Research Article

Rayleigh-Ritz Vibrational Analysis of Multiwalled Carbon Nanotubes Based on the Nonlocal Flügge Shell Theory

H. Rouhi,¹ M. Bazdid-Vahdati,² and R. Ansari²

¹Department of Engineering Science, Faculty of Technology and Engineering (East of Guilan), University of Guilan, Vajargah, Rudsar 44891-63157, Iran

²Department of Mechanical Engineering, University of Guilan, P.O. Box 3756, Rasht, Iran

Correspondence should be addressed to H. Rouhi; rouhi.hessam@gmail.com

Received 1 June 2015; Accepted 3 December 2015

Academic Editor: Federico Juan Sabina

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A nonlocal elastic shell model considering the small scale effects is developed to study the free vibrations of multiwalled carbon nanotubes subject to different types of boundary conditions. Based on the nonlocal elasticity and the Flügge shell theory, the governing equations are derived which include the interaction of van der Waals forces between adjacent and nonadjacent layers. To analytically solve the problem, the Rayleigh-Ritz method is employed. In the present analysis, different combinations of layerwise boundary conditions are taken into account. Some new intertube resonant frequencies and the associated noncoaxial vibrational modes are identified owing to incorporating circumferential modes into the shell model.

1. Introduction

Nanotechnology is quickly becoming one of the fastest developing fields of research. This is largely due to numerous applications of carbon-based nanostructures in different fields. Among such nanostructures, carbon nanotubes (CNTs) discovered by Iijima [1] in 1991 have attracted the attention of researchers worldwide [2–4].

The theoretical models based on continuum mechanics can be efficiently employed for the analysis of nanostructures. Applications of the classical (local) continuum modeling to the study of CNTs have been suggested in several papers [5–11]. Because the local continuum models are incapable of capturing the influence of small scale of nanostructures, they become controversial to implement for the analysis of such nanoscale systems. So as to consider the small size effects, some other researchers have proposed the application of nonlocal continuum mechanics to study nanostructured materials.

The use of nonlocal elasticity to nanotechnology was first proposed by Peddieson et al. [12]. After that, nonlocal continuum models have been utilized by many researchers owing to

their computational efficiency as well as their accuracy [13–22]. The vibrational analysis of CNTs based on the nonlocal continuum mechanics has been the focus of considerable research. Wang and Varadan [13] investigated the vibration of single-walled and double-walled CNTs (SWCNTs) based on a nonlocal beam model. It was shown that the nonlocal results are in good agreement with the experimental results. Li and Kardomateas [14] investigated the vibrations of multiwalled carbon nanotubes (MWCNTs) subject to simply supported boundary conditions via a nonlocal shell model. It was concluded that the beam models may be insufficient for analyzing the dynamics of nanotubes due to not considering the circumferential mode. Narendar and Gopalakrishnan [16] presented the influence of nonlocal scaling parameter on the wave propagation in MWCNTs using the nonlocal Timoshenko beam model. Arash and Ansari [20] utilized a nonlocal shell model to study the free vibration behavior of SWCNTs. The results indicated that although uncertainty exists in defining nanotube wall thickness, their developed nonlocal shell model is able to predict the results obtained from the molecular dynamics (MD) simulations.

Using shell models in the analysis of CNTs has attracted the attention of several researchers. There are different shell theories. The reader is referred to [23] for a comprehensive summary and discussion on shell theories. A pioneering work in this field is related to Love's paper [24]. Using the first approximation theory of Love, Flügge shell theory [25] was established. The Flügge shell theory is according to the Kirchhoff-Love assumption for thin elastic shells. By means of this theory, the kinematical relations of the middle surface of a cylindrical shell can be derived. Donnell's simplified theory can be also obtained by neglecting few terms in the equations of Flügge shell theory.

The Flügge shell theory is very accurate for thin shells. In addition, CNT is in fact a crystalline membrane with one atom thickness. From this viewpoint, the Flügge shell theory assumptions made in representing a CNT as a homogeneous thin shell can be satisfactory. Also, the Flügge shell theory is frequently used for the analysis of CNTs due to the relatively accurate results in spite of its theoretical simplicity. A marked increase in the volume of research papers dealing with the application of Flügge shell theory to different problems of CNTs stands as testimony to this claim [26–29]. In [27], the axisymmetric and beamlike vibrations of MWCNTs under simply supported boundary conditions were studied using a Flügge-type shell model.

