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Research Article

A Mixed Cooperative Dual to the Nash Equilibrium

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A mixed dual to the Nash equilibrium is defined for n-person games in strategic form. In a Nash equilibrium every player's mixed strategy maximizes his own expected payoff for the other $n-1$ players' strategies. Conversely, in the dual equilibrium every $n-1$ players have mixed strategies that maximize the remaining player's expected payoff. Hence this dual equilibrium models mutual support and cooperation to extend the Berge equilibrium from pure to mixed strategies. This dual equilibrium is compared and related to the mixed Nash equilibrium, and both topological and algebraic conditions are given for the existence of the dual. Computational issues are discussed, and it is shown that for each $n > 2$ there exists a game for which no dual equilibrium exists.

1. Introduction

The mathematical analysis of both competition and cooperation falls within the realm of game theory, whose systematic development began with von Neumann and Morgenstern [1]. For a game with n players, Nash [2] later assumed that the players are rational and hence selfish. He then defined an equilibrium in which every player's strategy maximizes his payoff for the other $n-1$ players' strategies. Modern game theory [3–5] has long required that any rational solution concept for a game should be a Nash equilibrium. However, in games known as social dilemmas [6, 7] selfish behavior conflicts with group interests; and individual players do better by cooperating. The less well-known Berge equilibrium was introduced for pure strategies in [8] and formalized in [9]. A Berge equilibrium is a pure strategy profile in which every $n-1$ players have strategies that maximize the remaining player's payoff. It has been increasingly studied as a model of mutual support and cooperation as in [10–15].

In this paper we consider n-person games in strategic form and extend the Berge equilibrium from pure to mixed strategies to provide a dual to the Nash equilibrium. In this dual equilibrium, every $n-1$ players have strategies that maximize the remaining player's expected payoff. The Nash equilibrium and the dual equilibrium thus model opposite decision criteria for choosing the n player's actions, regardless of whether these actions are independently selected by the players, are coordinated by the players, or are even prescribed. For example, the results here are valid if an arbiter assigns actions to the players as in [16].

In Section 2 the dual equilibrium is formally defined and related to the Nash equilibrium. In Section 3 topological and algebraic conditions are established for the existence of a dual equilibrium. In Section 4 computation complexity issues are discussed. In Section 5 it is shown that for each $n > 2$ there exists a game for which no dual equilibrium exists. Conclusions are stated in Section 6.

2. Relationships between the Dual and Nash Equilibria

The following definitions and notation are used. Let $G_n = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ be an n-person game in strategic form, where $I = \{1, \ldots, n\}$ is the index set for the n players, S_i is the finite set of m_i actions for player i, and $u_i(s_1, \ldots, s_n)$ is the von Neumann-Morgenstern utility of player i for a pure strategy profile $s = (s_1, \ldots, s_n) \in \bigtimes_{j\in I} S_j = S$. Write $u(s)$ for $(u_1(s), \ldots, u_n(s))$ and denote the set of mixed strategies for player i by ΔS_i. A mixed-strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \bigtimes_{j\in I} \Delta S_j = \Delta S$ is an n-tuple of individual randomized strategies, where $\sigma_i(s_i)$ is the probability that player i uses the pure strategy $s_i \in S_i$. For any $\tau_i \in \Delta S_i$,

abbreviate the strategy profile $(\sigma_1, \ldots, \sigma_{i-1}, \tau_i, \sigma_{i+1}, \ldots, \sigma_n)$ by (τ_i, σ_{-i}). Similarly write $I_{-i} = I - \{i\}$, $S_{-i} = \bigtimes_{j \in I_{-i}} S_j$, and $\Delta S_{-i} = \bigtimes_{j \in I_{-i}} \Delta S_j$. For $\forall i \in I$, extend $u_i : S \to R^1$ from S to ΔS by defining the expected utility function $u_i(\sigma) = \sum_{(s_1,\ldots,s_n) \in S} \prod_{j \in I} \sigma_j(s_j) u_i(s_1, \ldots, s_n)$. Writing $\sigma_{-i}(s_{-i}) = \prod_{j \in I_{-i}} \sigma_j(s_j)$, $\forall s_{-i} \in S_{-i}$, and noting that a pure strategy is a special case of a mixed strategy, we can derive the following useful identities:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}), \tag{1}$$

$$u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})\, u_i(s_i, s_{-i}), \tag{2}$$

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})\, u_i(\sigma_i, s_{-i}), \tag{3}$$

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, s_{-i}). \tag{4}$$

The Nash equilibrium is now formally defined for comparison with the dual equilibrium.

