Research Article

An Efficient Method to Find Solutions for Transcendental Equations with Several Roots

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This paper presents a method for obtaining a solution for all the roots of a transcendental equation within a bounded region by finding a polynomial equation with the same roots as the transcendental equation. The proposed method is developed using Cauchy's integral theorem for complex variables and transforms the problem of finding the roots of a transcendental equation into an equivalent problem of finding roots of a polynomial equation with exactly the same roots. The interesting result is that the coefficients of the polynomial form a vector which lies in the null space of a Hankel matrix made up of the Fourier series coefficients of the inverse of the original transcendental equation. Then the explicit solution can be readily obtained using the complex fast Fourier transform. To conclude, the authors present an example by solving for the first three eigenvalues of the 1D transient heat conduction problem.

1. Introduction

The determination of roots of transcendental functions is a problem commonly encountered in a broad spectrum of engineering applications, such as heat transfer (e.g., [1–4]), dynamics and control (e.g., [5]), sound and vibration (e.g., [6, 7]), and quantum mechanics (e.g., [8]). There exist a wide variety of numerical methods that are useful for approximating the solution to any desired degree of accuracy. General descriptions of common root finding techniques are available in most textbooks in the area of numerical analysis, for example, Chapra and Canale [9].

Explicit methods have also been developed to find roots of transcendental equations. Explicit solutions can provide more insight into the problem under consideration. They are also useful, for example, in the development of analytical derivatives for uncertainty analyses and sensitivity studies or for checking convergence of approximate root finding techniques [3]. Leathers and McCormick [10] addressed a methodology for obtaining explicit solutions to several transcendental problems arising in heat transfer. The general approach used was based on methods of Muskhelishvili [11]

and was developed by Burniston and Siewert [12]. This approach “depends on formulating an appropriate Riemann problem of complex variable theory and then expressing the solution of the transcendental equation in terms of a canonical solution of that problem” [10]. Luck and Stevens [13] formulated an explicit expression for a single root of any analytic transcendental function. Their method is based on Cauchy's integral theorem and uses only basic concepts of complex integration to determine the root of a function by locating the singularity of the reciprocal of the function. It has now been found that this idea was initially proposed by Jackson [14, 15]. The current paper extends their idea presenting a method for finding several roots of transcendental equations within a bounded region by transforming the problem into an equivalent problem of finding roots of a polynomial equation with exactly the same roots. It follows that an explicit solution is readily available through the complex fast Fourier transform (cfft) when dealing with up to four roots at a time. For five or more roots the solution is implicit according to the Abel-Ruffini theorem [16]. The implicitness of finding roots of fifth- and higher-order polynomials is not really an issue as algorithms

designed for this task are readily available (McNamee [17–20], McNamee and Pan [21]) or larger problems can be divided into smaller problems by analyzing nonoverlapping bounded regions with four roots or less in each region.

2. Method Development

Suppose $f(x)$ is a transcendental function with N roots within an interval of width $2 \cdot R$ and center h such that $h - R < x < h + R$. Let $p(x)$ be an N th-order polynomial with roots located at exactly the same locations as the roots of $f(x)$ in the interval $h - R < x < h + R$. The ratio $p(x)/f(x)$ is an analytic function in the interval $h - R < x < h + R$ because the poles of $1/f(x)$ in the interval are cancelled out by the roots of $p(x)$. The next step in the analysis is to normalize the variable of interest. Let

$$y = \frac{x - h}{R} \quad (1)$$

such that $-1 < y < 1$. Functions $f(x)$ and $p(x)$ can be recast as

$$\begin{aligned} f(x) &= f(R \cdot y + h) = F(y), \\ p(x) &= p(R \cdot y + h) = P(y), \end{aligned} \quad (2)$$

where

$$P(y) = \sum_{k=0}^N (c_k y^k) \quad (3)$$

and N is the number of roots in the interval.

