Research Article

Block Backward Differentiation Formulas for Fractional Differential Equations

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This paper concerns the numerical approximation of Fractional Initial Value Problems (FIVPs). This is achieved by constructing k -step continuous BDFs. These continuous schemes are developed via the interpolation and collocation approach and are used to obtain the discrete k -step BDF and $(k - 1)$ additional methods which are applied as numerical integrators in a block-by-block mode for the integration of FIVP. The properties of the methods are established and regions of absolute stability of the methods are plotted in the complex plane. Numerical tests including large systems arising from the semidiscretization of one-dimensional fractional Burger's equation show that the methods are highly accurate and efficient.

1. Introduction

In what follows, we consider the FIVP of the following form:

$$\begin{aligned} {}_c D_{x_0}^{\alpha} y(x) &= f(x, y(x)), \\ y(x_0) &= y_0, \end{aligned} \quad (1)$$

where $0 < \alpha < 1$ is the fractional order and ${}_c D_{x_0}^{\alpha}$ (in the sequel we will simply use D^{α}) denotes the Caputo α derivative operator which is defined as

$$D^{\alpha} y(x) = \frac{1}{\Gamma(1 - \alpha)} \int_{x_0}^x (x - s)^{-\alpha} y'(s) ds. \quad (2)$$

The mathematical modeling of several physical phenomena results in fractional differential equations of form (1) and it plays an important role in various branches of science and engineering. Applications of FDEs are found in chemistry, electronics, circuit theory, seismology, signal processing, control theory, and so on. Also, these FDEs serve as a generalization of their corresponding ordinary differential equations (ODEs). For a brief history and introduction to fractional calculus, we refer the reader to [1–3].

We have adopted the Caputo's definition of derivatives of noninteger order (which is a modification of the Riemann-Liouville definition) since it can be coupled with initial conditions having a clear physical meaning. The existence and the uniqueness of the solution of (1) have been given in Diethelm and Ford [4].

The development of well suited methods for numerically approximating FDEs has received great attention over the past few decades. This is due to the occurrence of FDEs in several models. Several methods have been proposed and analyzed for the numerical approximation of this important class of problems (see Lubich [5–7], Garrappa [8], Galeone and Garrappa [9–11], and the references therein). These authors have independently developed Fractional Linear Multistep Methods (FLMMs) using convolution quadratures. Lubich [6] proposed formulas of the following form:

$$y_n = f(t_n) + h^{\alpha} \sum_{j=0}^n \omega_{n-j}^{(\alpha)} g(t_j, y_j) + h^{\alpha} \sum_{j=0}^m \omega_{nj} g(t_j, y_j), \quad (3)$$

$$nh \in \mathbb{I},$$

where ω_n^{α} and ω_{nj} are the convolution and starting quadrature weights, respectively, and are independent of the stepsize h .

One major difficulty in the FLMMs (3) is in evaluating the convolution weights ω_n^α . Most of the methods rely on the J.C.P Miller formula for the computation of these weights. In order to avoid this major drawback, we give a different approach in the construction of the FLMMs. This approach is based on interpolation and collocation as was discussed by Onumanyi et al. [12].

The main aim of this paper is to present and investigate a class of fractional BDF methods which generalize the BDF methods for ODEs. The fractional BDF methods are developed using the interpolation and collocation approach and the regions of absolute stability of the methods are plotted via the boundary locus method.

The paper is organized as follows: in Section 2, we discuss the development of the fractional BDF methods. Section 3 details the convergence of the methods while, in Section 4, we give the stability properties and implementation of the methods. In Section 5, we give five numerical examples to elucidate our theoretical results. Finally, we give some concluding remarks in Section 6.