The majority of the previous studies are limited to CNTs with all edges simply supported for which exact analytical solution exists. The boundary conditions can significantly affect the mechanical behavior of nanostructures. In addition, for MWCNTs, different combinations of layerwise boundary conditions are possible to be considered. In this regard, Xu et al. [30] commented that “the relevance of the existing model in which both tubes have the same boundary conditions for the vibration of double-walled CNTs is questionable.”

In this work, a nonlocal Flügge shell model is developed to study the vibrations of MWCNTs with different boundary conditions in a layerwise manner. The analytical Rayleigh-Ritz method is applied to the variational statement derived from the nonlocal Flügge-type vibration equations. Some new noncoaxial intertube resonances for MWCNTs are also predicted in this paper which is a matter of great technical importance in the study of vibrations of nested nanotubes.

2. Elastic Shell Model in Nonlocal Elasticity

Based on Eringen's nonlocal theory [31, 32], the nonlocal stress tensor σ at point x is given by

$$\sigma(x) = \int_V \lambda(|x - x'|, \mu) \mathbf{t}(x') dV(x'), \quad \forall x \in V, \quad (1)$$

where $\mathbf{t}$ denotes the macroscopic stress tensor and $\lambda(|x - x'|, \mu)$ is the nonlocal modulus or attenuation function whose arguments are the Euclidean distance $|x - x'|$ and a material constant $\mu = e_0 a / l$ with a as an internal characteristic length, l as an external characteristic length (e.g., crack length,

FIGURE 1: Schematic of an MWCNT.

wavelength), and e_0 as material constant. The differential form of (1) is

$$\left(1 - (e_0 a)^2 \nabla^2\right) \sigma = \mathbf{t}, \quad (2)$$

where $e_0 a$ is the nonlocal parameter. Hooke's generalized law states that

$$\mathbf{t} = \mathbf{S} : \epsilon \quad (3)$$

in which $\mathbf{S}$ is the fourth-order elasticity tensor and “:” denotes the double dot product. Using (3) leads to

$$\sigma_{xx} - (e_0 a)^2 \nabla^2 \sigma_{xx} = \frac{E}{1 - \nu^2} (\epsilon_{xx} + \nu \epsilon_{\theta\theta}), \quad (4a)$$

$$\sigma_{\theta\theta} - (e_0 a)^2 \nabla^2 \sigma_{\theta\theta} = \frac{E}{1 - \nu^2} (\epsilon_{\theta\theta} + \nu \epsilon_{xx}), \quad (4b)$$

$$\sigma_{x\theta} - (e_0 a)^2 \nabla^2 \sigma_{x\theta} = \frac{E}{2(1 + \nu)} \gamma_{x\theta}, \quad (4c)$$

where E is Young's modulus of the material and ν is the Poisson ratio. x and θ are longitudinal and angular circumferential coordinates. Each tube of MWCNTs is described as an individual cylindrical shell of radius R , length L , and thickness h , as shown in Figure 1. According to the classic shell theory, the three-dimensional displacement components u_x , u_θ , and u_z in x , θ , and z directions, respectively, are of the form

$$u_x(x, \theta, z, t) = u(x, \theta, t) - z \frac{\partial w}{\partial x}(x, \theta, t), \quad (5a)$$

$$u_\theta(x, \theta, z, t) = v(x, \theta, t) - z \frac{\partial w}{\partial \theta}(x, \theta, t), \quad (5b)$$

$$u_z(x, \theta, z, t) = w(x, \theta, z, t), \quad (5c)$$

where u , v , and w are the reference surface displacements. The kinematical relations are given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \gamma_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{1}{R} \frac{\partial v}{\partial \theta} + \frac{w}{R} \\ \frac{\partial v}{\partial x} + \frac{1}{R} \frac{\partial u}{\partial \theta} \end{Bmatrix} + z \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{1}{R^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \\ -\frac{2}{R} \frac{\partial^2 w}{\partial x \partial \theta} + \frac{1}{R} \frac{\partial v}{\partial x} - \frac{1}{R^2} \frac{\partial u}{\partial \theta} \end{Bmatrix}. \quad (6)$$

The stress and moment resultants can be obtained by

$$N = (N_{xx}, N_{\theta\theta}, N_{x\theta})^T dz = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{\theta\theta}, \sigma_{x\theta})^T dz, \quad (7a)$$

$$\begin{aligned} M &= (M_{xx}, M_{\theta\theta}, M_{x\theta})^T dz \\ &= \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{\theta\theta}, \sigma_{x\theta})^T z dz, \end{aligned} \quad (7b)$$

where h is the shell thickness.