Definition 1 (NE). The strategy profile σ^* is a Nash equilibrium (NE) of G_n if and only if

$$u_i(\sigma^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma^*_{-i}), \quad \forall i \in I. \tag{5}$$

Definition 2 (DE). The strategy profile σ^* is a dual equilibrium (DE) of G_n if and only if

$$u_i(\sigma^*) = \max_{\sigma_{-i} \in \Delta S_{-i}} u_i(\sigma^*_i, \sigma_{-i}), \quad \forall i \in I. \tag{6}$$

Designating a strategy profile σ^* satisfying (6) as a dual equilibrium may be justified as follows. Equations (5) and (6) involve systems of optimization problems defining NEs and DEs. These systems may be regarded as symmetric dual systems in the sense that Definition 1 becomes Definition 2 and vice versa when the subscripts i and $-i$ are interchanged on the right (but not left) sides of (5) and (6). Thus if (5) is considered the primal system and (6) the dual system, then the dual system of the dual system is the primal system. This duality is manifested in the fact that in Definition 1 every player's NE strategy selfishly maximizes his own payoff for the other $n-1$ players' NE strategies, while the opposite situation occurs in Definition 2. For a DE, every $n-1$ players have DE strategies that maximize the remaining player's payoff. Thus a DE exhibits the *musketeer property* in [17] of "all for one, one for all." Alternately, in an NE no individual player can improve his expected payoff with a unilateral change in strategy, while in a DE no individual player's payoff can be improved by any change in the remaining players' DE strategies.

The DE is the mixed extension of the Berge equilibrium (BE) of [11–14] and the mutual-max outcome of [18]. For a BE, which involves only pure strategies, mutual cooperation occurs in a single game. For a mixed-strategy DE, the probability $\sigma_i(s_i)$ can be interpreted in two ways [19]. The first interpretation is the fraction of time in the long run that player i would choose to play pure strategy s_i in a series of repeated games in which the n mutually cooperative players invoke decision criterion (6). The second interpretation of $\sigma_i(s_i)$ considers a mixed-strategy equilibrium as a steady state in a large population with n subpopulations. In this case $\sigma_i(s_i)$ is the fraction of subpopulation i preferring pure strategy s_i according to decision criteria (6).

Some results relating DE and NE are now established. Let the function $d : I \to I$ be one-to-one on the index set I. Then d is a derangement of I if and only if $d(i) \neq i$, $\forall i \in I$. Thus associated with $G_n = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ is the deranged game $G_n(d) = (I, (S_i)_{i \in I}, (v_i)_{i \in I})$, where $v_i = u_{d(i)}$. In other words, for $G_n(d)$ the expected utility function of any player i is $u_{d(i)}$. In addition, for a strategy profile σ of G_n, let $\sigma(d)$ be its deranged strategy profile $(\sigma_1(d), \ldots, \sigma_n(d))$, where $\sigma_j(d)$ is the unique σ_i for which $d(i) = j$. Obviously if σ is a DE of G_n, then the deranged strategy profile $\sigma(d)$ is also a DE of the deranged game $G_n(d)$.

Theorem 3. *If σ^* is a dual equilibrium for any n-person game G_n, then $\sigma^*(d)$ is a Nash equilibrium for G_n for any derangement d of I.*

Proof. It is first proved that if σ^* is a DE of a game G_n, then σ^* is an NE of $G_n(d)$ for any derangement d of I. Let $i \in I$ and d be an arbitrary derangement such that $d(i) = j$. From (6)

$$u_j\left(\sigma^*_j, \sigma_{-j}\right) \leq u_j(\sigma^*), \quad \forall \sigma_{-j} \in \Delta S_{-j}. \tag{7}$$

Expanding (7) gives that

$$u_j\left(\sigma_1, \ldots, \sigma_{j-1}, \sigma^*_j, \sigma_{j+1}, \ldots, \sigma_n\right) \leq u_j(\sigma^*), \quad \forall \sigma_k \in \Delta S_k,\ k \in I_{-j}. \tag{8}$$

In (8) set $\sigma_k = \sigma^*_k$ for $k \in I$, $k \neq i, j$. Hence, for $j = d(i)$,

$$u_j(\sigma^*_1, \ldots, \sigma^*_{i-1}, \sigma_i, \sigma^*_{i+1}, \ldots, \sigma^*_n) = u_j(\sigma_i, \sigma^*_{-i}) \leq u_j(\sigma^*), \quad \forall \sigma_i \in \Delta S_i. \tag{9}$$

From (9), for $\forall i \in I$ and any derangement d of I,

$$u_{d(i)}(\sigma_i, \sigma^*_{-i}) \leq u_{d(i)}(\sigma^*), \quad \forall \sigma_i \in \Delta S_i. \tag{10}$$

Thus from (10) and (5), σ^* is an NE for $G_n(d)$. It follows that if σ^* is a DE of any n-person game G_n, then σ^* is an NE of $G_n(d)$ for any derangement d of I.