2. Fractional BDFs

We will construct a k -step Continuous Fractional BDF (CFBDF) which will be used to obtain the discrete fractional BDF (FBDF). The CFBDF has the following general form:

$$U(x) = \sum_{j=0}^{k-1} \gamma_j(x) y_{n+j} + h^\alpha \beta_k(x) f_{n+k}, \quad (4)$$

where $\gamma_j(x)$ and $\beta_k(x)$ are continuous coefficients. We assume that $y_{n+j} = U(x_n + jh)$ is the numerical approximation to the analytical solution $y(x_{n+j})$ and $f_{n+j} = D^\alpha U(x_n + jh)$ is an approximation to $D^\alpha y(x_{n+j})$. The CFBDF is constructed from its equivalent form by requiring that the exact solution $y(x)$ is locally approximated by function (4) on the interval $[x_n, x_{n+k}]$. Next, we discuss the construction of the CFBDF in the following theorem.

Theorem 1. Let (4) satisfy the following conditions:

$$\begin{aligned} U(x_{n+j}) &= y_{n+j}, \quad j = 0(1)(k-1), \\ D^\alpha U(x_{n+k}) &= f_{n+k}. \end{aligned} \quad (5)$$

Then continuous representation (4) is equivalent to

$$U(x) = \sum_{j=0}^k \frac{\det(W_j)}{\det(W)} P_j(x), \quad (6)$$

where we define matrix W as

$$W = \begin{pmatrix} P_0(x_n) & \cdots & P_k(x_n) \\ P_0(x_{n+1}) & \cdots & P_k(x_{n+1}) \\ \vdots & \vdots & \vdots \\ P_0(x_{n+k-1}) & \cdots & P_k(x_{n+k-1}) \\ D^\alpha P_0(x_{n+k}) & \cdots & D^\alpha P_k(x_{n+k}) \end{pmatrix}. \quad (7)$$

W_j is obtained by replacing the j th column of W by V where T denotes the transpose, $P_j(x) = x^j$, $j = 0(1)k$, are basis functions, and V is a vector given by

$$V = (y_n, y_{n+1}, y_{n+2}, \dots, y_{n+k-1}, f_{n+k})^T. \quad (8)$$

Proof. We require that method (4) be defined by the assumed polynomial basis functions:

$$\gamma_j(x) = \sum_{i=0}^k \gamma_{i+1,j} P_i(x), \quad j = 0(1)(k-1), \quad (9)$$

$$h^\alpha \beta_k(x) = \sum_{i=0}^k h^\alpha \beta_{i+1,k} P_i(x),$$

where $\gamma_{i+1,j}$ and $h^\alpha \beta_{i+1,k}$ are coefficients to be determined.

Substituting (9) into (4), we have

$$U(x) = \sum_{i=0}^k \sum_{j=0}^{k-1} \gamma_{i+1,j} P_i(x) y_{n+j} + \sum_{i=0}^k h^\alpha \beta_{i+1,k} P_i(x) f_{n+k} \quad (10)$$

which may be written as

$$U(x) = \sum_{i=0}^k \left\{ \sum_{j=0}^{k-1} \gamma_{i+1,j} y_{n+j} + h^\alpha \beta_{i+1,k} f_{n+k} \right\} P_i(x) \quad (11)$$

and expressed as

$$U(x) = \sum_{i=0}^k \ell_i P_i(x), \quad (12)$$

where

$$\ell_i = \sum_{j=0}^{k-1} \gamma_{i+1,j} y_{n+j} + h^\alpha \beta_{i+1,k} f_{n+k}. \quad (13)$$

By imposing condition (5) on (12), we obtain a system of $(k+1)$ equations, which can be expressed as $W = LV$ where $L = (\ell_0, \ell_1, \dots, \ell_k)^T$ is a vector of $(k+1)$ undetermined coefficients. Using Cramer's rule, the elements of L can be obtained and are given by

$$\ell_i = \frac{\det(W_j)}{\det(W)}, \quad j = 0(1)k, \quad (14)$$

where W_j is obtained by replacing the j th column of W by V . We rewrite (12) using the newly found elements of L as

$$U(x) = \sum_{j=0}^k \frac{\det(W_j)}{\det(W)} P_j(x). \quad (15)$$

□

Remark 2. Continuous scheme (4) which is equivalent to (6) is evaluated at x_{n+k} to obtain the k -step FBDF of the following form:

$$y_{n+k} - \sum_{j=0}^{k-1} \gamma_{kj} y_{n+j} = h^\alpha \beta_{kk} f_{n+k}. \quad (16)$$