Moreover, the stress and moment resultants are obtained as [25]

$$\begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} - (e_0 a)^2 \nabla^2 \begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} = \begin{bmatrix} \frac{Eh}{1-\nu^2} \frac{\partial}{\partial x} & \frac{\nu Eh}{1-\nu^2} \frac{1}{R} \frac{\partial}{\partial \theta} & \frac{\nu Eh}{1-\nu^2} \frac{1}{R} \\ \frac{\nu Eh}{1-\nu^2} \frac{\partial}{\partial x} & \frac{Eh}{1-\nu^2} \frac{1}{R} \frac{\partial}{\partial \theta} & \frac{Eh}{1-\nu^2} \frac{1}{R} \\ \frac{Eh}{2(1+\nu)} \frac{1}{R} \frac{\partial}{\partial \theta} & \frac{Eh}{2(1+\nu)} \frac{\partial}{\partial x} & 0 \\ 0 & D \frac{\nu}{R^2} \frac{\partial}{\partial \theta} & -D \left(\frac{\partial^2}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2}{\partial \theta^2} \right) \\ 0 & D \frac{1}{R^2} \frac{\partial}{\partial \theta} & -D \left(\nu \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \right) \\ 0 & \frac{D}{R} (1-\nu) \frac{\partial}{\partial x} & -\frac{D}{R} (1-\nu) \frac{\partial^2}{\partial x \partial \theta} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (8)$$

in which D is the bending rigidity.

FIGURE 2: Cross-sectional view of an MWCNT under the vdW interactions.

The governing equations on the basis of the Flügge shell theory are given as [25]

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R} \frac{\partial N_{\theta x}}{\partial \theta} = \rho h \frac{\partial^2 u}{\partial t^2}, \quad (9a)$$

$$\frac{1}{R} \frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{1}{R^2} \frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{1}{R} \frac{\partial M_{x\theta}}{\partial x} = \rho h \frac{\partial^2 v}{\partial t^2}, \quad (9b)$$

$$\begin{aligned} \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} + \frac{2}{R} \frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} - \frac{N_{\theta\theta}}{R} + p \\ = \rho h \frac{\partial^2 w}{\partial t^2}, \end{aligned} \quad (9c)$$

where p is the pressure exerted on tube i through the van der Waals (vdW) interaction forces, as indicated in Figure 2. The vdW model is formulated as [33]

$$(p_{vdW})_i = w_i \sum_{j=1}^N c_{ij} - \sum_{j=1}^N c_{ij} w_j, \quad (i = 1, 2, \dots, N), \quad (10)$$

in which vdW coefficients c_{ij} representing the pressure increment contributing to layer i from layer j are given by

$$c_{ij} = \left[\frac{1001\pi\epsilon\sigma^{12}}{3a^4} E_{ij}^{13} - \frac{1120\pi\epsilon\sigma^6}{9a^4} E_{ij}^7 \right] R_j, \quad (11)$$

where $a = 1.42 \text{ \AA}$ is the C-C bond length, ϵ is the depth of the potential, σ is a parameter that is determined by the equilibrium distance and R_j is the radius of j th layer, and E_{ij}^m denotes the elliptic integral defined as

$$E_{ij}^m = (R_j + R_i)^{-m} \int_0^{\pi/2} \frac{d\theta}{[1 - K_{ij} \cos^2 \theta]^{m/2}}, \quad (12)$$

here m is an integer and the coefficient K_{ij} is given by

$$K_{ij} = \frac{4R_j R_i}{(R_j + R_i)^2}. \quad (13)$$

3. Field Equations

For i th layer of an MWCNT, by the use of (8), (9a), (9b), and (9c) can be stated in terms of the three field variables ($u^{(i)}, v^{(i)}, w^{(i)}$, ($i = 1, 2, \dots, N$)):

