Research Article

Some Ostrowski Type Inequalities for Harmonically (s, m) -Convex Functions in Second Sense

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The authors introduce the concept of harmonically (s, m) -convex functions in second sense and establish some Ostrowski type inequalities of these classes of functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping differentiable in I° and $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] \quad (1)$$

for $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality [1], which gives an upper bound for the approximation of the integral average $(1/(b-a)) \int_a^b f(t) dt$ by the value $f(x)$ at the point $x \in [a, b]$. For some results which generalize, improve, and extend inequality (1), we refer the reader to recent papers (see [2, 3]) and monograph of [4].

In [5], İşcan introduced class of harmonically convex functions. This class of functions is defined as follows.

Definition 1. Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (2)$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (2) is reversed, then f is said to be harmonically concave.

In [6], İşcan introduced the concept of harmonically s -convex function in second sense and established a variant of Ostrowski type inequalities which hold for these classes of functions as follows.

Definition 2. A function $f : I \subset \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ is said to be harmonically s -convex in second sense, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t^s f(y) + (1-t)^s f(x) \quad (3)$$

for all $x, y \in I$ and $t \in [0, 1]$ for some fixed $s \in (0, 1]$. If inequality in (3) is reversed, then f is said to be harmonically s -concave.

Remark 3. Note that, for $s = 1$, harmonically s -convexity reduces to ordinary harmonic convexity.

Theorem 4. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s -convex on $[a, b]$ for $q \geq 1$, then, for all $x \in [a, b]$, we have

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left\{ (x-a)^2 \cdot (\lambda_1(a, x, s, q, q) |f'(x)|^q + \lambda_2(a, x, s, q, q) |f'(a)|^q)^{1/q} + (b-x)^2 \right.$$

$$\cdot \left(\lambda_3(b, x, s, q, q) \left| f'(x) \right|^q + \lambda_4(b, x, s, q, q) \left| f'(b) \right|^q \right)^{1/q} \Big\}, \quad (4)$$

where

$$\begin{aligned} \lambda_1(a, x, s, \vartheta, \rho) &= \frac{\beta(\rho + s + 1, 1)}{x^{2\vartheta}} \\ &\cdot {}_2F_1\left(2\vartheta, \rho + s + 1; \rho + s + 2, 1 - \frac{a}{x}\right), \\ \lambda_2(a, x, s, \vartheta, \rho) &= \frac{\beta(\rho + 1, 1)}{x^{2\vartheta}} \cdot {}_2F_1\left(2\vartheta, \rho + 1; \rho + s + 2, 1 - \frac{a}{x}\right), \\ \lambda_3(b, x, s, \vartheta, \rho) &= \frac{\beta(1, \rho + s + 1)}{b^{2\vartheta}} \cdot {}_2F_1\left(2\vartheta, 1; \rho + s + 2, 1 - \frac{x}{b}\right), \\ \lambda_4(b, x, s, \vartheta, \rho) &= \frac{\beta(s + 1, \rho + 1)}{b^{2\vartheta}} \\ &\cdot {}_2F_1\left(2\vartheta, s + 1; \rho + s + 2, 1 - \frac{x}{b}\right). \end{aligned} \quad (5)$$

β is Euler Beta function defined by

$$\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (6)$$

$x, y > 0,$

and ${}_2F_1$ is hypergeometric function (see [4]) defined by

$$\begin{aligned} {}_2F_1(a, b; c, z) &= \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad (7) \\ c &> b > 0, \quad |z| < 1. \end{aligned}$$

Corollary 5. In Theorem 4, additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left\{ (x-a)^2 \right. \\ &\cdot \left(\lambda_1(a, x, s, q, q) + \lambda_2(a, x, s, q, q) \right)^{1/q} + (b-x)^2 \\ &\cdot \left. \left(\lambda_3(b, x, s, q, q) + \lambda_4(b, x, s, q, q) \right)^{1/q} \right\} \end{aligned} \quad (8)$$

holds.