Also, we emphasize that continuous scheme (6) is used to obtain $U'(x)$ and evaluated at x_{n+i} , $i = 1(1)(k-1)$, to obtain

$$\begin{aligned} h^\alpha f_{n+1} - \sum_{j=0}^{k-1} \gamma_{1j} y_{n+j} &= h^\alpha \beta_{1k} f_{n+k} \\ h^\alpha f_{n+2} - \sum_{j=0}^{k-1} \gamma_{2j} y_{n+j} &= h^\alpha \beta_{2k} f_{n+k} \\ &\vdots \\ h^\alpha f_{n+k-1} - \sum_{j=0}^{k-1} \gamma_{(k-1)j} y_{n+j} &= h^\alpha \beta_{(k-1)k} f_{n+k} \end{aligned} \quad (17)$$

which, together with (16), forms the Block FBDF (BFBDF) which may be written in the following form:

$$A\bar{Y} - h^\alpha BF(\bar{Y}) + C = 0, \quad (18)$$

where $\bar{Y} = [y_{n+1}, y_{n+2}, \dots, y_{n+k}]^T$, $F(\bar{Y}) = [f_{n+1}, f_{n+2}, \dots, f_{n+k}]^T$, A and B are the coefficients of the formulas in (16) and (17), and C is a vector of initial conditions.

Remark 3. We note that it is possible to construct method (6) using other bases such exponential and trigonometric functions. However, (6) is constructed using polynomial basis functions, since methods produced via polynomial basis functions are easier to analyze. In fact, using other bases for the construction of (6) has the disadvantage of introducing additional parameters which makes the analysis of the methods produced more cumbersome. Moreover, the polynomial basis functions are appropriate for the construction of this class of methods since other bases can be written as polynomials in x via the Taylor series expansion.

3. Convergence of Methods

In this section, we will discuss the convergence of the methods in the following theorem.

Theorem 4. Let $f(x, y)$ be Lipschitz continuous with respect to y in a region $\mathbb{D}$ defined by $a \leq x \leq b$ and $-\infty < y < \infty$, where a and b are finite. Let (16) and (17) be constructed in such a way that $\beta_{ik} = O(n^{\alpha-1})$; then (18) is said to be convergent if, for all initial value problems (1), we have that

$$|\bar{Y} - Y| \leq K \cdot x^{\alpha-1} h^p, \quad (x = nh). \quad (19)$$

for all $x \in [a, b]$ and where constant K does not depend on h and p is the order of the method and $h > 0$ is sufficiently small.

Proof. The exact form of the system formed by (16) and (17) is given by

$$AY - h^\alpha BF(Y) + C + L(h) = 0, \quad (20)$$

where $L(h)$ is the truncation error vector of the formula in (18), $Y = [y(x_{n+1}), y(x_{n+2}), \dots, y(x_{n+k})]^T$, and $F(Y) = [f(x_{n+1}, y(x_{n+1})), f(x_{n+2}, y(x_{n+2})), \dots, f(x_{n+k}, y(x_{n+k}))]^T$.

The approximate form of the system is given by

$$A\bar{Y} - h^\alpha BF(\bar{Y}) + C = 0, \quad (21)$$

where $\bar{Y}$ is the approximate solution of vector Y .

Subtracting (20) from (21), we obtain the following error system:

$$AE = h^\alpha B [F(\bar{Y}) - F(Y)] + L(h), \quad (22)$$

where $E = \bar{Y} - Y$. Let K_1 be the Lipschitz constant of f ; then

$$\begin{aligned} AE &\leq h^\alpha B \cdot K_1 \cdot E + L(h), \\ E &\leq (A - h^\alpha B \cdot K_1)^{-1} L(h). \end{aligned} \quad (23)$$

