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Research Article

Error Estimation of Functions by Fourier-Laguerre Polynomials Using Matrix-Euler Operators

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Various investigators have studied the degree of approximation of a function using different summability (Cesáro means of order α: C_α, Euler E_q, and Nörlund N_p) means of its Fourier-Laguerre series at the point $x = 0$ after replacing the continuity condition in Szegö theorem by much lighter conditions. The product summability methods are more powerful than the individual summability methods and thus give an approximation for wider class of functions than the individual methods. This has motivated us to investigate the error estimation of a function by $(T \cdot E_q)$-transform of its Fourier-Laguerre series at frontier point $x = 0$, where T is a general lower triangular regular matrix. A particular case, when T is a Cesáro matrix of order 1, that is, C_1, has also been discussed as a corollary of main result.

Dedicated to Professor Bani Singh

1. Introduction

Let $\sum u_n := \sum_{n=0}^{\infty} u_n$ be given infinite series with the sequence of its $(n+1)$th partial sums $\{s_n\}$. Define $E_n^q = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^k s_k$, $\forall q > 0$. If $\lim_{n\to\infty} E_n^q = S_1$, then the series $\sum u_n$ is said to be E_q-summable to S_1.

Let $T \equiv (a_{n,k})$ be an infinite triangular matrix with real constants. The sequence-to-sequence transformation $t_n = \sum_{k=0}^{n} a_{n,k} s_k$, $\forall n \geq 0$, defines the T-transform of the sequence $\{s_n\}$. The series $\sum u_n$ is said to be T-summable to S_2 if $\lim_{n\to\infty} t_n = S_2$. Throughout this paper, $T \equiv (a_{n,k})$ has nonnegative entries with row sums one. T is said to be regular if it is limit preserving over the space of convergent sequences. Thus, T behaves as a linear operator.

The $(T \cdot E_q)$-transform of $\{s_n\}$, denoted by $(TE)_{n,q}$, are defined by

$$(TE)_{n,q} = \sum_{r=0}^{n} a_{n,r} E_r^q = \sum_{r=0}^{n} a_{n,r} \frac{1}{(1+q)^r} \sum_{k=0}^{r} \binom{r}{k} q^k s_k. \quad (1)$$

If $(TE)_{n,q} \to S$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(T \cdot E_q)$-summable to S. The regularity of $(T \cdot E_q)$ method follows from the regularity of E_q method as well as T-method and thus the matrix $(T \cdot E_q)$ behaves as a linear operator. Some important particular cases of the matrix-Euler operator are as follows:

(i) If $a_{n,k} = 1/(n-k+1)\log(n+1)$, then we get $(H_1 \cdot E_q)$ operator.

(ii) Let $\{p_n\}$ be a sequence of real, nonnegative numbers such that $P_n = \sum_{k=0}^{n} p_k \neq 0$, $P_{-1} = 0 = p_{-1}$, and $p_0 \neq 0$. If $a_{n,k} = p_{n-k}/P_n$, then we get $(N_p \cdot E_q)$ operator. A special case in which $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$; then $(N_p \cdot E_q)$ operator further reduces to $(C_\alpha \cdot E_q)$ operator.

(iii) If $a_{n,k} = p_{n-k} q_k / R_n$, where $R_n = \sum_{k=0}^{n} p_{n-k} q_k \neq 0$, then we get $(N, p_n, q_n) \cdot E_q$ operator.

(iv) If $q = 1$ in above cases, then we get $(H_1 \cdot E_1)$, $(N_p \cdot E_1)$, $(C_\alpha \cdot E_1)$, and $(N, p_n, q_n) \cdot E_1$ operators, respectively.

(v) If we take identity matrix I instead of Euler matrix E_q, then $(T \cdot E_q)$ operators reduce to T-operators which further reduce to Cesáro C_α, Euler E_q, Harmonic H_1,

and Nörlund N_p operators with suitable choice of $a_{n,k}$ as above.

Remark 1. The product summability methods are more powerful than the individual summability methods; for example, the infinite series $1-4\sum_{n=1}^{\infty}(-3)^{n-1}$ is neither C_1-summable nor E_1-summable. However, it can be shown easily that the above series is $(C_1 \cdot E_1)$-summable [1, page 11]. Thus, the product summability methods give an approximation for wider class of functions than the individual methods. Some more examples and recent results on product summability methods can be seen in [2, 3].

Remark 2. As in [4], for operator $(T \cdot C_1)$, the product summability operator $(T \cdot E_q)$ behaves as double digital filter and thus plays an important role in signal theory.