Theorem 6. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s -convex on $[a, b]$ for $q \geq 1$, then, for all $x \in [a, b]$, one has

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} \left(\frac{1}{2} \right)^{1-1/q} \left\{ (x \right. \\ &- a)^2 \left(\lambda_1(a, x, s, q, 1) \left| f'(x) \right|^q \right. \\ &+ \lambda_2(a, x, s, q, 1) \left| f'(a) \right|^q \Big)^{1/q} + (b-x)^2 \\ &\cdot \left. \left(\lambda_3(b, x, s, q, 1) \left| f'(x) \right|^q \right. \right. \\ &+ \lambda_4(b, x, s, q, 1) \left| f'(b) \right|^q \Big)^{1/q} \Big\}, \end{aligned} \quad (9)$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Corollary 7. In Theorem 6, additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then the inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left(\frac{1}{2} \right)^{1-1/q} \\ &\cdot \left\{ (x-a)^2 \left(\lambda_1(a, x, s, q, 1) + \lambda_2(a, x, s, q, 1) \right)^{1/q} \right. \\ &+ (b-x)^2 \left. \left(\lambda_3(b, x, s, q, 1) + \lambda_4(b, x, s, q, 1) \right)^{1/q} \right\} \end{aligned} \quad (10)$$

holds.

Theorem 8. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s -convex on $[a, b]$ for $q \geq 1$, then, for all $x \in [a, b]$, one has

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} \left\{ \lambda_5^{1-1/q}(a, x) (x \right. \\ &- a)^2 \left(\lambda_1(a, x, s, 1, 1) \left| f'(x) \right|^q \right. \\ &+ \lambda_2(a, x, s, 1, 1) \left| f'(a) \right|^q \Big)^{1/q} + \lambda_5^{1-1/q}(b, x) (b \\ &- x)^2 \left(\lambda_3(b, x, s, 1, 1) \left| f'(x) \right|^q \right. \\ &+ \lambda_4(b, x, s, 1, 1) \left| f'(b) \right|^q \Big)^{1/q} \Big\}, \end{aligned} \quad (11)$$

where

$$\lambda_5(\theta, x) = \frac{1}{x-\theta} \left\{ \frac{1}{\theta} - \frac{\ln x - \ln \theta}{x-\theta} \right\} \quad (12)$$

and $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Corollary 9. In Theorem 8, additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then the inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left\{ \lambda_5^{1-1/q}(a, x) \right. \\ &\cdot (x-a)^2 (\lambda_1(a, x, s, 1, 1) + \lambda_2(a, x, s, 1, 1))^{1/q} \\ &+ \lambda_5^{1-1/q}(b, x) (b-x)^2 \\ &\cdot (\lambda_3(b, x, s, 1, 1) + \lambda_4(b, x, s, 1, 1))^{1/q} \left. \right\} \end{aligned} \quad (13)$$

holds.

Theorem 10. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s -convex on $[a, b]$ for $q > 1$, $1/p + 1/q = 1$, then, for all $x \in [a, b]$, one has

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} \left(\frac{1}{1+p} \right)^{1/p} \left\{ (x \right. \\ &- a)^2 (\lambda_1(a, x, s, q, 0) |f'(x)|^q \\ &+ \lambda_2(a, x, s, q, 0) |f'(a)|^q)^{1/q} + (b-x)^2 \\ &\cdot (\lambda_3(b, x, s, q, 0) |f'(x)|^q \\ &+ \lambda_4(b, x, s, q, 0) |f'(b)|^q)^{1/q} \left. \right\}, \end{aligned} \quad (14)$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Corollary 11. In Theorem 10, additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then the inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left(\frac{1}{1+p} \right)^{1/p} \\ &\cdot \left\{ (x-a)^2 (\lambda_1(a, x, s, q, 0) + \lambda_2(a, x, s, q, 0))^{1/q} \right. \\ &+ (b-x)^2 (\lambda_3(b, x, s, q, 0) + \lambda_4(b, x, s, q, 0))^{1/q} \left. \right\} \end{aligned} \quad (15)$$

holds.