We next use the fact that if σ^* is a DE of G_n, the deranged strategy profile $\sigma^*(d)$ is also a DE of the deranged game $G_n(d)$. Then from the first part of the proof, $\sigma^*(d)$ is an NE of the original game G_n obtained by applying the inverse derangement d^{-1} of d to $G_n(d)$. □

Corollary 4. *G_n has at least as many NEs as DEs.*

Proof. Let d be the cyclic derangement given by $d(i) = i + 1$, $i = 1, \ldots, n-1$, and $d(n) = 1$. Since d is one-to-one, it follows from Theorem 3 that there is a one-to-one correspondence between the DEs of G_n and a subset of the NEs for G_n. Hence there are at least as many NEs as DEs. □

Table 1: Payoff matrix for G_2.

		Player 2	
		s_2	t_2
Player 1	s_1	(3, 5)	(2, 8)
	t_1	(2, 6)	(5, 4)

In Section 5 it is shown that there can be more NEs than DEs. The converse of Theorem 3 is true in general only for two-person games, in which case an argument similar to that for Theorem 3 yields the following result of [20], where the zero-sum case is also considered.

Corollary 5. *A strategy profile σ^* is a DE for the two-person game where Player 1 has utility function u_1 and Player 2 has utility function u_2 if and only if σ^* is an NE for the dual two-person game where Player 1 has utility function u_2 and Player 2 has utility function u_1.*

Example 6. Consider G_2, with the payoff matrix of Table 1, where Player 1 has pure strategies $S_1 = \{s_1, t_1\}$ and Player 2 has pure strategies $S_2 = \{s_2, t_2\}$. For simplicity, write $\sigma_1(s_1) = p_1$, $\sigma_1(t_1) = p_2$ for Player 1 and $\sigma_2(s_2) = q_1$, $\sigma_2(t_2) = q_2$ for Player 2. We apply Corollary 5. Using standard procedures for finding all NEs of two-person games [3], we obtain all DEs of G_2 by determining all NEs associated with the payoff matrix obtained by interchanging the payoffs in Table 1. The unique DE has $p_1 = 0.75$, $p_2 = 0.25$ and $q_1 = 0.80$, $q_2 = 0.20$, with expected payoffs (2.75, 5.60) with respect to the original payoff matrix of Table 1. By comparison, the unique NE of G_2 has $p_1 = 0.40$, $p_2 = 0.60$ and $q_1 = 0.75$, $q_2 = 0.25$, with expected payoffs (2.75, 5.65). Thus, the expected payoff vector for the NE dominates that of the DE in a Pareto sense. In general, the expected DE payoff $u_i(\sigma^*)$ for any player i in G_n may be larger than, smaller than, or equal to the expected NE payoff for i. In this regard, neither mutual cooperation nor selfishness is necessarily better than the other.

3. Existence Conditions for Dual Equilibria

In this section sufficient conditions are presented for the existence of a DE for G_n. For any two-person game, Corollary 5 implies that a DE exists since an NE exists [2]. For $n > 2$, however, the existence of a DE is more complicated. The following preliminaries are needed.

Definition 7. A correspondence $\Phi : X \mapsto 2^Y$ is a set-valued mapping Φ such that $\Phi(x)$ is a subset of Y, $\forall x \in X$. The domain of Φ is $\{x : x \in X, \Phi(x) \neq \varnothing\}$, the range is $\bigcup_{x\in X} \Phi(x)$, and the graph is $\{(x, y) : x \in X, y \in Y, y \in \Phi(x)\}$.

Definition 8. Let X and Y be topological vector spaces. The correspondence $\Phi : X \mapsto 2^Y$ is upper semicontinuous at $x \in X$ if and only if whenever the sequence $\langle x^k, y^k\rangle_{k=1}^{\infty}$ in $X \times Y$ has $\lim_{k\to\infty}(x^k, x^k) = (x, y) \in X \times Y$, then $y \in \Phi(x)$.

Definition 9. Let X be a topological vector space, and let C be a nonempty convex set in X. The function $f : X \to R^1$ is quasiconcave on C if and only if the level set $L_\alpha = \{x \in C : f(x) \geq \alpha\}$ is a convex set for all real α.

Result 1 (Kakutani fixed point theorem [21]). Let C be a nonempty compact convex set in a finite-dimensional topological vector space X, and let $\Phi : C \mapsto 2^C$. If Φ is upper semicontinuous at $\forall x \in C$ and if $\Phi(x)$ is a nonempty convex subset of C, then $\exists x^* \in C$ for which $x^* \in \Phi(x^*)$. The point x^* is called a fixed point of Φ on C.