The Fourier-Laguerre expansion of a function $f(x) \in L_1[0,\infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n(f) L_n^{\alpha}(x), \tag{2}$$

where

$$a_n(f) = \left\{\Gamma(\alpha+1)\binom{n+\alpha}{\alpha}\right\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{\alpha}(y)\, dy \tag{3}$$

and $L_n^{\alpha}(x)$ denotes the nth degree Laguerre polynomial of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\omega^n = (1-\omega)^{-\alpha-1} e^{-(x\omega/(\omega-1))}, \tag{4}$$

provided the integral in (3) exists. The elementary properties of Laguerre polynomials can be seen in [5, 6]. Let $s_n(f;x) = \sum_{k=0}^{n} a_k(f) L_k^{\alpha}(x)$ denote the partial sums, called Fourier-Laguerre polynomials of degree n, of the first $(n+1)$ terms of the Fourier-Laguerre series of f in (2). At the point $x=0$,

$$\begin{aligned} s_n(f;0) &= \sum_{k=0}^{n} a_k(f) L_k^{\alpha}(0) \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} f(y) \sum_{k=0}^{n} L_k^{\alpha}(y)\, dy \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{\alpha+1}(y)\, dy, \end{aligned} \tag{5}$$

since $L_n^{\alpha}(0) = \binom{n+\alpha}{\alpha}$ and $\sum_{k=0}^{n} L_k^{\alpha}(x) = L_n^{\alpha+1}(x)$. Thus, using $s_n(f;0)$ and (1), we get

$$\begin{aligned} (TE)_{n,q}(f;0) &= \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} \frac{q^k}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_k^{\alpha+1}(y)\, dy. \end{aligned} \tag{6}$$

We write

$$\phi(y) = \frac{e^{-y} y^{\alpha} [f(y) - f(0)]}{\Gamma(\alpha+1)}. \tag{7}$$

2. Main Results

Various investigators such as Gupta [7], Singh [8], Beohar and Jadiya [9], Lal and Nigam [10], and Nigam and Sharma [11] have studied the degree of approximation of a function using different summability (C_α, E_q, and N_p) methods of series (2) at the point $x=0$ after replacing the continuity condition in Szegö theorem [12] by much lighter conditions. The main aim of this paper is to generalize these earlier results in view of Remark 1. We prove the following.

Theorem 3. *Let $T \equiv (a_{n,k})$ be an infinite lower triangular regular matrix with nonnegative entries. Then, the degree of approximation of a function f by its Fourier-Laguerre expansion (2) at the point $x=0$ using matrix-Euler operators is given by*

$$(TE)_{n,q}(f;0) - f(0) = o(\xi(n)), \tag{8}$$

provided that

$$\Phi(t) = \int_0^t |\phi(y)|\, dy = o\left(t^{\alpha+1}\xi\left(\frac{1}{t}\right)\right), \quad t \longrightarrow 0, \tag{9}$$

$$\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\, dy = o\left(n^{-(2\alpha+1)/4}\xi(n)\right), \tag{10}$$

$$\int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)|\, dy = o(\xi(n)), \quad n \longrightarrow \infty, \tag{11}$$

where δ is a fixed positive constant, $\alpha \in (-1,-1/2)$, and $\xi(t)$ is a positive monotonic increasing function of t such that $\xi(n) \to \infty$ as $n \to \infty$.

Corollary 4 (see [13]). *The degree of approximation of a function f by its Fourier-Laguerre expansion (2) at the point $x=0$ using $[C_1E_q]$ $(q \geq 1)$-means is given:*

$$[C_1E_q]_n(f;0) - f(0) = o(\xi(n)), \tag{12}$$

provided (9), (10), and (11) and supplementary conditions on δ, α, and $\xi(t)$ hold as in Theorem 3.

Proof. If $a_{n,k} = (n+1)^{-1}$, then $(T \cdot E_q)$ operator reduces to $(C_1 \cdot E_q)$ operator. Hence, the proof is completed. □

3. Lemmas

Lemma 5 (see [6, page 177]). *Let α be arbitrary and real and c and ω fixed positive constants and let $n \to \infty$. Then,*

$$L_n^{\alpha}(x) = \begin{cases} O(n^{\alpha}), & 0 \leq x \leq cn^{-1}; \\ x^{-(2\alpha+1)/4} O\left(n^{(2\alpha-1)/4}\right), & cn^{-1} \leq x \leq \omega. \end{cases} \tag{13}$$