Let $P = -h^\alpha B \cdot K_1$, so that

$$E \leq (A + P)^{-1} L(h) = O((nh)^{\alpha-1}) \cdot Kh^p. \quad (24)$$

$((A + P)^{-1})$ exists since it is a monotone matrix (see [13]). $\square$

Definition 5. BFAMM (18) is said to be

- (a) stable if and only if, for any $y_0 \in \mathbb{R}^m$, there exists $K_2 > 0$ such that the solution of (1) satisfies $\|y(x_n)\| \leq K_2$ for $n \geq 1$,
- (b) asymptotically stable if and only if, for any $y_0 \in \mathbb{R}^m$, the solution of (1) satisfies $\|y(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

4. Stability Properties of the Methods

To study the stability properties of BFBDF (18), we consider the following linear test problem:

$$\begin{aligned} D^\alpha y(x) &= \lambda y(x), \quad \lambda \in \mathbb{C}, \quad 0 < \alpha < 1, \\ y(x_0) &= y_0, \end{aligned} \quad (25)$$

whose exact solution can be expressed in terms of the Mittag-Leffler function $E_\alpha(x) = \sum_{k=0}^{\infty} (x^k / \Gamma(\alpha k + 1))$, as $y(x) = E_\alpha(\lambda(x - x_0)^\alpha) y_0$.

We rewrite (18) in the following form:

$$A_0 Y_\gamma + A_1 Y_{\gamma-1} = B_0 F_\gamma, \quad (26)$$

where $Y_\gamma = [y_{n+1}, y_{n+2}, \dots, y_{n+k}]^T$, $Y_{\gamma-1} = [y_{n-k-1}, \dots, y_{n-1}, y_n]^T$, and $F_\gamma = [f_{n+1}, f_{n+2}, \dots, f_{n+k}]^T$. Applying (26) to the linear test equation, we have the following linear recurrence relation:

$$\begin{aligned} Y_\gamma &= M(q) Y_{\gamma-1}, \\ q &= \lambda h^\alpha, \end{aligned} \quad (27)$$

where

$$M(q) = (A_0 - qB_0)^{-1} A_1, \quad (28)$$

FIGURE 1: The region of stability of the BFBDF for $k = 1$ is to the left of the dividing line and is symmetric about the real axis.

where $M(q)$ is the amplification matrix which determines the stability of the method.

We are interested in investigating the values of $h^\alpha \lambda$ for which the numerical solution of (1) given by (18) asymptotically vanishes as the true solution. We give below the following definitions.

Definition 6. Stability domain Ω of BFBDF (18) is the set of all $q = h^\alpha \lambda \in \mathbb{C}$ such that linear recurrence (18) is asymptotically stable.

Definition 7. The stability domain of BFBDF (18) is given by the following set:

$$\Omega = \{q : |\rho(q)| < 1\}, \quad (29)$$

where $\rho(q)$ is the spectral radius of $M(q)$.

Figures 1, 2, 3, and 4 show the plot of the stability domain Ω of BFBDF (18) for some α for $k = 1$ to 4.

Remark 8. The 1-step BFBDF is A-stable for all values of $\alpha \in (0, 1)$ while the 2-step and 3-step BFBDF are A-stable for values of $\alpha \geq 0.75$. Also, the 4-step BFBDF is A-stable for $\alpha \geq 0.80$.

4.1. Implementation. Conventionally, FBDF (16) requires k initial conditions which are usually provided using a one-step method (like the Runge-Kutta method). However, in

this paper, we construct additional methods from continuous scheme (15) which are implemented together with FBDF (16) to obtain approximate solutions to the exact solution of (1) without requiring starting values and predictors. For instance, if $n = 0$ and $k = 2$, then $(y_1, y_2)^T$ are simultaneously obtained over the interval $[x_0, x_2]$ as y_0 is known from the IVP. Similarly, if $n = 1$ and $k = 3$, then $(y_4, y_5, y_6)^T$ are simultaneously obtained over the interval $[x_3, x_6]$ and y_3 is known from the previous block and so on, until we reach the final subinterval $[x_{N-k}, x_N]$.

5. Numerical Examples

We validate our theoretical results from the previous sections by considering the following examples, which were solved using the BFBDF using a written code in Mathematica. The maximum errors are obtained for different step sizes in the interval of integration. We have solved two scalar examples, one system, one linear, and one nonlinear heat-type fractional differential equation.

