_t, y_t) \geq \frac{a}{2}y^2 + bx^2 + \frac{1}{2}(y+ax)^2. \quad (22)$$

Then, there exists a positive constant β such that

$$V_1(x_t, y_t) \geq \beta\left(x^2+y^2\right). \quad (23)$$

Also, since $|g'(y)| \leq l$, $g(0) = 0$, and by using the mean-value theorem, we get $g(y) \leq ly$. So we can rewrite (14) in the following form:

$$\begin{aligned} V_1(x_t, y_t) \leq &\frac{l}{2}y^2 + bx^2 + \frac{1}{2}(y+ax)^2 \\ &+ \gamma\int_{-h}^{0}\int_{t+s}^{t} y^2(\phi)\,d\phi\,ds. \end{aligned} \quad (24)$$

It follows that, by using the inequality $2uv \leq u^2+v^2$, we have

$$\begin{aligned} V_1(x_t, y_t) \leq &\frac{l}{2}y^2 + bx^2 + \frac{1}{2}y^2 + \frac{a}{2}\left(x^2+y^2\right) \\ &+ \frac{a^2}{2}x^2 + \gamma\int_{t-h}^{t}(\theta-t+h)\,y^2(\theta)\,d\theta. \end{aligned} \quad (25)$$

Therefore, we obtain

$$V_1(x_t, y_t) \le \frac{a^2+a+2b}{2}\|x\|^2 + \frac{a+l+1}{2}\|y\|^2 + \frac{\gamma h^2}{2}\|y\|^2 = \frac{a^2+a+2b}{2}\|x\|^2 + \frac{a+l+1+\gamma h^2}{2}\|y\|^2. \tag{26}$$

Then, there exists a positive constant ν satisfying

$$V_1(x_t, y_t) \le \nu\left(x^2+y^2\right). \tag{27}$$

Thus, from the results (19), (23), and (27), we note that all the conditions of Theorem 4 are satisfied, and then the zero solution of (3) is stochastically asymptotically stable. This completes the proof of Theorem 5. □

In the next section, we shall state and prove our main second result on boundedness of (3) with $p(t, x(t), \dot{x}(t), x(t-h)) \not\equiv 0$.

3. Further Preliminaries and Boundedness Result

Consider the n-dimensional stochastic delay differential equation (SDDE):

$$dx(t) = f(t, x(t), x(t-\tau))\,dt + g(t, x(t), x(t-\tau))\,dB(t), \quad t \ge 0, \tag{28}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$. In order to solve the equation, we need to know the initial data and we assume that they are given by initial value of $\{x(\theta) : -\tau \le \theta \le 0\}$, $x_0 \in \mathscr{C}([-\tau, 0]; \mathbb{R}^n)$.

In [23], Liu and Raffoul use Lyapunov second method to determine sufficient conditions for stochastic boundedness of system (28). The theorems in [23] will make significant contribution to the theory of stochastic differential equations, when dealing with equations that might contain unbounded terms.

Definition 6. A solution $x(t; t_0, x_0)$ of (28) is said to be stochastically bounded or bounded in probability, if it satisfies

$$E^{x_0}\|x(t; t_0, x_0)\| \le C(\|x_0\|, t_0), \quad \forall t \ge t_0, \tag{29}$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on t_0 and x_0. We say that the solutions of (28) are uniformly stochastically bounded, if C is independent of t_0.

Assumption 7. We assume that for any solution $x(t)$ of (28) and for any fixed $0 \le t_0 \le T < \infty$, the following condition holds:

$$E^{x_0}\left\{\int_{t_0}^{T} V_{x_i}^2(t, x(t))\, g_{ik}^2(t, x(t))\,dt\right\} < \infty, \quad 1 \le i \le n, \quad 1 \le k \le m. \tag{30}$$

Assumption 8. A special case of the general condition (30) is the following condition: assume that there exists a function $\chi(t)$ such that

$$\left|V_{x_i}(t, x(t))\, g_{ik}(t, x(t))\right| \le \chi(t), \quad x \in \mathbb{R}^n, \quad 1 \le i \le n, \quad 1 \le k \le m, \tag{31}$$

and for any fixed $0 \le t_0 \le T < \infty$,

$$\int_{t_0}^{T} \chi^2(t)\,dt < \infty. \tag{32}$$

Here, we will present the main theorems used in our main boundedness result.