Theorem 12. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s -convex on $[a, b]$ for $q > 1$, $1/p + 1/q = 1$, then, for all $x \in [a, b]$, one has

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\ \leq \frac{ab}{b-a} \left\{ (\lambda_1(a, x, 0, p, p))^{1/p} (x-a)^2 \right. \end{aligned}$$

$$\begin{aligned} &\cdot \left(\frac{|f'(x)|^q + |f'(a)|^q}{s+1} \right)^{1/q} \\ &+ (\lambda_3(b, x, 0, p, p))^{1/p} (b-x)^2 \\ &\cdot \left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right)^{1/q} \left. \right\}, \end{aligned} \quad (16)$$

where λ_1 and λ_3 are defined as in Theorem 4.

Corollary 13. In Theorem 12, additionally, if $|f'(x)| \leq M$, $x \in [a, b]$, then the inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left(\frac{2}{s+1} \right)^{1/q} \\ &\cdot \left\{ (\lambda_1(a, x, 0, p, p))^{1/p} (x-a)^2 \right. \\ &+ (\lambda_3(b, x, 0, p, p))^{1/p} (b-x)^2 \left. \right\} \end{aligned} \quad (17)$$

holds.

In [7], İşcan introduced the concept of harmonically (α, m) -convex functions and established some Hermite-Hadamard type inequalities for this class of function. This class of functions is defined as follows.

Definition 14. The function $f : (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically (α, m) -convex, where $\alpha \in [0, 1]$ and $m \in (0, 1]$, if

$$\begin{aligned} f\left(\frac{mxy}{mty + (1-t)x}\right) &= f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \\ &\leq t^\alpha f(x) + m(1-t^\alpha) f(y) \end{aligned} \quad (18)$$

for all $x, y \in (0, \infty)$ and $t \in [0, 1]$. If the inequality in (18) is reversed, then f is said to be harmonically (α, m) -concave.

In [8], Park considered the class of (s, m) -convex functions in second sense. This class of functions is defined as follows.

Definition 15. For some fixed $s \in (0, 1]$ and $m \in [0, 1]$, a mapping $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be (s, m) -convex in the second sense on I if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) \quad (19)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Now, we introduce the concept of harmonically (s, m) -convex functions in second sense, which generalize the notion of harmonically convex and harmonically s -convex functions in second sense introduced by İşcan in [5, 6], as follows.

Definition 16. The function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically (s, m) -convex in second sense, where $s \in (0, 1]$ and $m \in (0, 1]$ if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) = f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \leq t^s f(x) + m(1-t)^s f(y), \quad (20)$$

$\forall x, y \in I$ and $t \in [0, 1]$.

Remark 17. Note that, for $s = 1$, harmonic (s, m) -convexity reduces to harmonic m -convexity and for $m = 1$ harmonic (s, m) -convexity reduces to harmonic s -convexity in second sense (see [6]) and for $s, m = 1$ harmonic (s, m) -convexity reduces to ordinary harmonic convexity (see [5]).

Proposition 18. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a function:

- (a) if f is (s, m) -convex function in second sense and nondecreasing, then f is a harmonically (s, m) -convex function in second sense.
- (b) If f is a harmonically (s, m) -convex function in second sense and nonincreasing, then f is (s, m) -convex function in second sense.

Remark 19. According to Proposition 18, every nondecreasing (s, m) -convex function in second sense is also a harmonically (s, m) -convex function in second sense.

Example 20 (see [9]). Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$; then, function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & x = 0, \\ bx^2 + c, & x > 0, \end{cases} \quad (21)$$

is a nondecreasing s -convex function in second sense for $b \geq 0$ and $0 \leq c \leq a$. Hence, by Proposition 18, f is a harmonically $(s, 1)$ -convex function.