Example 1. We consider the following problem given in [14]:

$$D^\alpha y(x) = -y + x^2 - x + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{x^{1-\alpha}}{\Gamma(2-\alpha)},$$

$$0 < \alpha < 1, \quad 0 \leq x \leq 1,$$

FIGURE 2: The region of stability of the BFBDF for $k = 2$ is to the left of the dividing line and is symmetric about the real axis; the square and plus symbols represent the zeros and poles of the spectral radius of $M(q)$, respectively.

FIGURE 3: The region of stability of the BFBDF for $k = 3$ is to the left of the dividing line and is symmetric about the real axis; the square and plus symbols represent the zeros and poles of the spectral radius of $M(q)$, respectively.

$$y(0) = 0,$$

$$\text{Exact: } y(x) = x^2 - x.$$

Tables 1, 2, and 3 show the numerical results for Example 1 using different values of α . It is evident from the table that the BFBDF performs favorably well with smaller number of steps.

(30)

FIGURE 4: The region of stability of the BFBDF for $k = 4$ is to the left of the dividing line and is symmetric about the real axis; the square and plus symbols represent the zeros and poles of spectral radius of $M(q)$, respectively.

Example 2. We also consider the following FIVP with variable coefficients:

$$D^\alpha y(x) - xy = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} x^\alpha - x - x^{2\alpha+1},$$

$$0 < \alpha < 1, \quad 0 \leq x \leq 1, \quad (31)$$

$$y(0) = 1,$$

$$\text{Exact: } y(x) = 1 + x^{2\alpha}.$$

This example was chosen to show the performance of the BFBDF on FIVP with variable coefficients. Tables 4, 5, and 6

show the numerical results for Example 2 using different values of α . It is evident from the table that the BFBDF performs favorably well with smaller number of steps.

Example 3. We also consider the system of FIVP given in [15]:

$$D^\alpha y_1(x) = y_1(x) + y_2(x),$$

$$D^\alpha y_2(x) = -y_1(x) + y_2(x),$$

$$0 < \alpha < 1, \quad 0 \leq x \leq 1,$$

$$y_1(0) = 0,$$

$$y_2(0) = 1,$$

TABLE 1: Maximum errors using the BFBDF $k = 2$ for Example 1.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$76.661e - 16$	$8.882e - 16$	$3.1091e - 15$
20	$4.441e - 16$	$9.992e - 16$	$1.998e - 15$
40	$9.437e - 16$	$1.499e - 15$	$9.992e - 16$
80	$2.220e - 15$	$1.499e - 15$	$3.386e - 15$

TABLE 2: Maximum errors using the BFBDF $k = 3$ for Example 1.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$8.882e - 16$	$3.071e - 15$	$2.442e - 15$
20	$2.648e - 15$	$2.387e - 15$	$2.054e - 15$
40	$1.724e - 15$	$6.085e - 15$	$2.866e - 15$
80	$3.109e - 15$	$1.263e - 14$	$1.379e - 14$

TABLE 3: Maximum errors using the BFBDF $k = 4$ for Example 1.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$3.997e - 15$	$9.869e - 15$	$2.663e - 12$
20	$3.425e - 15$	$8.040e - 14$	$2.988e - 12$
40	$8.527e - 14$	$5.887e - 13$	$3.077e - 12$
80	$1.337e - 12$	$4.041e - 12$	$9.291e - 12$

TABLE 4: Maximum errors using the BFBDF $k = 2$ for Example 2.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$4.723e - 01$	$1.243e - 15$	$2.563e - 02$
20	$5.122e - 01$	$6.661e - 15$	$2.900e - 02$
40	$5.281e - 01$	$8.660e - 15$	$2.931e - 02$
80	$5.320e - 02$	$1.776e - 14$	$2.894e - 02$

TABLE 5: Maximum errors using the BFBDF $k = 3$ for Example 2.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$8.047e - 02$	$1.688e - 14$	$7.124e - 03$
20	$1.322e - 01$	$5.240e - 14$	$9.719e - 03$
40	$1.587e - 01$	$6.239e - 14$	$1.036e - 02$
80	$1.796e - 01$	$4.352e - 13$	$1.074e - 02$

Exact: $y_1(x) = e^x \sin(x)$, $y_2(x) = e^x \cos(x)$

for $\alpha = 1$.