Theorem 9. *Assume there exists a function $V(t, x)$ in $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfying Assumption 7, such that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$*

(i) $\|x\|^{q_1} \le V(t, x) \le \|x\|^{q_2}$,

(ii) $\mathscr{L}V(t, x) \le -\lambda(t)\|x\|^r + \eta(t)$,

(iii) $V(t, x) - V^{r/q_2}(t, x) \le \mu$,

where λ, $\eta \in C(\mathbb{R}^+; \mathbb{R}^+)$, q_1, q_2, and r are positive constants, $q_1 \ge 1$, and μ is a nonnegative constant. Then, all solutions of (28) satisfy

$$E^{x_0}\|x(t; t_0, x_0)\| \le \left\{V(t_0, x_0)\, e^{-\int_{t_0}^{t}\lambda(s)ds} + \int_{t_0}^{t}\left(\mu\lambda(u) + \eta(u)\right) e^{-\int_{u}^{t}\lambda(s)ds}du\right\}^{1/q_1}, \tag{33}$$

for all $t \ge t_0$.

Theorem 10. *Assume there exists a function $V(t, x)$ in $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfying Assumption 7, such that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$*

(i) $\|x\|^{q_1} \le V(t, x)$,

(ii) $\mathscr{L}V(t, x) \le -\lambda(t)V^{q_2}(t, x) + \eta(t)$,

(iii) $V(t, x) - V^{q_2}(t, x) \le \mu$,

where $\lambda, \eta \in C(\mathbb{R}^+; \mathbb{R}^+)$, q_1 and q_2 are two positive constants, $q_1 \ge 1$, and μ is a nonnegative constant. Then, all solutions of (28) satisfy

$$E^{x_0}\|x(t; t_0, x_0)\| \le \left\{V(t_0, x_0)\, e^{-\int_{t_0}^{t}\lambda(s)ds} + \int_{t_0}^{t}\left(\mu\lambda(u) + \eta(u)\right) e^{-\int_{u}^{t}\lambda(s)ds}du\right\}^{1/q_1}, \tag{34}$$

for all $t \ge t_0$.

Corollary 11. *(1) Assume that the hypotheses of Theorem 9 hold. In addition,*

$$\int_{t_0}^{t} \{\mu\lambda(u) + \eta(u)\} e^{-\int_u^t \lambda(s)ds} du \le \mathscr{M}, \quad \forall t \ge t_0 \ge 0, \tag{35}$$

for some positive constant $\mathscr{M}$; then, all the solutions of (28) are uniformly stochastically bounded.

(2) Assume that the hypotheses of Theorem 10 hold. If condition (35) is satisfied, then all the solutions of (28) are stochastically bounded.

The following theorem is the second main result for (3) with $p \not\equiv 0$.

Theorem 12. *Let the conditions (i) and (ii) of Theorem 5 be satisfied. In addition, we assume that*

(iii) $\sigma^2 < (ab(2b-1) + b(2ab-1))/(3l+2b+1)$;

(iv) $|p(t, x(t), \dot{x}(t), x(t-h))| \le m$, *for some* $m > 0$.

Then, all the solutions of (3) are uniformly stochastically bounded, provided that

$$h < \min\left\{\frac{ab(2b-1) + b(2ab-1) - (3l+2b+1)\sigma^2}{2ab(2b+1)}, \frac{3a(2a-b) - b}{2b(6l+4b+2ab+a+2)}\right\}. \tag{36}$$

Proof of Theorem 12. Equation (3) has the following equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(y) - bx + b\int_{t-h}^{t} y(s)\,ds \\ &\quad - \sigma x\dot{\omega}(t) + p(t, x, y, x(t-h)). \end{aligned} \tag{37}$$