Proposition 21. Let $s \in [0, 1]$, $m \in (0, 1]$, and $f : [a, mb] \subset (0, \infty) \rightarrow \mathbb{R}$ be an increasing function and $g : [a, mb] \rightarrow [a, mb]$ and $g(x) = mab/(a + mb - x)$, $a < mb$. Then, f is harmonically (s, m) -convex in second sense on $[a, mb]$ if and only if $f \circ g$ is (s, m) -convex in second sense on $[a, mb]$.

The following result of the Hermite-Hadamard type holds.

Theorem 22. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s, m) -convex function in second sense with $s \in [0, 1]$ and $m \in (0, 1]$. If $0 < a < b < \infty$ and $f \in L[a, b]$, then one has the following inequality:

$$\begin{aligned} & \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ & \leq \min \left[\frac{f(a) + mf(b/m)}{s+1}, \frac{f(b) + mf(a/m)}{s+1} \right]. \end{aligned} \quad (22)$$

Corollary 23. If one takes $m = 1$ in Theorem 22, then one gets

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{s+1}. \quad (23)$$

Corollary 24. If one takes $s = 1$ in Theorem 22, then one gets

$$\begin{aligned} & \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ & \leq \min \left[\frac{f(a) + mf(b/m)}{2}, \frac{f(b) + mf(a/m)}{2} \right]. \end{aligned} \quad (24)$$

In this paper, we obtain similar inequalities (1) for harmonically (s, m) -convex functions and establish some new results of Ostrowski type inequalities for harmonically (s, m) -convex functions such that results given in [6] by İşcan are obtained for particular value of m .

2. Main Results

For finding some new inequalities of Ostrowski type for the functions whose derivatives are harmonically (s, m) -convex in second sense, we need the following lemma.

Lemma 25. Let $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then

$$\begin{aligned} & f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du = \frac{ab}{b-a} \left\{ (x-a)^2 \right. \\ & \cdot \int_0^1 \frac{t}{(ta + (1-t)x)^2} f' \left(\frac{ax}{(ta + (1-t)x)} \right) dt \\ & - (b-x)^2 \\ & \cdot \left. \int_0^1 \frac{t}{(tb + (1-t)x)^2} f' \left(\frac{bx}{(tb + (1-t)x)} \right) dt \right\}. \end{aligned} \quad (25)$$

Theorem 26. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, b/m]$ for $q \geq 1$ with $s \in [0, 1]$, then

$$\begin{aligned} & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left\{ (x-a)^2 \right. \\ & \cdot \left(\lambda_1(a, x, s, q, q) |f'(x)|^q \right. \\ & + m\lambda_2(a, x, s, q, q) \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} + (b-x)^2 \\ & \cdot \left(\lambda_3(b, x, s, q, q) |f'(x)|^q \right. \\ & + m\lambda_4(b, x, s, q, q) \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} \Big\}, \end{aligned} \quad (26)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are defined as in Theorem 4.

Proof. From Lemma 25 and using power mean inequality, we have

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} \left\{ (x-a)^2 \right. \\ &\cdot \int_0^1 \frac{t}{(ta + (1-t)x)^2} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right| dt + (b \\ &- x)^2 \int_0^1 \frac{t}{(tb + (1-t)x)^2} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right| dt \Big\} \\ &\leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 1 dt \right)^{1-1/q} \end{aligned}$$

$$\begin{aligned} &\cdot \left(\int_0^1 \frac{t^q}{(ta + (1-t)x)^{2q}} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right|^q dt \right)^{1/q} \\ &+ \frac{ab(b-x)^2}{b-a} \left(\int_0^1 1 dt \right)^{1-1/q} \\ &\cdot \left(\int_0^1 \frac{t^q}{(tb + (1-t)x)^{2q}} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right|^q dt \right)^{1/q}. \end{aligned} \quad (27)$$