Topological conditions for the existence of a dual equilibrium are now derived. For each player $i \in I$ and each $\sigma_i \in \Delta S_i$, denote the best support correspondence $\Gamma_i : \Delta S_i \mapsto \Delta S$ by $\Gamma_i(\sigma_i) = \{(\sigma_i, \sigma^*_{-i}) \in \Delta S : \sigma^*_{-i} \in \arg\max_{\sigma_{-i}\in\Delta S_{-i}} u_i(\sigma_i, \sigma_{-i})\}$. From [3] the sets ΔS, ΔS_i, and ΔS_{-i} can be identified, respectively, with nonempty, convex, and compact subsets of the finite-dimensional Euclidean spaces R^m for $m = \sum_{i\in I} m_i$, R^{m_i}, and $R^{m_{-i}}$ for $m_{-i} = \sum_{j\in I_{-i}} m_j$. For each $i \in I$, $u_i(\sigma_i, \sigma_{-i})$ is continuous with respect to $\sigma_{-i} \in \Delta S_{-i}$ and $\sigma_i \in \Delta S_i$, so the sets $\arg\max_{\sigma_{-i}\in\Delta S_{-i}} u_i(\sigma_i, \sigma_{-i})$ and $\Gamma_i(\sigma_i)$ are nonempty. Hence the correspondence $\Gamma : \Delta S \mapsto \Delta S$ given by $\Gamma(\sigma^*) = \bigcap_{i\in N} \Gamma_i(\sigma^*_i)$ is nonempty if and only if σ^* satisfies (6). The following lemma has thus been established.

Lemma 10. *A strategy profile σ^* is a DE for the game G_n if and only if σ^* is a fixed point of Γ on ΔS.*

Result 1 and Lemma 10 are next used to prove a topological sufficient condition.

Theorem 11. *If $\Gamma(\sigma)$ is a nonempty convex set, $\forall\sigma \in \Delta S$, then there exists a DE σ^* for the game G_n.*

Proof. We show that Γ is upper semicontinuous on the nonempty, compact, and convex set ΔS. Consider convergent sequences $\langle\sigma^k\rangle_{k=1}^{\infty}$ and $\langle\tau^k\rangle_{k=1}^{\infty}$ in ΔS, where $\lim_{k\to\infty}\sigma^k = \sigma$, $\lim_{k\to\infty}\tau^k = \tau$, and $\tau^k \in \Gamma(\sigma^k)$, $\forall k \in \{1, 2, 3, \ldots\}$. For $\forall i \in I$ it follows from the definition of Γ that $u_i(\sigma^k_i, \tau^k_{-i}) \geq u_i(\sigma^k_i, \sigma_{-i})$, $\forall\sigma_{-i} \in \Delta S_{-i}$. But the expected utility function u_i is continuous on ΔS, so $u_i(\sigma_i, \tau_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$, $\forall\sigma_{-i} \in \Delta S_{-i}$. Hence $\tau \in \Gamma_i(\sigma_i,)$, $\forall i \in N$, and so $\tau \in \Gamma(\sigma)$. Thus Γ is upper semicontinuous on ΔS. By assumption $\Gamma(\sigma)$ is also a nonempty convex set of ΔS, $\forall\sigma \in \Delta S$. Hence from Result 1 there exists $\sigma^* \in \Delta S$ for which $\sigma^* \in \Gamma(\sigma^*)$, and so σ^* is a DE by Lemma 10. □

More restrictive sufficient conditions are given as a corollary.

Corollary 12. *If $u_i(\sigma_i, \sigma_{-i})$, $\forall i \in I$, is a quasiconcave function of σ_{-i} for each fixed $\sigma_i \in \Delta S_i$ and if $\Gamma(\sigma)$ is nonempty, $\forall\sigma \in \Delta S$, then there exists a DE σ^* for the game G_n.*

Proof. Fix $i \in N$ and $\sigma_i \in \Delta S_i$. Since $u_i(\sigma_i, \sigma_{-i})$ is a quasiconcave on ΔS_{-i}, it follows that the level set $L_\alpha = \{\sigma_{-i} \in \Delta S_{-i} : u_i(\sigma_i, \sigma_{-i}) \geq \alpha\}$ is a convex set in ΔS_{-i} for all real α. In

particular, setting $\alpha^* = \max_{\sigma_{-i} \in \Delta S_{-i}} u_i(\sigma_i, \sigma_{-i})$ implies that $u_i(\sigma_i, \lambda\sigma'_{-i} + (1-\lambda)\sigma''_{-i}) \geq \max_{\sigma_{-i} \in \Delta S_{-i}} u_i(\sigma_i, \sigma_{-i})$, $\forall \sigma'_{-i}, \sigma''_{-i} \in \Delta S_{-i}$, $\forall \lambda \in [0,1]$. Thus $(\sigma_i, \lambda\sigma'_{-i} + (1-\lambda)\sigma''_{-i}) \in \Gamma_i(\sigma_i)$, and $\Gamma_i(\sigma_i)$ is a nonempty convex set, $\forall i \in N, \forall \sigma_i \in \Delta S_i$. Hence, for $\forall \sigma \in \Delta S$, $\Gamma(\sigma) = \bigcap_{i \in N} \Gamma_i(\sigma_i)$ is convex in addition to being nonempty by assumption, so Theorem 11 applies. □

