Research Article

Error Estimation of Functions by Fourier-Laguerre Polynomials Using Matrix-Euler Operators

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Various investigators have studied the degree of approximation of a function using different summability (Cesàro means of order \( \alpha \); Euler \( E_q \) and Nörlund \( N_p \)) means of its Fourier-Laguerre series at the point \( x = 0 \) after replacing the continuity condition in Szegö theorem by much lighter conditions. The product summability methods are more powerful than the individual summability methods and thus give an approximation for wider class of functions than the individual methods. This has motivated us to investigate the error estimation of a function by \((T \cdot E_q)\)-transform of its Fourier-Laguerre series at frontier point \( x = 0 \), where \( T \) is a general lower triangular regular matrix. A particular case, when \( T \) is a Cesàro matrix of order 1, that is, \( C_1 \), has also been discussed as a corollary of main result.

1. Introduction

Let \( \sum u_n := \sum_{n=0}^{\infty} u_n \) be given infinite series with the sequence of its \((n + 1)\)th partial sums \( \{s_n\} \). Define \( E_q^n = \left(1 + q\right)^n \sum_{k=0}^{n} \binom{n}{k} q^k s_k, \forall q > 0 \). If \( \lim_{n \to \infty} E_q^n = S \), then the series \( \sum u_n \) is said to be \( E_q \)-summable to \( S \).

Let \( T \equiv (a_{n,k}) \) be an infinite triangular matrix with real constants. The sequence-to-sequence transformation \( t_n = \sum_{k=0}^{n} a_{n,k} s_k, \forall n \geq 0 \), defines the \( T \)-transform of the sequence \( \{s_n\} \). The series \( \sum u_n \) is said to be \( T \)-summable to \( S \) if \( \lim_{n \to \infty} t_n = S \). Throughout this paper, \( T \equiv (a_{n,k}) \) has nonnegative entries with row sums one. \( T \) is said to be regular if it is limit preserving over the space of convergent sequences. Thus, \( T \) behaves as a linear operator.

The \((T \cdot E_q)\)-transform of \( \{s_n\} \), denoted by \((TE)_{n,q}\), are defined by

\[
(TE)_{n,q} = \sum_{r=0}^{n} a_{n,r} E_q^r = \sum_{r=0}^{n} a_{n,r} \left(1 + q\right)^r \sum_{k=0}^{r} \binom{r}{k} q^k s_k.
\]

If \((TE)_{n,q} \to S\) as \( n \to \infty \), then the series \( \sum_{n=0}^{\infty} u_n \) is said to be \((T \cdot E_q)\)-summable to \( S \). The regularity of \((T \cdot E_q)\) method follows from the regularity of \( E_q \) method as well as \( T \)-method and thus the matrix \((T \cdot E_q)\) behaves as a linear operator. Some important particular cases of the matrix-Euler operator are as follows:

(i) If \( a_{n,k} = 1/(n - k + 1) \log(n + 1) \), then we get \((H_1 \cdot E_q)\) operator.

(ii) Let \( \{p_n\} \) be a sequence of real, nonnegative numbers such that \( P_n = \sum_{k=0}^{n} p_k \neq 0, P_{-1} = 0 = p_{-1} \), and \( p_0 \neq 0 \). If \( a_{n,k} = p_{n-k}/P_n \), then we get \((N_p \cdot E_q)\) operator. A special case in which \( p_n = \binom{n-1}{\alpha-1} \), \( \alpha > 0 \); then \((N_p \cdot E_q)\) operator further reduces to \((C_\alpha \cdot E_q)\) operator.

(iii) If \( a_{n,k} = p_{n-k}q_k/R_n \), where \( R_n = \sum_{k=0}^{n} p_{n-k}q_k \neq 0 \), then we get \((N, p_n, q_n) \cdot E_q\) operator.

(iv) If \( q = 1 \) in above cases, then we get \((H_1 \cdot E_1)\), \((N_p \cdot E_1)\), \((C_\alpha \cdot E_1)\), and \((N, p_n, q_n) \cdot E_1\) operators, respectively.

(v) If we take identity matrix \( I \) instead of Euler matrix \( E_q \), then \((T \cdot E_q)\) operators reduce to \( T \)-operators which further reduce to Cesàro \( C_\alpha \), Euler \( E_q \), Harmonic \( H_1 \),
2. Main Results

Various investigators such as Gupta [7], Singh [8], Beohar and Jadiya [9], Lal and Nigam [10], and Nigam and Sharma [11] have studied the degree of approximation of a function using different summability \((C_α, E_q, N_p)\) methods of series (2) at the point \(x = 0\) after replacing the continuity condition in Szegö theorem [12] by much lighter conditions. The main aim of this paper is to generalize these earlier results in view of Remark 1. We prove the following.