Since $|f'|^q$ is harmonically (s, m) -convex function in second sense, we have

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t^q [t^s |f'(x)|^q + m(1-t)^s |f'(a/m)|^q]}{(ta + (1-t)x)^{2q}} dt \right)^{1/q} \\ &+ \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t^q [t^s |f'(x)|^q + m(1-t)^s |f'(b/m)|^q]}{(tb + (1-t)x)^{2q}} dt \right)^{1/q}. \end{aligned} \quad (28)$$

It is easy to check that

$$\begin{aligned} &\int_0^1 \frac{t^{q+s}}{(ta + (1-t)x)^{2q}} dt \\ &= \frac{\beta(q+s+1, 1)}{x^{2q}} \\ &\cdot {}_2F_1 \left(2q, q+s+1, q+s+2; 1 - \frac{a}{x} \right), \end{aligned} \quad (29)$$

$$\begin{aligned} &\int_0^1 \frac{t^{q+s}}{(tb + (1-t)x)^{2q}} dt \\ &= \frac{\beta(1, q+s+1)}{b^{2q}} \cdot {}_2F_1 \left(2q, 1, q+s+2; 1 - \frac{x}{b} \right), \end{aligned} \quad (30)$$

$$\begin{aligned} &\int_0^1 \frac{t^q (1-t)^s}{(ta + (1-t)x)^{2q}} dt \\ &= \frac{\beta(q+1, s+1)}{x^{2q}} \\ &\cdot {}_2F_1 \left(2q, q+1, q+s+2; 1 - \frac{a}{x} \right), \end{aligned} \quad (31)$$

$$\begin{aligned} &\int_0^1 \frac{t^q (1-t)^s}{(tb + (1-t)x)^{2q}} dt \\ &= \frac{\beta(s+1, q+1)}{b^{2q}} \\ &\cdot {}_2F_1 \left(2q, s+1, q+s+2; 1 - \frac{a}{x} \right). \end{aligned} \quad (32)$$

This completes the proof. $\square$

Remark 27. If we take $m = 1$ in Theorem 26, we get Theorem 4.

Corollary 28. In Theorem 26, additionally, if $|f'(x)| \leq M$, $x \in [a, b/m]$, then inequality

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} M \left\{ (x-a)^2 \right. \\ &\cdot (\lambda_1(a, x, s, q, q) + m\lambda_2(a, x, s, q, q))^{1/q} \\ &+ (b-x)^2 \\ &\cdot (\lambda_3(b, x, s, q, q) + m\lambda_4(b, x, s, q, q))^{1/q} \Big\} \end{aligned} \quad (33)$$

holds.

Remark 29. If we take $m = 1$ in Corollary 28, we get Corollary 5.

Theorem 30. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, b/m]$ for $q \geq 1$ with $s \in [0, 1]$, then, for all $x \in [a, b]$, one has

$$\begin{aligned} \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq \frac{ab}{b-a} \left(\frac{1}{2} \right)^{1-1/q} \left\{ (x \right. \\ &- a)^2 \left(\lambda_1(a, x, s, q, 1) |f'(x)|^q \right. \\ &+ m\lambda_2(a, x, s, q, 1) \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} + (b-x)^2 \\ &\cdot \left(\lambda_3(b, x, s, q, 1) |f'(x)|^q \right. \\ &+ m\lambda_4(b, x, s, q, 1) \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} \Big\}, \end{aligned} \quad (34)$$

where, $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Proof. From Lemma 25, power mean inequality, and harmonic (s, m) -convexity in second sense of $|f'|^q$ on $[a, b/m]$, we have