$$l_{11}^{(i)} u^{(i)} + l_{12}^{(i)} v^{(i)} + l_{13}^{(i)} w^{(i)} = \rho h \ddot{u}^{(i)} - (e_0 a)^2 \rho h \left(\ddot{u}_{xx}^{(i)} + \frac{1}{R_i^2} \ddot{u}_{\theta\theta}^{(i)} \right), \quad (14a)$$

$$l_{21}^{(i)} u^{(i)} + l_{22}^{(i)} v^{(i)} + l_{23}^{(i)} w^{(i)} = \rho h \ddot{v}^{(i)} - (e_0 a)^2 \rho h \left(\ddot{v}_{xx}^{(i)} + \frac{1}{R_i^2} \ddot{v}_{\theta\theta}^{(i)} \right), \quad (14b)$$

$$l_{31}^{(i)} u^{(i)} + l_{32}^{(i)} v^{(i)} + l_{33}^{(i)} w^{(i)} = \rho h \ddot{w}^{(i)} + w^{(i)} \sum_{j=1, j \neq i}^N c_{ij} - \sum_{j=1, j \neq i}^N c_{ij} w^{(j)} - (e_0 a)^2 \left[\rho h \left(\ddot{w}_{xx}^{(i)} + \frac{1}{R_i^2} \ddot{w}_{\theta\theta}^{(i)} \right) \right]$$

$$+ \left(w_{xx}^{(i)} + \frac{1}{R_i^2} w_{\theta\theta}^{(i)} \right) \sum_{j=1, j \neq i}^N c_{ij} - \sum_{j=1, j \neq i}^N c_{ij} \left(w_{xx}^{(j)} + \frac{1}{R_i^2} w_{\theta\theta}^{(j)} \right) \right], \quad (14c)$$

where $l_{pq}^{(i)}$ ($p, q = 1, 2, 3$) are the partial operators, which are given in the Appendix.

4. Solution Using the Rayleigh-Ritz Method

4.1. Variational Form of the Field Equations. According to the semi-inverse method [34–36], variational trial-functional $\Pi(u^{(1)}, u^{(2)}, \dots, u^{(N)}, v^{(1)}, v^{(2)}, \dots, v^{(N)}, w^{(1)}, w^{(2)}, \dots, w^{(N)})$ is constructed as

$$\Pi = \sum_{i=1}^N \Pi^{(i)}(u^{(i)}, v^{(i)}, w^{(1)}, w^{(2)}, \dots, w^{(N)}), \quad (15)$$

where

$$\begin{aligned} \Pi^{(i)}(u^{(i)}, v^{(i)}, w^{(1)}, w^{(2)}, \dots, w^{(N)}) \\ = \Pi_K^{(i)}(u^{(i)}, v^{(i)}, w^{(i)}) + \Pi_M^{(i)}(u^{(i)}, v^{(i)}, w^{(i)}) \\ + \Pi_{vdW}^{(i)}(w^{(1)}, w^{(2)}, \dots, w^{(N)}) \end{aligned} \quad (16)$$