(32)

This example was chosen to show that the BFBDF performs well on a system as demonstrated for $\alpha = 1$. We note that for $\alpha = 0.85$ the solutions produced by the BFBDF are in agreement with the expected behavior of the exact solution. Details of the results are displayed in Table 7 and Figures 5 and 6.

Example 4. We also consider the following one-dimensional fractional heat-like problem given in [16]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}, \quad 0 < \alpha < 1, \quad 0 \leq x \leq 1 \quad (33)$$

TABLE 6: Maximum errors using the BFBDF $k = 4$ for Example 2.

N	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
10	$3.761e - 01$	$1.165e - 11$	$5.805e - 03$
20	$9.320e - 02$	$1.075e - 11$	$5.008e - 03$
40	$1.123e - 01$	$5.349e - 11$	$5.606e - 03$
80	$1.242e - 01$	$2.826e - 10$	$5.740e - 03$

TABLE 7: Maximum errors using the BFBDF $\alpha = 1$ for Example 3.

N	$k = 2$	$k = 3$	$k = 4$
5	$6.647e - 02$	$6.033e - 03$	$3.991e - 03$
10	$5.276e - 03$	$7.498e - 04$	$1.382e - 04$
20	$9.839 - 04$	$9.430e - 05$	$2.814e - 06$
40	$2.073e - 04$	$1.174e - 05$	$1.727e - 07$
80	$4.717e - 05$	$1.446e - 06$	$6.489e - 06$

subject to the initial/boundary conditions $u(x, 0) = x^2$, $u(0, t) = 0$, $u(1, t) = e^t$, and $t \geq 0$. The exact solution $u(x, t) = x^2(1+t^\alpha/\Gamma(\alpha+1)+t^{2\alpha}/\Gamma(2\alpha+1)+t^{3\alpha}/\Gamma(3\alpha+1)+\dots)$.

In order to solve this PDE using the BFBDF, we carry out the semidiscretization of the spatial variable x using the second-order finite difference method to obtain the following first-order system in the second variable t :

$$\frac{\partial^\alpha u_m}{\partial t^\alpha} - x_m^2 \left(\frac{(u_{m+1} - 2u_m + u_{m-1}))}{(\Delta x)^2} \right) = g_m, \quad m = 1, \dots, M-1, \quad (34)$$

$$u(x_m, 0) = x_m^2,$$

where $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0, 1, \dots, M$, $\mathbf{u} = [u_1(t), \dots, u_M(t)]^T$, $\mathbf{g} = [g_1(t), \dots, g_M(t)]^T$, $u_m(t) \approx u(x_m, t)$, and $g_m(t) \approx g(x_m, t) = 0$, which can be written in the following form:

$$\mathbf{u}^\alpha = \mathbf{f}(\mathbf{t}, \mathbf{u}), \quad (35)$$

subject to the boundary conditions $\mathbf{u}(t_0) = \mathbf{u}_0$, where $\mathbf{f}(\mathbf{t}, \mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{g}$ and $\mathbf{A}$ is $M \times M$ matrix arising from the semidiscretized system and $\mathbf{g}$ is a vector of constants.

This example was chosen to show that the BFBDF performs well on a one-dimensional fractional heat-like problem with variable coefficients as demonstrated for $\alpha = 1$. We note that for $\alpha = 0.75$ the solutions produced by the BFBDF are in agreement with the expected behavior of the exact solution. Details of the results are displayed in Figures 7 and 8.

Example 5. We consider the following fractional Burger's equation given in [17]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u^2}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < \alpha \leq 1, \quad 0 < x < 1 \quad (36)$$

subject to the initial/boundary conditions $u(x, 0) = x$, $u(0, t) = 0$, $u(1, t) = 1/(1+t)$, and $t \geq 0$. The exact solution $u(x, t) = x/(1+t)$, $\alpha = 1$.