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left\{ (x-a)^2 \int_0^1 \frac{t}{(ta + (1-t)x)^2} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right| dt \right. \\
 & \quad + (b-x)^2 \int_0^1 \frac{t}{(tb + (1-t)x)^2} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right| dt \Big\} \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 t dt \right)^{1-1/q} \\
 & \quad \cdot \left(\int_0^1 \frac{t}{(ta + (1-t)x)^{2q}} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right|^q dt \right)^{1/q} + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 t dt \right)^{1-1/q} \\
 & \quad \cdot \left(\int_0^1 \frac{t}{(tb + (1-t)x)^{2q}} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right|^q dt \right)^{1/q} \\
 & \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t [t^s |f'(x)|^q + m(1-t)^s |f'(a/m)|^q]}{(ta + (1-t)x)^{2q}} dt \right)^{1/q} \\
 & \quad + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t [t^s |f'(x)|^q + m(1-t)^s |f'(b/m)|^q]}{(tb + (1-t)x)^{2q}} dt \right)^{1/q} \leq \frac{ab}{b-a} \left(\frac{1}{2} \right)^{1-1/q} \\
 & \quad \cdot \left\{ (x-a)^2 \left(\lambda_1(a, x, s, q, 1) |f'(x)|^q + m\lambda_2(a, x, s, q, 1) \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \right. \\
 & \quad \left. + (b-x)^2 \left(\lambda_3(b, x, s, q, 1) |f'(x)|^q + m\lambda_4(b, x, s, q, 1) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right\}. \tag{35}
 \end{aligned}$$

This completes the proof. $\square$

Remark 31. If we take $m = 1$ in Theorem 30, we get Theorem 6.

Corollary 32. In Theorem 30, additionally, if $|f'(x)| \leq M$, $x \in [a, b/m]$, then inequality

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} M \left(\frac{1}{2} \right)^{1-1/q} \\
 & \quad \cdot \left\{ (x-a)^2 (\lambda_1(a, x, s, q, 1) + m\lambda_2(a, x, s, q, 1))^{1/q} \right. \\
 & \quad + (b-x)^2 \\
 & \quad \left. \cdot (\lambda_3(b, x, s, q, 1) + m\lambda_4(b, x, s, q, 1))^{1/q} \right\} \tag{36}
 \end{aligned}$$

holds.

Remark 33. If we take $m = 1$ in Corollary 32, then we get Corollary 7.

Theorem 34. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense

on $[a, b/m]$ for $q \geq 1$ with $s \in [0, 1]$, then, for all $x \in [a, b]$, one has

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left\{ \lambda_5^{1-1/q}(a, x) (x \right. \\
 & \quad - a)^2 \left(\lambda_1(a, x, s, 1, 1) |f'(x)|^q \right. \\
 & \quad + m\lambda_2(a, x, s, 1, 1) \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} + \lambda_5^{1-1/q}(b, x) \\
 & \quad \cdot (b-x)^2 \left(\lambda_3(b, x, s, 1, 1) |f'(x)|^q \right. \\
 & \quad \left. + m\lambda_4(b, x, s, 1, 1) \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} \right\}, \tag{37}
 \end{aligned}$$

where

$$\lambda_5(\theta, x) = \frac{1}{\theta - x} \left\{ \frac{1}{x - \theta} - \frac{\ln x - \ln a}{x - \theta} \right\}, \tag{38}$$

and $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Proof. From Lemma 25, power mean inequality, and harmonic (s, m) -convexity in second sense of $|f'|^q$ on $[a, b/m]$, we have

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left\{ (x-a)^2 \right. \\
 & \cdot \int_0^1 \frac{t}{(ta + (1-t)x)^2} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right| dt \\
 & + (b-x)^2 \\
 & \cdot \int_0^1 \frac{t}{(tb + (1-t)x)^2} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right| dt \Big\} \\
 & \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t}{(ta + (1-t)x)^2} dt \right)^{1-1/q} \\
 & \cdot \left(\int_0^1 \frac{t}{(ta + (1-t)x)^2} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right|^q dt \right)^{1/q} \\
 & + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t}{(tb + (1-t)x)^2} dt \right)^{1-1/q} \\
 & \cdot \left(\int_0^1 \frac{t}{(tb + (1-t)x)^2} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right|^q dt \right)^{1/q} \\
 & \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t}{(ta + (1-t)x)^2} dt \right)^{1-1/q} \\
 & \cdot \left(\int_0^1 \frac{t [t^s |f'(x)|^q + m(1-t)^s |f'(a/m)|^q]}{(ta + (1-t)x)^{2q}} dt \right)^{1/q} \\
 & + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t}{(tb + (1-t)x)^2} dt \right)^{1-1/q} \\
 & \cdot \left(\int_0^1 \frac{t [t^s |f'(x)|^q + m(1-t)^s |f'(b/m)|^q]}{(tb + (1-t)x)^{2q}} dt \right)^{1/q}.
 \end{aligned} \tag{39}$$