Lemma 6 (see [6, page 241]). *Let α and λ be arbitrary and real, $a > 0$, and $0 < \eta < 4$. Then, for $n \to \infty$,*

$$\max_x e^{-x/2} x^{\lambda} |L_n^{\alpha}(x)| \sim n^{Q}, \tag{14}$$

where

$$Q = \begin{cases} \max\left(\lambda - \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{4}\right), & \alpha \le x \le (4-\eta)n; \\ \max\left(\lambda - \frac{1}{3}, \frac{\alpha}{2} - \frac{1}{4}\right), & x \ge a. \end{cases} \tag{15}$$

4. Proof of the Main Results

In view of the orthogonality of Laguerre polynomials [6, page 100] and (6) and (7),

$$\begin{aligned} (TE)_{n,q}(f;0) - f(0) &= \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \sum_{k=0}^{r} \binom{r}{k} q^k \\ &\cdot \int_0^\infty \phi(y) L_k^{\alpha+1}(y)\,dy = \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k \left(\int_0^{c/n} + \int_{c/n}^{\delta} + \int_{\delta}^{n} + \int_{n}^{\infty}\right) \phi(y) \\ &\cdot L_k^{\alpha+1}(y)\,dy = \sum_{j=1}^{4} I_j, \text{ say,} \end{aligned} \tag{16}$$

where

$$\begin{aligned} |I_1| &\le \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k \int_0^{c/n} |\phi(y)| \left|L_k^{\alpha+1}(y)\right| dy \\ &= \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k O\left(k^{\alpha+1}\right) o\left(\frac{c^{\alpha+1}\xi(n/c)}{n^{\alpha+1}}\right) = O\left(n^{\alpha+1}\right) \\ &\cdot o\left(\frac{\xi(n/c)}{n^{\alpha+1}}\right) \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \sum_{k=0}^{r} \binom{r}{k} q^k \\ &= o\left(\xi\left(\frac{n}{c}\right)\right) = o(\xi(n)), \end{aligned} \tag{17}$$

in view of Lemma 5 (first part) and condition (9), and

$$\begin{aligned} |I_2| &\le \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k \int_{c/n}^{\delta} |\phi(y)| \left|L_k^{\alpha+1}(y)\right| dy \\ &= \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k O\left(k^{(2\alpha+1)/4}\right) \int_{c/n}^{\delta} |\phi(y)|\, y^{-(2\alpha+3)/4} dy \\ &= O\left(n^{(2\alpha+1)/4}\right) \int_{c/n}^{\delta} |\phi(y)|\, y^{-(2\alpha+3)/4} dy = o(\xi(n)), \end{aligned} \tag{18}$$

in view of Lemma 5 (second part) and condition (9), integrating by parts and using the argument as in [11, page 6]. Alternatively, using

$$(1+q)^{-r} \sum_{k=0}^{r} \binom{r}{k} q^k k^{(2\alpha+1)/4} \le \left(1 + \frac{2}{k}\right) r^{(2\alpha+1)/4}, \quad \left(0 < q \le K^{-1},\ \alpha \ge -\frac{1}{2}\right), \tag{19}$$

for I_2, (18) can be proved as in [13, pages 37-38]. Now

$$\begin{aligned} |I_3| &\le \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \sum_{k=0}^{r} \binom{r}{k} q^k \\ &\cdot \int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| \\ &\cdot e^{-y/2} y^{(2\alpha+3)/4} \left|L_k^{\alpha+1}(y)\right| dy = \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k O\left(k^{(2\alpha+1)/4}\right) \\ &\cdot \int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\,dy = O\left(n^{(2\alpha+1)/4}\right) \\ &\cdot o\left(n^{-(2\alpha+1)/4}\xi(n)\right) = o(\xi(n)), \end{aligned} \tag{20}$$

using Lemma 6 and condition (10), and

$$\begin{aligned} |I_4| &\le \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k \int_{n}^{\infty} e^{y/2} y^{-(3\alpha+5)/6} |\phi(y)| \\ &\cdot e^{-y/2} y^{(3\alpha+5)/6} \left|L_k^{\alpha+1}(y)\right| dy = \sum_{r=0}^{n} a_{n,r}(1+q)^{-r} \\ &\cdot \sum_{k=0}^{r} \binom{r}{k} q^k O\left(k^{(\alpha+1)/2}\right) \\ &\cdot \int_{n}^{\infty} e^{y/2} y^{-(3\alpha+5)/6} |\phi(y)|\,dy = o(\xi(n)), \end{aligned} \tag{21}$$

using Lemma 6 and condition (11). Combining (17)–(21) and putting them into (16), this completes the proof of Theorem 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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