Consider the Lyapunov functional $V(x_t, y_t)$ as

$$V(x_t, y_t) = V_1(x_t, y_t) + V_2(x_t, y_t), \tag{38}$$

where $V_1(x_t, y_t)$ is defined as (14) and $V_2(x_t, y_t)$ is defined as

$$V_2(x_t, y_t) = a^2bx^2 + 2\int_0^y g(\eta)\,d\eta + 2abxy + by^2 + b^2x^2. \tag{39}$$

From (14), (37), and (iv), we find

$$\begin{aligned} \mathscr{L}V_1(x_t, y_t) \le &-\left\{\frac{b(a-1)}{2} - \frac{l+1}{2}\sigma^2 - \frac{abh}{2}\right\}x^2 \\ &-\left\{\frac{a(a-b) + a^2 - b}{2} - \left(\frac{bl}{2} + \frac{b}{2} + \gamma\right)h\right\}y^2 \\ &+\left(\frac{bl}{2} + \frac{ab}{2} + \frac{b}{2} - \gamma\right)\int_{t-h}^{t} y^2(s)\,ds \\ &+ am|x| + (l+1)m|y|. \end{aligned} \tag{40}$$

Also, from (39), (37), (i), and (iv), we obtain

$$\begin{aligned} \mathscr{L}V_2(x_t, y_t) \le &-\left\{ab(2b-1) - (l+b)\sigma^2 - ab^2h\right\}x^2 \\ &-\{a(2a-b) - b(l+b)h\}y^2 \\ &+\left(bl + b^2 + ab^2\right)\int_{t-h}^{t} y^2(s)\,ds \\ &+ 2abm|x| + 2(l+b)m|y|. \end{aligned} \tag{41}$$

Therefore, from (38), (40), and (41), we obtain

$$\begin{aligned} &\mathscr{L}V(x_t, y_t) \\ &\le -\left\{\frac{ab(2b-1) + b(2ab-1)}{2} - \frac{3l+2b+1}{2}\sigma^2 - \frac{ab(2b+1)}{2}h\right\}x^2 \\ &\quad -\left[\frac{3a(2a-b) - b}{2} - \left\{\frac{b(3l+2b+1)}{2} + \gamma\right\}h\right]y^2 \\ &\quad -\left\{\gamma - \frac{b(3l+2b+2ab+a+1)}{2}\right\}\int_{t-h}^{t} y^2(s)\,ds \\ &\quad + am(2b+1)|x| + (3l+2b+1)m|y|. \end{aligned} \tag{42}$$

If we take $\gamma = b(3l+2b+2ab+a+1)/2 > 0$, we can write (42) in the following form:

$$\begin{aligned} &\mathscr{L}V(x_t, y_t) \\ &\le -\left\{\frac{ab(2b-1) + b(2ab-1)}{2} - \frac{3l+2b+1}{2}\sigma^2 - \frac{ab(2b+1)}{2}h\right\}x^2 \\ &\quad -\left\{\frac{3a(2a-b) - b}{2} - \frac{b(6l+4b+2ab+a+2)}{2}h\right\}y^2 \\ &\quad + am(2b+1)|x| + (3l+2b+1)m|y|. \end{aligned} \tag{43}$$

Therefore, if

$$h < \min\left\{\frac{ab(2b-1) + b(2ab-1) - (3l+2b+1)\sigma^2}{2ab(2b+1)}, \frac{3a(2a-b) - b}{2b(6l+4b+2ab+a+2)}\right\}, \tag{44}$$

then we get

$$\begin{aligned}\mathscr{L}V(x_t,y_t) &\le -\vartheta\left(x^2+y^2\right)+k\vartheta\left(|x|+|y|\right)\\ &= -\frac{\vartheta}{2}\left(x^2+y^2\right)\\ &\quad -\frac{\vartheta}{2}\left\{(|x|-k)^2+\left(|y|-k\right)^2\right\}+\vartheta k^2\\ &\le -\frac{\vartheta}{2}\left(x^2+y^2\right)+\vartheta k^2, \quad \forall k,\vartheta>0.\end{aligned} \tag{45}$$