It is easy to check that

$$\begin{aligned}
 \int_0^1 \frac{t}{(ta + (1-t)x)^2} dt &= \frac{1}{x-a} \left\{ \frac{1}{a} - \frac{\ln x - \ln a}{x-a} \right\} \\
 \int_0^1 \frac{t}{(tb + (1-t)x)^2} dt &= \frac{1}{b-x} \left\{ \frac{\ln x - \ln a}{x-a} - \frac{1}{b} \right\}.
 \end{aligned} \tag{40}$$

Hence, by use of (29), (30), (31), and (32) for $q = 1$ and (40) in (39), we get the desired result. $\square$

Remark 35. If we take $m = 1$ in Theorem 34, we get Theorem 8.

Corollary 36. In Theorem 34, additionally, if $|f'(x)| \leq M$, $x \in [a, b/m]$, then inequality

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} M \left\{ \lambda_5^{1-1/q}(a, x) \right. \\
 & \cdot (x-a)^2 (\lambda_1(a, x, s, 1, 1) + m\lambda_2(a, x, s, 1, 1))^{1/q} \\
 & + \lambda_5^{1-1/q}(b, x) (b-x)^2 \\
 & \cdot (\lambda_3(b, x, s, 1, 1) + m\lambda_4(b, x, s, 1, 1))^{1/q} \Big\}
 \end{aligned} \tag{41}$$

holds.

Remark 37. If we take $m = 1$ in Corollary 36, we get Corollary 9.

Theorem 38. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, b/m]$ for $q > 1$, $1/p + 1/q = 1$, with $s \in [0, 1]$, then

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left(\frac{1}{p+1} \right)^{1/p} \left\{ (x \right. \\
 & - a)^2 \left(\lambda_1(a, x, s, q, 0) |f'(x)|^q \right. \\
 & + m\lambda_2(a, x, s, q, 0) \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} + (b-x)^2 \\
 & \cdot \left(\lambda_3(b, x, s, q, 0) |f'(x)|^q \right. \\
 & + m\lambda_4(b, x, s, q, 0) \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} \Big\},
 \end{aligned} \tag{42}$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are defined as in Theorem 4.

Proof. From Lemma 25, Hölder's inequality, and harmonic (s, m) -convexity of $|f'|^q$ on $[a, b/m]$, one has

$$\begin{aligned}
 & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 t^p dt \right)^{1/p} \left(\int_0^1 \frac{1}{(ta + (1-t)x)^{2q}} \left| f' \left(\frac{ax}{ta + (1-t)x} \right) \right|^q dt \right)^{1/q} \\
 & + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 t^p dt \right)^{1/p} \left(\int_0^1 \frac{1}{(tb + (1-t)x)^{2q}} \left| f' \left(\frac{bx}{tb + (1-t)x} \right) \right|^q dt \right)^{1/q} \leq \frac{ab(x-a)^2}{b-a} \left(\frac{1}{p+1} \right)^{1/p}
 \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \left(\int_0^1 \frac{[t^s |f'(x)|^q + m(1-t)^s |f'(a/m)|^q]}{(ta + (1-t)x)^{2q}} dt \right)^{1/q} \right. \\
& + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{[t^s |f'(x)|^q + m(1-t)^s |f'(b/m)|^q]}{(tb + (1-t)x)^{2q}} dt \right)^{1/q} \Bigg\} \leq \frac{ab}{b-a} \left(\frac{1}{p+1} \right)^{1/p} \\
& \cdot \left\{ (x-a)^2 \left(\lambda_1(a, x, s, q, 0) |f'(x)|^q + m\lambda_2(a, x, s, q, 0) \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \right. \\
& + (b-x)^2 \left(\lambda_3(b, x, s, q, 0) |f'(x)|^q + m\lambda_4(b, x, s, q, 0) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \Bigg\}.
\end{aligned} \tag{43}$$

This completes the proof. $\square$

Remark 39. If we take $m = 1$ in Theorem 38, we get Theorem 10.