FIGURE 5: Graphical evidence for Example 3, $N = 10$.

In order to solve this PDE using the BFBDF, we carry out the semidiscretization of the spatial variable x using the second-order finite difference method to obtain the following first-order system in the second variable t :

$$\frac{\partial^\alpha u_m}{\partial t^\alpha} - x_m^2 \left(\frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta x)^2} \right) = g_m, \quad m = 1, \dots, M-1, \quad (37)$$

$$u(x_m, 0) = x_m^2,$$

where $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0, 1, \dots, M$, $\mathbf{u} = [u_1(t), \dots, u_M(t)]^T$, $\mathbf{g} = [g_1(t), \dots, g_M(t)]^T$, $u_m(t) \approx u(x_m, t)$, and $g_m(t) \approx g(x_m, t) = 0$, which can be written in the following form:

$$\mathbf{u}^\alpha = \mathbf{f}(\mathbf{t}, \mathbf{u}), \quad (38)$$

subject to the boundary conditions $\mathbf{u}(t_0) = \mathbf{u}_0$, where $\mathbf{f}(\mathbf{t}, \mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{g}$ and $\mathbf{A}$ is $M \times M$ matrix arising from the semidiscretized system and $\mathbf{g}$ is a vector of constants.

This example was chosen to show that the BFBDF performs well on the Burger's equation as demonstrated for $\alpha = 1$. We note that for $\alpha = 0.75$ the solutions produced by the BFBDF are in agreement with the expected behavior of the exact solution. Details of the results are displayed in Figures 9 and 10.

5.1. Block versus Predictor-Corrector Implementations. In order to demonstrate the superiority of the block implementation of the BFBDF over its predictor-corrector (PC) implementation, we have solved Example 1 for $k = 1, 2, 3, 4$ and the results are displayed in Table 8. We constructed Fractional Explicit Adams Methods and used them as predictors for the FBDF. In Table 8, we observe that as the step length decreases the PC mode gives inaccurate approximations, while the BFBDF still retains its high accuracy. With these observations, we conclude that the BFBDF can be used as

TABLE 8: Comparison of exact errors for Example 1 ($\alpha = 0.75$).

N	$k = 1$		$k = 2$		$k = 3$		$k = 4$	
	Block	PC	Block	PC	Block	PC	Block	PC
10	$9.434e-02$	$8.054e-02$	$3.107e-15$	$1.611e-12$	$2.442e-15$	$6.158e-16$	$2.663e-12$	$5.480e-10$
20	$1.209e-01$	$1.143e-01$	$1.887e-15$	$4.733e-10$	$2.054e-15$	$1.409e-15$	$2.988e-12$	$4.590e-07$
40	$1.344e-01$	$1.306e-01$	$1.110e-15$	$7.388e-09$	$2.866e-15$	$5.560e-14$	$3.077e-12$	$6.973e-05$
80	$1.412e-01$	$1.389e-01$	$3.776e-15$	$3.203e-06$	$1.379e-14$	$4.492e-13$	$7.365e-12$	$5.914e-03$

(a) BFBDF for $\alpha = 0.85$, $k = 2$ (b) BFBDF for $\alpha = 0.85$, $k = 3$ (c) BFBDF for $\alpha = 0.85$, $k = 4$ FIGURE 6: Graphical evidence for Example 3, $N = 10$.

a general purpose method for the integration of fractional differential equations.

6. Conclusion

We have constructed a family of fractional linear multistep methods via the interpolation and collocation techniques.

The methods developed are implemented as block methods for the numerical approximation of FIVPs. The stability properties of the methods are discussed and numerical examples are given to show that the methods are accurate and efficient, even when applied to large systems arising from the semidiscretization of one-dimensional fractional heat-like partial differential equations.

FIGURE 7: Graphical evidence for Example 4, $N = 10$.

FIGURE 8: Graphical evidence for Example 4, $N = 10$.

FIGURE 9: Graphical evidence for Example 5, $N = 10$.

FIGURE 10: Graphical evidence for Example 5, $N = 10$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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