**Theorem 3.** Let \(T \equiv (a_{n,k})\) be an infinite lower triangular regular matrix with nonnegative entries. Then, the degree of approximation of a function \(f\) by its Fourier-Laguerre expansion (2) at the point \(x = 0\) using matrix-Euler operators is given by

\[
(TE)_{n,q}(f;0) - f(0) = o(\xi(n)),
\]

provided that

\[
\Phi(t) = \int_0^{|t^{|\frac{1}{4}}(\frac{1}{1})|} |\phi(y)| dy = o\left(\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} \\
\end{array}\right) \left[\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} \\
\end{array}\right], t \to 0,
\]

\[
\int_{\frac{1}{\delta}}^{\frac{1}{n}} \frac{e^{y^2/2}}{y^{-(2a+3)/4}} |\phi(y)| dy = o\left(\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} \\
\end{array}\right) \left[\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} \\
\end{array}\right] (\xi(n)), \quad n \to \infty,
\]

where \(\delta\) is a fixed positive constant, \(\alpha \in (-1, -1/2)\), and \(\xi(t)\) is a positive monotonic increasing function of \(t\) such that \(\xi(n) \to \infty\) as \(n \to \infty\).

**Corollary 4** (see [13]). The degree of approximation of a function \(f\) by its Fourier-Laguerre expansion (2) at the point \(x = 0\) using \([C_1 E_q]\) \((q \geq 1)\)-means is given:

\[
[C_1 E_q]_{n}(f;0) - f(0) = o(\xi(n)),
\]

provided (9), (10), and (11) and supplementary conditions on \(\alpha, \xi(t)\) hold as in Theorem 3.

**Proof.** If \(a_{n,k} = (n + 1)^{-1}\), then \((T \cdot E_q)\) operator reduces to \((C_1 \cdot E_q)\) operator. Hence, the proof is completed. \(\square\)

3. Lemmas

**Lemma 5** (see [6, page 177]). Let \(\alpha\) be arbitrary and real and \(c\) and \(\omega\) fixed positive constants and let \(n \to \infty\). Then,

\[
L_n^\alpha(x) = \begin{cases}
O(n^\alpha), & 0 \leq x \leq cn^{-1}; \\
\alpha^{-(2a+1)/4}O(n^{(2a-1)/4}), & cn^{-1} \leq x \leq \omega.
\end{cases}
\]

**Lemma 6** (see [6, page 241]). Let \(\alpha\) and \(\lambda\) be arbitrary and real, \(\alpha > 0\), and \(0 < \eta < 4\). Then, for \(n \to \infty\),

\[
\max_{x} e^{-x/2} x^{\lambda} |L_n^\alpha(x)| \sim n^{\alpha^2}.
\]
where
\[
Q = \begin{cases} 
\max \left( \lambda - \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{4} \right), & x \leq (4 - \eta) n; \\
\max \left( \lambda - \frac{1}{3}, \frac{\alpha}{2} - \frac{1}{4} \right), & x \geq a.
\end{cases}
\]  

(15)

4. Proof of the Main Results

In view of the orthogonality of Laguerre polynomials [6, page 100] and (6) and (7),
\[
(TE)_{n,q} (f; 0) - f (0) = \sum_{r=0}^{n} a_{n,r} \left( 1 + q \right)^{-r} \sum_{k=0}^{r} \binom{r}{k} q^{k} \cdot \int_{0}^{\infty} \phi (y) L_{k}^{\alpha+1} (y) \, dy = \sum_{r=0}^{n} a_{n,r} \left( 1 + q \right)^{-r} \cdot \sum_{k=0}^{r} \binom{r}{k} q^{k} \cdot O (\xi (n)) = o (\xi (n)),
\]

(16)

using Lemma 6 and condition (10), and
\[
\int_{0}^{\infty} e^{y/2} y^{-(3\alpha+5)/6} |\phi (y)| \, dy = O \left( n^{(2\alpha+1)/4} \right)
\]

(17)

using Lemma 6 and condition (11). Combining (17)–(21) and putting them into (16), this completes the proof of Theorem 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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