4. Computational Complexity

For the DE to be useful as a solution concept, it must be computable. Since NEs can be difficult to obtain [22, 23], we briefly discuss the computational complexity of DEs as compared to NEs. Let $N = n\prod_{j \in I} m_j$ = (# players) (# all possible joint actions) denote the size of the input data for the game $G_n = (I, (S_i)_{i \in N}, (u_i)_{i \in N})$. It is shown in [22] that $O(N)$ is the computational complexity on a deterministic Turing machine for checking whether a pure NE exists for G_n and then computing all such NEs. An algorithm described in [20] uses a regret matrix to compute all pure NEs of G_2 and readily extends to G_n. A similar algorithm to compute all Berge equilibria (pure DEs) was proposed for G_n in [24] using a disappointment matrix. Both algorithms have computational complexity $O(N)$. The general problem of finding a mixed NE for G_n, which is well known to exist, involves solving the following algebraic equivalence for (5), which is proved in [25] using (1) and (2).

Result 2. The strategy profile σ^* is an NE of G_n if and only if

$$\sum_{(s_1,\ldots,s_n) \in S} \left(\prod_{j \in I} \sigma_j^*(s_j) \right) u_i(s_1, \ldots, s_n) = \max_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \in I_{-i}} \sigma_j^*(s_j) \right) u_i(s_1, \ldots, s_n), \quad \forall i \in I. \tag{11}$$

For a given σ^* and $\forall i \in I$, (11) can be checked by total enumeration in $O(N)$ to determine if σ^* is an NE. The problem of computing an NE for G_n is thus a member of the complexity class NP, yet obtaining an NE σ^* from (11) by solving for $\sigma_j^*(s_j)$ may not be feasible for large n. Indeed, it is shown in [23] that this problem is PPAD-complete. In other words, this problem is believed to be computationally very difficult but may not be NP-complete. The next theorem gives an algebraic equivalence for (6) similar to (11) for DEs.

Theorem 13. *The strategy profile σ^* is a DE of G_n if and only if*

$$\sum_{(s_1,\ldots,s_n) \in S} \prod_{j \in I} \sigma_j^*(s_j)\, u_i(s_1, \ldots, s_n) = \max_{s_{-i} \in S_{-i}} \sum_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_1, \ldots, s_n), \quad \forall i \in I. \tag{12}$$

Proof. From (3) and (6), σ^* is a DE if and only if

$$u_i(\sigma^*) = \max_{\sigma_{-i} \in \Delta S_{-i}} u_i(\sigma_i^*, \sigma_{-i}) = \max_{\sigma_{-i} \in \Delta S_{-i}} \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})\, u_i(\sigma_i^*, s_{-i}), \quad \forall i \in I. \tag{13}$$

Fix $i \in I$ and let $\hat{s}_{-i}$ maximize $u_i(\sigma_i^*, s_{-i})$ over S_{-i}. This maximum exists since $u_i(\sigma_i^*, s_{-i})$ is continuous in s_{-i} on the closed and bounded set S_{-i}. Then by inspection, $\max_{\sigma_{-i} \in \Delta S_{-i}} \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(\sigma_i^*, s_{-i})$ on the right side of (13) is maximized over $\sigma_{-i}(s_{-i}) \in \Delta S_{-i}$ when $\sigma_{-i}(s_{-i}) = \prod_{j \in I_{-i}} \sigma_j(s_j) = 1$ for $s_{-i} = \hat{s}_{-i}$ and $\sigma_{-i}(s_{-i}) = 0$ for $s_{-i} \neq \hat{s}_{-i}$. Hence

$$u_i(\sigma^*) = \max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i}), \quad \forall i \in I, \tag{14}$$

and (12) follows from (4) and (14). □

For a given σ^* and $\forall i \in I$, (12) can also be checked by total enumeration in $O(N)$ to determine if σ^* is a DE. Hence the problem of computing a DE for G_n is also a member of the complexity class NP. Nonetheless, for $n > 2$, using (12) to obtain a DE or else determine that one does not exist is computationally harder than using (11) to find an NE. For DEs, the maximization for each $i \in I$ on the right side of (12) is over S_{-i}, as opposed to being over S_i in (11) for NEs. This increased difficulty results from the fact that determining a DE σ^* from (6) requires that the $n-1$ components σ_{-i}^* for each of the n maximizations of (6) must match, whereas in (5) only the single component σ_i^* for all the n maximizations of (5) must match. Thus it appears more likely for the DE computational problem to be NP-complete than the NE computational problem. Theorem 11 further suggests this fact since stringent conditions are required there to guarantee the existence of a DE, whereas an NE always exists. Indeed, it is next shown that a DE may not exist.