in which

$$\begin{aligned} \Pi_K^{(i)}(u^{(i)}, v^{(i)}, w^{(i)}) &= \frac{1}{2} \int_0^T \int_{\Omega} \left(\left[\frac{Eh}{1-\nu^2} u_{,x}^{(i)} + \frac{1}{R_i} \frac{\nu Eh}{1-\nu^2} (v_{,\theta}^{(i)} + w^{(i)}) \right] u_{,x}^{(i)} + \frac{Eh}{2(1+\nu)} \left(\frac{u_{,\theta}^{(i)}}{R_i} + v_{,x}^{(i)} \right)^2 \right. \\ &\quad + \left[\frac{\nu Eh}{1-\nu^2} u_{,x}^{(i)} + \frac{1}{R_i} \frac{Eh}{1-\nu^2} (v_{,\theta}^{(i)} + w^{(i)}) \right] \frac{1}{R_i} (v_{,\theta}^{(i)} + w^{(i)}) + D \left(w_{,xx}^{(i)} + \frac{\nu}{R_i^2} w_{,\theta\theta}^{(i)} \right) w_{,xx}^{(i)} + \frac{D}{R_i^2} \left(\left(\frac{1}{R_i^2} \right) w_{,\theta\theta}^{(i)} + \nu w_{,xx}^{(i)} \right) w_{,\theta\theta}^{(i)} \\ &\quad \left. + \frac{2(1-\nu)D}{R_i^2} (w_{,x\theta}^{(i)} - v_{,x}^{(i)})^2 - \frac{2D}{R_i^2} \left(\nu w_{,xx}^{(i)} + \frac{1}{R_i^2} w_{,\theta\theta}^{(i)} \right) v_{,\theta}^{(i)} \right) d\Omega dt, \\ \Pi_M^{(i)}(u^{(i)}, v^{(i)}, w^{(i)}) &= \frac{1}{2} \int_0^T \int_{\Omega} \rho h \left((\dot{u}^{(i)})^2 - (e_0 a)^2 \left[\left(\dot{u}_{,xx}^{(i)} + \frac{1}{R_i^2} \dot{u}_{,\theta\theta}^{(i)} \right) \dot{u}^{(i)} \right] + (\dot{v}^{(i)})^2 - (e_0 a)^2 \left[\left(\dot{v}_{,xx}^{(i)} + \frac{1}{R_i^2} \dot{v}_{,\theta\theta}^{(i)} \right) \dot{v}^{(i)} \right] \right. \\ &\quad \left. + (\dot{w}^{(i)})^2 - (e_0 a)^2 \left[\left(\dot{w}_{,xx}^{(i)} + \frac{1}{R_i^2} \dot{w}_{,\theta\theta}^{(i)} \right) \dot{w}^{(i)} \right] \right) d\Omega dt, \\ \Pi_{vdW}^{(i)}(w^{(1)}, w^{(2)}, \dots, w^{(N)}) &= \frac{1}{2} \\ &\quad \cdot \sum_{j=1}^N \int_0^T \int_{\Omega} \left(c_{ij} (w^{(i)} - w^{(j)})^2 - (e_0 a)^2 \left[w^{(i)} \left(w_{,xx}^{(i)} + \frac{1}{R_i^2} w_{,\theta\theta}^{(i)} \right) + w^{(j)} \left(w_{,xx}^{(j)} + \frac{1}{R_i^2} w_{,\theta\theta}^{(j)} \right) \right] \right) d\Omega dt. \end{aligned} \quad (17)$$

TABLE 1: Values of α_i , λ_m , and ζ_m for SS, CC, FF, CS, CF, and FS boundary conditions.

Boundary condition	α_i ($i = 1, \dots, 4$)	Values for λ_m	ζ_m
SS	$\alpha_1 = 0, \alpha_2 = 0$ $\alpha_3 = 0, \alpha_4 = -1$	$m\pi$	1
CC	$\alpha_1 = 1, \alpha_2 = -1$ $\alpha_3 = 1, \alpha_4 = -1$	$\cos \lambda_m \cosh \lambda_m = 1$	$\frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}$
FF	$\alpha_1 = 1, \alpha_2 = 1$ $\alpha_3 = 1, \alpha_4 = 1$	$\cos \lambda_m \cosh \lambda_m = 1$	$\frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}$
CS	$\alpha_1 = 1, \alpha_2 = -1$ $\alpha_3 = 1, \alpha_4 = -1$	$\tan \lambda_m = \tanh \lambda_m$	$\frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}$
CF	$\alpha_1 = 1, \alpha_2 = -1$ $\alpha_3 = 1, \alpha_4 = -1$	$\cos \lambda_m \cosh \lambda_m = -1$	$\frac{\sinh \lambda_m - \sin \lambda_m}{\cosh \lambda_m + \cos \lambda_m}$
FS	$\alpha_1 = 1, \alpha_2 = 1$ $\alpha_3 = 1, \alpha_4 = 1$	$\tan \lambda_m = \tanh \lambda_m$	$\frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}$