Corollary 40. In Theorem 38, additionally, if $|f'(x)| \leq M$, $x \in [a, b/m]$, then inequality

$$\begin{aligned}
& \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} dx \right| \leq \frac{ab}{b-a} M \left(\frac{1}{p+1} \right)^{1/p} \\
& \cdot \left\{ (x-a)^2 (\lambda_1(a, x, s, q, 0) + m\lambda_2(a, x, s, q, 0))^{1/q} \right. \\
& + (b-x)^2 \\
& \cdot (\lambda_3(b, x, s, q, 0) + m\lambda_4(b, x, s, q, 0))^{1/q} \Bigg\}
\end{aligned} \tag{44}$$

holds.

Remark 41. If we take $m = 1$ in Corollary 40, we get Corollary 11.

Theorem 42. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, b/m]$ for $q > 1$, $1/p + 1/q = 1$, with $s \in [0, 1]$, then one has the following inequality:

$$\begin{aligned}
& \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\
& \leq \frac{ab}{b-a} \left\{ \lambda_1(a, x, 0, p, p) (x-a)^2 \right. \\
& \cdot \left(\frac{|f'(x)|^q + m|f'(a/m)|^q}{s+1} \right)^{1/q} \\
& + \lambda_3(b, x, 0, p, p) (b-x)^2 \\
& \cdot \left(\frac{|f'(x)|^q + m|f'(b/m)|^q}{s+1} \right)^{1/q} \Bigg\},
\end{aligned} \tag{45}$$

where, λ_1 and λ_3 are defined as in Theorem 4.

Proof. From Lemma 25, Hölder's inequality, and harmonic (s, m) -convexity of $|f'|^q$ on $[a, b/m]$, we have

$$\begin{aligned}
& \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\
& \leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t^p}{(ta + (1-t)x)^{2p}} dt \right)^{1/p} \\
& \cdot \left(\int_0^1 [t^s |f'(x)|^q + m(1-t)^s |f'(a/m)|^q] dt \right)^{1/q} \\
& + \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t^p}{(tb + (1-t)x)^{2p}} dt \right)^{1/p} \\
& \cdot \left(\int_0^1 [t^s |f'(x)|^q + m(1-t)^s |f'(b/m)|^q] dt \right)^{1/q} \\
& \leq \frac{ab}{b-a} \left\{ \lambda_1(a, x, 0, p, p) (x-a)^2 \right. \\
& \cdot \left(\frac{|f'(x)|^q + m|f'(a/m)|^q}{s+1} \right)^{1/q} \\
& + \lambda_3(b, x, 0, p, p) (b-x)^2 \\
& \cdot \left(\frac{|f'(x)|^q + m|f'(b/m)|^q}{s+1} \right)^{1/q} \Bigg\}.
\end{aligned} \tag{46}$$

$\square$

Remark 43. If we take $m = 1$ in Theorem 42, we get Theorem 12.

Corollary 44. In Theorem 42, additionally, if $|f'(x)| \leq M$, $x \in [a, b/m]$, then inequality

$$\begin{aligned}
& \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} M \left(\frac{2}{s+1} \right)^{1/q} \\
& \cdot \left\{ (\lambda_1(a, x, 0, p, p))^{1/p} (x-a)^2 \right. \\
& + (\lambda_3(b, x, 0, p, p))^{1/p} (b-x)^2 \Bigg\}
\end{aligned} \tag{47}$$

holds.

Remark 45. Corollary 44 is exactly as Corollary 13.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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