5. Mutual Cooperation Impossibility Theorem

The following result establishes that mutual cooperation in the form of a DE may be impossible when the number of players is greater than two. As noted above, a DE always exists when $n = 2$. For $n > 2$, however, existence depends on the payoff matrix.

Theorem 14. *For every $n > 2$ there exists a game $G_n = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ with no DE.*

Proof. The result is first proved for $n = 3$. Consider a game G_3 with the payoff matrix P_3 of Table 2, where Player 1 has pure strategies $S_1 = \{s_1, t_1\}$, Player 2 has pure strategies $S_2 = \{s_2, t_2\}$, and Player 3 has pure strategies $S_3 = \{s_3, t_3\}$. For simplicity, write $\sigma_1(s_1) = p_1$ and $\sigma_1(t_1) = p_2$ for Player 1, $\sigma_2(s_2) = q_1$ and $\sigma_2(t_2) = q_2$ for Player 2, and $\sigma_3(s_3) = r_1$ and $\sigma_3(t_3) = r_2$ for Player 3.

Table 2: Payoff matrix for G_3.

	Player 3			
	s_3		t_3	
Player 1	Player 2		Player 2	
	s_2	t_2	s_2	t_2
s_1	(1, 1, 0)	(0, 0, 0)	(0, 0, 1)	(0, 0, 0)
t_1	(0, 0, 0)	(0, 0, 1)	(0, 0, 0)	(1, 1, 0)

Assume that G_3 has a DE $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_3^*)$. Then it follows from (6) that there exist $p_i^*, q_i^*, r_i^*, i = 1, 2$, such that

$$\begin{aligned} u_1(\sigma^*) &= p_1^* q_1^* r_1^* + p_2^* q_2^* r_2^* = \max_{q_i, r_i} \{p_1^* q_1 r_1 \\ &+ p_2^* q_2 r_2 \colon q_1 + q_2 = 1;\ r_1 + r_2 = 1;\ q_1, q_2, r_1, r_2 \\ &\geq 0\}, \end{aligned} \tag{15}$$

$$\begin{aligned} u_2(\sigma^*) &= p_1^* q_1^* r_1^* + p_2^* q_2^* r_2^* = \max_{p_i, r_i} \{p_1 q_1^* r_1 \\ &+ p_2 q_2^* r_2 \colon p_1 + p_2 = 1;\ r_1 + r_2 = 1;\ p_1, p_2, r_1, r_2 \\ &\geq 0\}, \end{aligned} \tag{16}$$

$$\begin{aligned} u_3(\sigma^*) &= p_1^* q_1^* r_2^* + p_2^* q_2^* r_1^* = \max_{p_i, q_i} \{p_1 q_1 r_2^* \\ &+ p_2 q_2 r_1^* \colon p_1 + p_2 = 1;\ q_1 + q_2 = 1;\ p_1, p_2, q_1, q_2 \\ &\geq 0\}. \end{aligned} \tag{17}$$

Solving the maximization in (15) yields the only possible solutions $q_i^*, r_i^*, i = 1, 2$, for (15) as follows.

If $p_1^* > p_2^*$, the solution is

$$\begin{aligned} q_1^* &= r_1^* = 1 \\ q_2^* &= r_2^* = 0. \end{aligned} \tag{18}$$

If $p_1^* < p_2^*$, the solution is

$$\begin{aligned} q_1^* &= r_1^* = 0 \\ q_2^* &= r_2^* = 1. \end{aligned} \tag{19}$$

If $p_1^* = p_2^* = 1/2$, the two solutions are

$$\begin{aligned} q_1^* &= r_1^* = 1 \\ q_2^* &= r_2^* = 0, \end{aligned} \tag{20}$$

as well as

$$\begin{aligned} q_1^* &= r_1^* = 0 \\ q_2^* &= r_2^* = 1. \end{aligned} \tag{21}$$

Equations (18)–(21) therefore require that any DE in (6) must satisfy

$$q_i^*, r_i^* \in \{0, 1\}, \quad i = 1, 2. \tag{22}$$

Next utilizing (22) to solve the maximization in (16) yields the only two possible solutions $p_i^*, r_i^*, i = 1, 2$, for (16) as follows. If $q_1^* = 1$ and $q_2^* = 0$, the solution is

$$\begin{aligned} p_1^* &= r_1^* = 1 \\ p_2^* &= r_2^* = 0. \end{aligned} \tag{23}$$