Thus, condition (ii) of Theorem 9 is satisfied by taking $\lambda(t) = \vartheta/2$, $\eta(t) = \vartheta k^2$, and $r = 2$. Also, we can easily check that conditions (i)–(iii) of Theorem 9 with $q_1 = q_2 = 2$ and $\mu = 0$ are satisfied, using the same techniques which have already been demonstrated in proof of Theorem 5. With $\lambda(t) = \vartheta/2$, $\eta(t) = \vartheta k^2$, and $\mu = 0$ and using them in (35), we note that

$$\begin{aligned}&\int_{t_0}^{t}\left\{\mu\lambda(u)+\eta(u)\right\}e^{-\int_u^t\lambda(s)ds}du\\ &\quad = \vartheta k^2\int_{t_0}^{t}e^{-(\vartheta/2)\int_u^t ds}du \le 2k^2,\end{aligned} \tag{46}$$

for all $t \ge t_0 \ge 0$. Thus, condition (35) holds. Now, since

$$\begin{aligned}g^T &= (0-\sigma x),\\ V_x &= (V_1)_x+(V_2)_x\\ &= 2bx+ay+a^2x+2a^2bx+2aby+2b^2x,\\ V_y &= (V_1)_y+(V_2)_y\\ &= 3g(y)+ax+y+2abx+2by.\end{aligned} \tag{47}$$

Then, we have

$$\begin{aligned}\left|V_{x_i}(t,x(t))\,g_{ik}(t,x(t))\right| &\le \sigma\left[\left\{2ab+a+b+\frac{3l+1}{2}\right\}x^2\right.\\ &\quad \left.+\left\{\frac{3l+1}{2}+b\right\}y^2\right] := \chi(t).\end{aligned} \tag{48}$$

Hence, the conditions (31) and (32) are satisfied. So by Corollary 11(1), we find that all solutions of (3) are uniformly stochastically bounded and satisfy

$$E^{x_0}\left\|x(t;t_0,x_0)\right\| \le \left\{Cx_0^2+2k^2\right\}^{1/2}, \quad \forall t\ge t_0\ge 0, \tag{49}$$

where C is a constant. This completes the proof of Theorem 12. □

4. Illustrative Examples

In this section, we display two examples to illustrate the application of the results we obtained in the previous sections.

Example 1. As an application of Theorem 5, we consider the second-order stochastic delay differential equation

$$\ddot{x}(t)+4\dot{x}(t)+\sin\dot{x}(t)+2x(t-h)+\frac{1}{2}x(t)\dot{\omega}(t)=0. \tag{50}$$

The equivalent system of (50) is

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -(4y+\sin y)-2x+2\int_{t-h}^{t}y(s)\,ds-\frac{1}{2}x(t)\dot{\omega}(t).\end{aligned} \tag{51}$$

From (50), we have

$$g(y)=4y+\sin y, \qquad b=2, \qquad \sigma=\frac{1}{2}. \tag{52}$$

It is obvious that

$$\begin{aligned}\frac{g(y)}{y} &= 4+\frac{\sin y}{y}\ge 3=a, \quad \forall y,\ (y\ne 0),\\ \left|g'(y)\right| &= \left|4+\cos y\right|\le 5=l,\\ \frac{b(a-1)}{2}-\left(\frac{l+1}{2}\right)\sigma^2 &= \frac{10}{8}>0.\end{aligned} \tag{53}$$

Thus,

$$\begin{aligned}\mathscr{L}V_1(x_t,y_t) \le &-\left(\frac{10}{8}-\frac{6}{2}h\right)x^2-\left\{\frac{10}{2}-\left(\frac{12}{2}+\gamma\right)h\right\}y^2\\ &+\left(\frac{10}{2}+\frac{6}{2}+\frac{2}{2}-\gamma\right)\int_{t-h}^{t}y^2(s)\,ds.\end{aligned} \tag{54}$$

Let us choose $\gamma = b(l+a+1)/2 = 9 > 0$. Then, one can conclude for some positive constant α that

$$\mathscr{L}V_1(x_t,y_t)\le -\alpha\left(x^2+y^2\right), \tag{55}$$

provided that $h < \min\{5/24, 5/30\}$.