Now consider the function $P(y)/F(y)$ in the interval $-1 < y < 1$. If the analysis is extended into the complex plane (by substituting the real variable y with the complex variable z), integration around a circular path of radius 1 yields

$$\oint \frac{P(z)}{F(z)} dz = 0 \quad (4)$$

which is a consequence of Cauchy's theorem regarding the line integral of a function that is analytic on and inside of the contour defined by the closed path of integration.

Equation (4) can be used to find the coefficients c_k of polynomial $P(y)$ as follows:

$$\oint \frac{P(z)}{F(z)} dz = \oint \frac{\sum_{k=0}^N (c_k z^k)}{F(z)} dz = \sum_{k=0}^N \left(c_k \oint \frac{z^k}{F(z)} dz \right) = 0. \quad (5)$$

Note that, given a positive integer r , it is also true that

$$\oint \frac{z^r P(z)}{F(z)} dz = 0 \quad (6)$$

because the function being integrated remains analytic when multiplied by z^r . Following the approach presented in (5)

$$\oint \frac{z^r P(z)}{F(z)} dz = \sum_{k=0}^N \left(c_k \oint \frac{z^{k+r}}{F(z)} dz \right) = 0. \quad (7)$$

Following is the procedure for obtaining coefficients c_k . First, the integrals are evaluated by substituting

$$z = e^{j\theta}, \quad (8)$$

$$dz = j e^{j\theta} d\theta.$$

Let

$$G(\theta) = \frac{1}{F(e^{j\theta})}. \quad (9)$$

It follows that

$$\oint \frac{z^{r+k}}{F(z)} dz = \int_0^{2\pi} G(\theta) (e^{j\theta})^{r+k} j e^{j\theta} d\theta = j \int_0^{2\pi} G(\theta) e^{j(r+k+1)\theta} d\theta. \quad (10)$$

Another simplification can be implemented by defining

$$G_k = \int_0^{2\pi} G(\theta) e^{jk\theta} d\theta. \quad (11)$$

Note that G_k can be interpreted as the k th complex Fourier series coefficient of $G(\theta)$. Finally, substituting G_k into (7) and cancelling j yield

$$\sum_{k=0}^N (c_k G_{r+k+1}) = 0. \quad (12)$$

Equation (12) is the equation used to find coefficients $c_0, c_1, c_2, \dots, c_N$. Note that (12) is true for any positive integer value of r . This means that one can “create” as many equations as there are unknown coefficients c_k . Varying r from 0 to $N-1$, the following system of equations is obtained:

$$\begin{bmatrix} G_1 & G_2 & G_3 & G_4 & \dots & G_{N+1} \\ G_2 & G_3 & G_4 & G_5 & \dots & G_{N+2} \\ G_3 & G_4 & G_5 & G_6 & \dots & G_{N+3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ G_N & G_{N+1} & G_{N+2} & G_{N+3} & \dots & G_{2N} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = 0. \quad (13)$$

Because (13) represents a system of linearly dependent equations, that is, having fewer equations than unknowns, additional equations are necessary to obtain a solution. One possible equation consists of setting c_N equal to any real nonzero number, which allows moving the last column of the G matrix to the right hand side.

Equation (13) is worthy of comment. Matrix G is composed solely of Fourier series coefficients, implying that the equations do not depend on normalization factors used to compute the Fourier series coefficients. In practice, the coefficients of G are approximated through the complex fast Fourier transform (cfft). The cfft approximation is less accurate for the last column of the G matrix because this column corresponds to high-frequency terms. It follows that moving the last column of the G matrix to the right hand side of the equation results in a slightly more robust solution. Furthermore, the system of equations constitutes a Hankel matrix for which an inverse can be easily obtained [22]. Therefore, (13) can be quickly implemented and solved for the coefficients c_k . Finally, the symmetry or “structure” of (13) leads the authors to believe there is a deeper meaning to this system of equations.

FIGURE 1: Plot of $f(x)$.