If $q_1^* = 0$ and $q_2^* = 1$, the solution is

$$\begin{aligned} p_1^* &= r_1^* = 0 \\ p_2^* &= r_2^* = 1. \end{aligned} \tag{24}$$

Hence (22)–(24) require that

$$p_i^*, q_i^*, r_i^* \in \{0, 1\}, \quad i = 1, 2. \tag{25}$$

Thus from (18)–(25), the only two possible solutions $p_i^*, q_i^*, r_i^*, i = 1, 2$, jointly solving the maximizations in (15) and (16) are

$$\begin{aligned} p_1^* &= q_1^* = r_1^* = 1 \\ p_2^* &= q_2^* = r_2^* = 0, \end{aligned} \tag{26}$$

as well as

$$\begin{aligned} p_1^* &= q_1^* = r_1^* = 0 \\ p_2^* &= q_2^* = r_2^* = 1. \end{aligned} \tag{27}$$

But using (25) to solve the maximization in (17) gives the only two possible solutions $p_i^*, q_i^*, r_i^*, i = 1, 2$, solving the maximization in (17) to be

$$\begin{aligned} p_1^* &= q_1^* = r_2^* = 1 \\ p_2^* &= q_2^* = r_1^* = 0, \end{aligned} \tag{28}$$

as well as

$$\begin{aligned} p_1^* &= q_1^* = r_2^* = 0 \\ p_2^* &= q_2^* = r_1^* = 1. \end{aligned} \tag{29}$$

It now follows that the only possible solutions (26) and (27) for (15) and (16) differ from the only possible solutions (28) and (29) for (17), in contradiction to the assumption that G_3 has a DE. Hence G_3 does not have a DE, and the theorem is true for $n = 3$.

For $n > 3$ we generalize G_3 to a game G_n with no DE. Let $S_i = \{s_i, t_i\}$, $i = 1, \ldots, n$, and let the von Neumann-Morgenstern utility vector u on S be

$$\begin{aligned} u(s_1, s_2, \ldots, s_{n-1}, s_n) &= (1, 1, \ldots, 1, 0), \\ u(s_1, s_2, \ldots, s_{n-1}, t_n) &= (0, 0, \ldots, 0, 1), \\ u(t_1, t_2, \ldots, t_{n-1}, t_n) &= (1, 1, \ldots, 1, 0), \\ u(t_1, t_2, \ldots, t_{n-1}, s_n) &= (0, 0, \ldots, 0, 1), \\ u(s) &= (0, 0, \ldots, 0, 0) \ \text{ otherwise.} \end{aligned} \tag{30}$$

Assume that G_n has a DE $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$. In a manner similar to that giving equations (26) and (27) for $n = 3$, it can be shown from (30) that the joint maximizations for Players $1, \ldots, n$ in (6) yield the only possible two solutions

$$\begin{aligned} \sigma_i^*(s_i) &= 1 \\ \sigma_i^*(t_i) &= 0, \end{aligned} \tag{31}$$

for $1, \ldots n$, and

$$\begin{aligned} \sigma_i^*(s_i) &= 0 \\ \sigma_i^*(t_i) &= 1, \end{aligned} \tag{32}$$

for $1, \ldots n$. However, in a manner similar to that giving (28) and (29) for $n = 3$, it can also be shown from (30) that the maximization for Player n in (6) yields the only possible two solutions

$$\begin{aligned} \sigma_1^*(s_1) = \cdots = \sigma_{n-1}^*(s_{n-1}) = \sigma_n^*(t_n) &= 1 \\ \sigma_1^*(t_1) = \cdots = \sigma_{n-1}^*(t_{n-1}) = \sigma_n^*(s_n) &= 0, \end{aligned} \tag{33}$$

as well as

$$\begin{aligned} \sigma_1^*(s_1) = \cdots = \sigma_{n-1}^*(s_{n-1}) = \sigma_n^*(t_n) &= 0 \\ \sigma_1^*(t_1) = \cdots = \sigma_{n-1}^*(t_{n-1}) = \sigma_n^*(s_n) &= 1. \end{aligned} \tag{34}$$

Since (31) and (32) contradict (33) and (34), G_n has no DE, and the proof is complete. □

Theorem 14 states that mutually cooperative behavior as defined by the DE cannot always be achieved for three or more players, no matter how the strategies are selected by the players or for the players. The result is quite intuitive. Depending on the payoff matrix, it is not always possible for every player to accommodate every other player when $n > 2$.