4.2. Implementation of the Rayleigh-Ritz Method with Beam Functions. The field variables corresponding to the i th tube, that is, $u^{(i)}$, $v^{(i)}$, and $w^{(i)}$, can be expressed in the form

$$u^{(i)}(x, \theta, t) = A^{(i)} \frac{\partial \phi(x)}{\partial x} \cos(n\theta) \sin(\omega t), \quad (18a)$$

$$v^{(i)}(x, \theta, t) = B^{(i)} \phi(x) \sin(n\theta) \sin(\omega t), \quad (18b)$$

$$w^{(i)}(x, \theta, t) = C^{(i)} \phi(x) \cos(n\theta) \sin(\omega t), \quad (18c)$$

where $A^{(i)}$, $B^{(i)}$, and $C^{(i)}$ are the constant parameters denoting the amplitudes of the vibrations, n is the circumferential wave number, ω is the angular frequency of vibrating CNT, and $\phi(x)$ is the axial function which satisfies the geometric boundary conditions. The axial function $\phi(x)$ is chosen as the characteristics beam function as

$$\begin{aligned} \phi(x) = & \alpha_1 \cosh\left(\frac{\lambda_m x}{L}\right) + \alpha_2 \cos\left(\frac{\lambda_m x}{L}\right) \\ & - \zeta_m \left[\alpha_3 \sinh\left(\frac{\lambda_m x}{L}\right) + \alpha_4 \sin\left(\frac{\lambda_m x}{L}\right) \right] \end{aligned} \quad (19)$$

in which α_i ($i = 1, \dots, 4$) are constants with value 0, 1, or -1 depending on the tube ends, λ_m is the roots of the transcendental equations obtained from the CNT boundary conditions, and ζ_m is the parameters corresponding to λ_m . The geometric boundary conditions for clamped (C), free (F), and simply supported (S) boundary conditions can be expressed mathematically in terms of $\phi(x)$ as follows.

Clamped boundary condition is

$$\phi(x) = \phi'(x) = 0. \quad (20)$$

Free boundary condition is

$$\phi''(x) = \phi'''(x) = 0. \quad (21)$$

Simply supported boundary condition is

$$\phi(x) = \phi''(x) = 0. \quad (22)$$

Parameters α_i , λ_m , and ζ_m that are chosen according to the CNT boundary conditions are given in Table 1. Substituting (18a), (18b), and (18c) into (15) and then minimizing the energy functional Π with respect to unknown coefficients $A^{(i)}$, $B^{(i)}$, and $C^{(i)}$ result in the following algebraic equations:

$$\begin{aligned} \frac{\partial \Pi}{\partial A^{(i)}} &= 0, \\ \frac{\partial \Pi}{\partial B^{(i)}} &= 0, \\ \frac{\partial \Pi}{\partial C^{(i)}} &= 0, \end{aligned} \quad (23)$$

$(i = 1, 2, \dots, N).$

The above equations can be recast in the form of a generalized eigenvalue problem as

$$([\mathbf{K}] - \omega^2 [\mathbf{M}]) \{\mathbf{X}\} = 0, \quad (24)$$

where $[\mathbf{K}]$ is the stiffness matrix, $[\mathbf{M}]$ is the mass matrix of the CNT, and $\{\mathbf{X}\}$ is given by

$$\{\mathbf{X}\}^T = \{A^{(1)}, B^{(1)}, C^{(1)}, A^{(2)}, B^{(2)}, C^{(2)}, \dots, A^{(N)}, B^{(N)}, C^{(N)}\}. \quad (25)$$

By solving this eigenvalue problem, the natural frequencies of MWCNTs can be obtained and the associated eigenvectors yield the corresponding mode shapes.

5. Numerical Results and Discussion

The mechanical properties and thickness of each tube of MWCNTs are assumed to be $E = 1$ TPa, $D = 0.85$ eV,

FIGURE 3: First resonant frequencies from the present shell model and MD simulation [37] for clamped single-walled CNT ($R = 0.34$ nm).

$h = 0.34$ nm, $\nu = 0.27$, and $\rho = 2300$ Kg/m³. The configuration of layerwise boundary conditions, for example, will be denoted by (SS/CF/CC), where SS is related to the outermost tube and CC is related to the innermost tube. Also, for given intertube mode number p , for convenience, the frequency associated with the n th axial mode will be denoted by $\omega_n(p)$.

5.1. Validation of the Present Approach. For validation, the present results are compared with the MD results given in [37]. The first resonant frequencies of clamped and cantilever SWCNTs against nanotube aspect ratio (L/d) are depicted in Figures 3 and 4, respectively. The nonlocal parameter e_0a should be calibrated. The calibrated values for e_0a associated with clamped-clamped and clamped-free boundary conditions are 1.98 and 2 nm, respectively. It means that the calibrated value of e_0a depends on end conditions. Moreover, Figure 3 provides a comparison between the nonlocal shell model and its local counterpart for clamped end conditions. It is observed that the local shell model ($e_0a = 0$) overestimates the frequencies, particularly for small aspect ratios. As the aspect ratio increases, resonant frequencies decrease and the small scale effect diminishes so that the frequency envelopes tend to converge.