Since $4+\sin y/y \ge 3 = a > 0$, therefore we get

$$V_1(x_t,y_t)\ge \frac{3}{2}y^2+2x^2+\frac{1}{2}(y+3x)^2. \tag{56}$$

Then, there exists a positive constant β such that

$$V_1(x_t,y_t)\ge \beta\left(x^2+y^2\right). \tag{57}$$

Also, by using the fact that $|4+\cos y|\le 5 = l$, we obtain

$$V_1(x_t,y_t)\le \frac{9+3+4}{2}\|x\|^2+\frac{1+3+5+9h^2}{2}\|y\|^2. \tag{58}$$

So there exists a positive constant ν satisfying

$$V_1(x_t,y_t)\le \nu\left(x^2+y^2\right). \tag{59}$$

Thus, from the results (55), (57), and (59), we note that all the conditions of Theorem 4 are satisfied, so the zero solution of (50) is stochastically asymptotically stable.

Example 2. As an application of Theorem 12, consider the second-order stochastic delay differential equation

$$\begin{aligned}\ddot{x}(t) + 4\dot{x}(t) + \sin\dot{x}(t) + 2x(t-h) + \frac{1}{2}x(t)\dot{\omega}(t) \\ = p(t, x, \dot{x}, x(t-h)).\end{aligned} \tag{60}$$

The equivalent system of (60)

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -(4y + \sin y) - 2x + 2\int_{t-h}^{t} y(s)\,ds \\ &\quad - \frac{1}{2}x(t)\dot{\omega}(t) + p.\end{aligned} \tag{61}$$

From (60), we have

$$g(y) = 4y + \sin y, \qquad b = 2, \qquad \sigma = \frac{1}{2}. \tag{62}$$

It is obvious that

$$\begin{aligned}\frac{g(y)}{y} &= 4 + \frac{\sin y}{y} \geq 3 = a, \quad \forall y, \ (y \neq 0), \\ \left|g'(y)\right| &= \left|4 + \cos y\right| \leq 5 = l, \\ \frac{ab(2b-1) + b(2ab-1)}{3l + 2b + 1} &= 2 > \sigma^2 = \frac{1}{4}, \\ m &= 0.02.\end{aligned} \tag{63}$$

Thus,

$$\begin{aligned}\mathscr{L}V(x_t, y_t) \leq & -\left(\frac{35}{2} - \frac{30}{2}h\right)x^2 - \left(\frac{34}{2} - \frac{110}{2}h\right)y^2 \\ & + 0.3|x| + 0.4|y|.\end{aligned} \tag{64}$$

Since $\gamma = b(3l + 2b + 2ab + a + 1)/2 = 35 > 0$, provided that $h < \min\{35/60, 34/220\}$.

If we take $\vartheta = 8.5$ and $k = 0.047$. Thus, condition (ii) of Theorem 9 is satisfied by taking $\lambda(t) = 4.25$, $\eta(t) = 0.019$, and $r = 2$. Also, we can easily check that the conditions (i)–(iii) of Theorem 9 with $q_1 = q_2 = r = 2$ and $\mu = 0$ are satisfied, since

$$\int_{t_0}^{t} \{\mu\lambda(u) + \eta(u)\} e^{-\int_u^t \lambda(s)ds} du \leq 0.0044, \tag{65}$$

for all $t \geq t_0 \geq 0$. Thus, condition (35) holds. Now since

$$\left|V_{x_i}(t, x(t))\, g_{ik}(t, x(t))\right| \leq \frac{1}{2}\left[25x^2 + 10y^2\right] := \chi(t), \tag{66}$$

hence the conditions (31) and (32) are satisfied. So by Corollary 11, all solutions of (60) with $|p| \leq 0.02$ are uniformly stochastically bounded and satisfy

$$E^{x_0}\left\|x(t; t_0, x_0)\right\| \leq \left\{x_0^2 + 0.0044\right\}^{1/2}, \quad \forall t \geq t_0 \geq 0. \tag{67}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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