3. Example

To illustrate the current method, a simple yet helpful example is described below. The solution of the unsteady heat conduction equation yields the characteristic equation or eigenfunction of the following form [23]:

$$x \tan x = \text{Bi}. \quad (14)$$

Equation (14) is a transcendental equation with an infinite number of roots called characteristic values or eigenvalues. The characteristic equation in this case is implicit. Arbitrarily assuming $\text{Bi} = 1$, (14) can be rewritten as

$$x \tan x - 1 = 0. \quad (15)$$

Let

$$f(x) = x \tan x - 1. \quad (16)$$

The plot of $f(x)$ is shown on Figure 1.

The method described in the previous section is now applied to solve for the first three positive roots of (16). Figure 1 illustrates that the three roots occur in the interval $0 < x < 8$, suggesting $h = 4$, $R = 4$, and $N = 3$. Note that one simple way of approximating coefficients G_k is to discretize $G(\theta)$ for $\theta \in [0, 2\pi)$ and to use the complex fast Fourier transform (cfft) readily available in many commercial mathematical software packages (e.g., MATLAB and Mathcad), rather than performing the integral of (11). With $c_3 = 1$ and $N = 3$, (13) becomes

$$\begin{bmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_4 \\ G_3 & G_4 & G_5 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = - \begin{Bmatrix} G_4 \\ G_5 \\ G_6 \end{Bmatrix}. \quad (17)$$

Discretizing θ into 256 equidistant points, the cfft yields

$$\begin{bmatrix} 2.846 & -0.768 & 1.014 \\ -0.768 & 1.014 & -0.476 \\ 1.014 & -0.476 & 0.559 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0.476 \\ -0.559 \\ 0.324 \end{Bmatrix}. \quad (18)$$

Solving this system of equations gives

$$c = \begin{Bmatrix} -0.069 \\ -0.453 \\ 0.319 \\ 1 \end{Bmatrix} \quad (19)$$

FIGURE 2: Plot of $P(y)$ and $F(y)$ with locations of the roots.

which translates into

$$P(y) = -0.069 - 0.453y + 0.319y^2 + y^3. \quad (20)$$

Figure 2 plots the original transcendental function $F(y)$ and the polynomial $P(y)$. The plot clearly shows that the polynomial has the same roots as the original function $F(y)$; that is, $y = -0.785$, $y = -0.144$, and $y = 0.609$. The roots of a cubic polynomial can be found using the explicit algebraic expressions given by many authors, for example, Abramowitz and Stegun [24], Press et al. [25], and Weisstein [26]. Finally, undoing the variable normalization yields $x = 0.860$, $x = 3.426$, and $x = 6.437$ as the roots of $f(x)$ or eigenvalues of the original problem.

4. Conclusion

The current paper presents a method for finding simultaneously several roots of a transcendental equation $f(x)$ within a bounded region, by transforming the problem into an equivalent problem of finding roots of a polynomial equation $p(x)$ with exactly the same roots. The method requires that the ratio $p(x)/f(x)$ be analytic in the interval under study. The concept shown here is that the correlation between the roots of a transcendental function and the Fourier series coefficients of its inverse can be expressed with a polynomial and proven with Cauchy's integral theorem.

The advantage of the current method is the possibility of solving for several roots in one shot and is most evident when dealing with an implicit problem as shown in the previous example. In this case, an implicit problem was transformed into a simpler explicit one. The current method remains explicit when dealing with up to four roots. However, the implicitness of finding roots of higher-order polynomials is not really an issue as algorithms designed for this task are readily available or larger problems can be divided into smaller problems with four roots each by selecting appropriate intervals as illustrated in Section 3. In any case, the current method provides a relatively simple solution procedure for a group of problems encountered in many engineering applications.

Notation

Bi: Biot number
 c_k : k th coefficient of the polynomial $P(y)$
 f : Function of x
 F : Function of y
 G : Function of θ
 G_k : k th Fourier series coefficient of function G
 h : Center of a circle in the complex plane
 N : Number of roots
 p : Polynomial, function of x
 P : Polynomial, function of y
 r : Positive integer
 R : Radius of a circle in the complex plane
 x : Independent variable
 y : Normalized variable
 z : Complex variable
 θ : Angle in the complex plane.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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