5.2. Explanatory Examples

Example 1. The fundamental resonant frequencies of a triple-walled CNT obtained by the present nonlocal shell model versus nanotube aspect ratio are graphed in Figure 5. Two different layerwise boundary conditions have been considered. The values of nonlocal parameter are varied from 0 to 1 nm. From this figure, one can see that the effects of the boundary conditions and small scale become more

FIGURE 4: First resonant frequencies from the present shell model and MD simulation [37] for cantilever single-walled CNT ($R = 0.34$ nm).

FIGURE 5: Fundamental resonant frequency curves for a triple-walled CNT corresponding to various end conditions and nonlocal parameters ($R_1 = 8.5$ nm).

significant for shorter nanotubes. Moreover, it is observed that the frequencies decrease by increasing e_0a . As the ratio of length-to-innermost radius increases, resonant frequencies tend to decrease and the effects of small length scale and CNT end conditions diminish.

Example 2. Since the effective thicknesses of nanotubes are scattered in the range of 0.066–0.34 nm, Figure 6 is presented to demonstrate the influence of thickness variation on the fundamental resonant frequencies of a triple-walled CNT

FIGURE 6: Effect of the nanotube thickness variation on the fundamental resonant frequencies of a fully clamped triple-walled CNT ($R_1 = 0.35$ nm).

with clamped end conditions (CC/CC/CC). From this figure, the frequency difference can be observed due to this effect in both local and nonlocal models. Furthermore, resonant frequencies of CNTs with $h = 0.066$ nm are higher than those of CNTs with $h = 0.34$ nm when aspect ratio decreases.

Example 3. Figure 7 depicts the natural frequency of a simply supported triple-walled CNT versus circumferential mode number. Different values of nonlocal parameter have been considered ranging from 0 to 1.5 nm. Once again, it is found that the natural frequency diminishes with increasing nonlocal parameter. Physical interpretation is that the small scale effects in the nonlocal model make nanotubes more flexible. It is further observed that the magnitude of decrease in natural frequencies corresponding to higher circumferential modes is considerably higher than those corresponding to lower ones. This reveals that the influence of the small scale becomes more important for shorter wavelength at higher modes.

Example 4. Figure 8 shows the frequency ratio of a triple-walled (SS/SS/SS) CNT for several length-to-innermost radius ratios. It can be found that the effects of the small length scale are more significant for shorter length CNTs. For example, in the case of frequencies of nanotube with $L/R_1 = 5$ a relative error equal to 26.5% for nonlocal parameter $e_0a = 2$ nm is obtained. This relative error reduces to about 9% as L/R_1 increases by 20.

Example 5. To show the ability of the present shell model in predicting new intertube frequencies and corresponding noncoaxial vibrational modes, mode shapes of a simply supported triple-walled CNT are shown in Figure 9. As revealed

FIGURE 7: Variation of natural frequency with circumferential mode number for a triple-walled CNT with simply supported end conditions ($R_1 = 8.5$ nm, $L/R_1 = 5$).

FIGURE 8: Effect of the small length scale on the frequency ratio for a simply supported triple-walled CNT corresponding to various length-to-innermost radius ratios ($R_1 = 8.5$ nm).

in this figure, noncoaxial vibrational modes are predictable. Furthermore, noncoaxial vibrational modes may shift to the ones corresponding to higher circumferential mode numbers as the radius of MWCNT increases. The three-dimensional vibrational mode shape of a simply supported triple-walled CNT associated with the first intertube and the fifth axial and circumferential modes is also plotted in Figure 10. This figure is accompanied by a cross-sectional view in the middle of the nanotube.

FIGURE 9: Mode shapes of a simply supported triple-walled CNT predicted by the shell model ($R_1 = 2$ nm, $L/R_1 = 10$).

FIGURE 10: Three-dimensional mode shape associated with the first intertube and the fifth axial and circumferential modes for a triple-walled CNT with simply supported boundary conditions ($R_1 = 8.5$ nm, $L/R_1 = 5$).