6. Conclusions

The Berge equilibrium has been extended to a mixed dual equilibrium for the Nash equilibrium. In this duality, NEs embody selfishness for all players, as opposed to selflessness for DEs. The two concepts are closely related mathematically, however. Relabeling each player in a DE yields an NE for the original payoff matrix, from which it follows that a game has at least as many NEs as DEs. But an NE always exists for any $n = 1, 2, 3, \ldots$, while for $n > 2$ a DE may not exist—even on the average in the long run, even if the players try to be selfless. Mutual cooperation thus differs from the notion of compromise as defined in [16], which exists for any n. In particular, for a given payoff matrix, mutual cooperation is not always possible for strictly mathematical reasons as a consequence of sociological information about the players reflected in their joint von Neumann-Morgenstern utilities. Because of such issues, the DE computational problem appears more difficult than the NE one.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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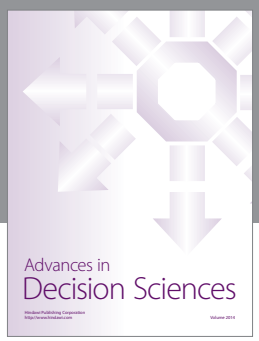

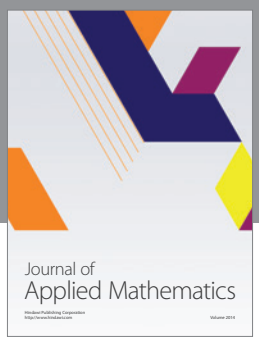

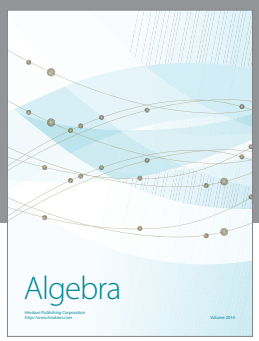

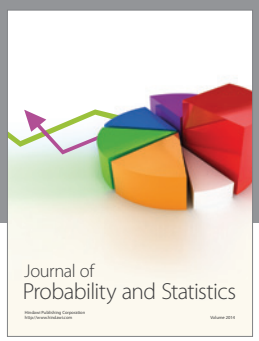

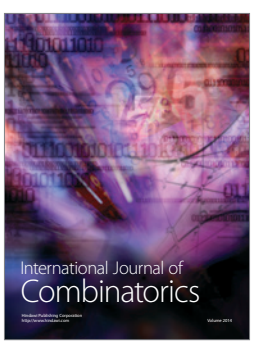

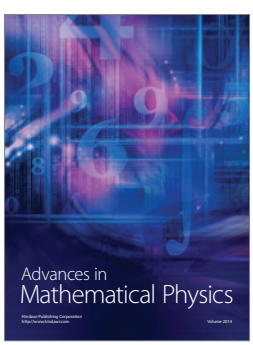

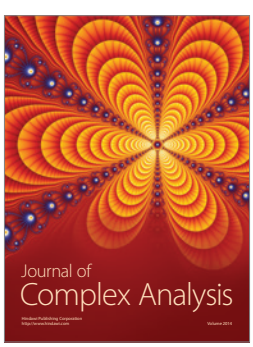

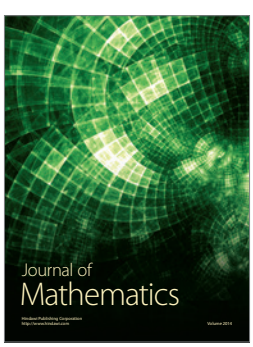

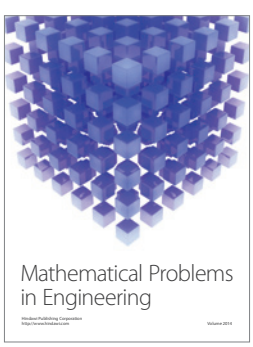

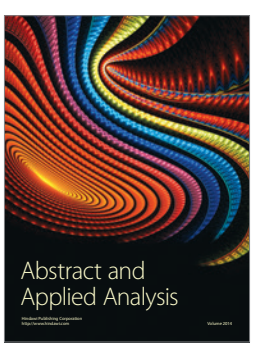

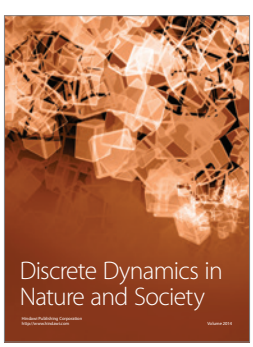

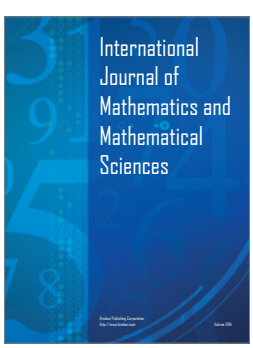

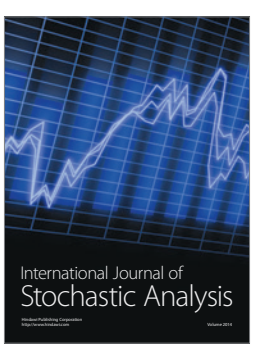

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