Example 6. This example provides a comparison between the present shell model and the beam model given by Xu et al. [30]. The first three natural frequencies of a double-walled CNT with different layerwise boundary conditions obtained by the present shell model and by the beam model [30] versus nanotube length are tabulated in Table 2. It is observed that, for long double-walled CNTs for which the beam-like

vibrations are dominant, the two models agree reasonably well. However, the results obtained by the beam model for double-walled CNTs of finite length are overestimated due to not taking circumferential modes into consideration. Moreover, this reduction in the size of nanotubes brings the effects of small scale and boundary conditions into focus and accordingly makes the present nonlocal shell model more preferable than the local beam counterpart.

Example 7. Presented graphically in Figure 11 are the frequency ratios related to a simply supported CNT with different number of walls. It is seen that the effects of the small length scale are dependent on the number of tubes so that the relative error in resonant frequencies decreases by increasing the number of walls. For $e_0 a = 2$ nm as an example, the relative errors corresponding to the double-walled and five-walled CNTs are approximately 19% and 12.5%, respectively.

6. Concluding Remarks

Considering various layerwise boundary conditions, this paper probed the free vibrations of MWCNTs based on a nonlocal elastic shell model. Using the Flügge shell theory, the displacement field equations coupled by vdW forces were derived. The variational form of the Flügge type equations was constructed to which the analytical Rayleigh-Ritz method was applied. Among the more significant conclusions

TABLE 2: Comparison between the present shell model and the beam model given in [30] for the first three natural frequencies of a double-walled CNT with different boundary conditions ($R_1 = 0.35$ nm, $e_0 a = 0$).

Frequencies (THz)		Length (nm)							
		14	16	18	20	24	26	28	30
CC/SS	ω_1^a	0.8760	0.6876	0.5531	0.4539	0.3208	0.2750	0.2383	0.2084
	ω_1^b	1.05	0.81	0.64	0.52	0.36	0.31	0.27	0.23
	ω_2^a	2.2109	1.7654	1.4385	1.1925	0.8548	0.7365	0.6407	0.5623
	ω_2^b	2.84	2.20	1.75	1.43	1.00	0.85	0.73	0.64
	ω_3^a	3.8679	3.1476	2.6038	2.1848	1.5944	1.3828	1.2098	1.0666
	ω_3^b	5.18	4.17	3.36	2.76	1.94	1.66	1.43	1.25
CC/FF	ω_1^a	0.9290	0.7276	0.5843	0.4790	0.3381	0.2897	0.2509	0.2194
	ω_1^b	1.04	0.8	0.53	0.51	0.36	0.31	0.26	0.23
	ω_2^a	2.3230	1.8484	1.5027	1.2437	0.8898	0.7661	0.6662	0.5844
	ω_2^b	2.84	2.2	1.75	1.42	0.99	0.85	0.73	0.64
	ω_3^a	4.0609	3.2857	2.7080	2.2666	1.6491	1.4290	1.2493	1.1008
	ω_3^b	5.14	4.16	3.36	2.76	1.94	1.66	1.43	1.25
CC/CC	ω_1^a	0.9277	0.7269	0.5839	0.4788	0.3380	0.2896	0.2509	0.2193
	ω_1^b	1.08	—	—	—	—	—	—	—
	ω_2^a	2.3176	1.8455	1.5010	1.2427	0.8893	0.7658	0.6660	0.5842
	ω_2^b	2.94	—	—	—	—	—	—	—
	ω_3^a	4.0420	3.2756	2.7021	2.2630	1.6475	1.4279	1.2484	1.1002
	ω_3^b	5.49	—	—	—	—	—	—	—

^aBased on the present shell model.

^bBased on the beam model given by Xu et al. [30].

FIGURE 11: Effect of the small length scale on the frequency ratio for a simply supported CNT with different number of walls ($R_N = 8.5$ nm, $L/R_N = 10$).

to be obtained, the following findings may be summarized from the present study:

- (i) The efficiency of the present shell model was checked by the MD simulation and nonlocal parameters were calibrated for clamped and clamped-free SWCNTs.
- (ii) The small scale effects in the nonlocal continuum model reduce the frequencies of CNT as compared to the predictions of classical model.
- (iii) It was shown that the significance of the small size effects on the natural frequencies of MWCNTs is dependent on the geometric parameters of CNTs, vibrational modes, boundary conditions, and number of walls.
- (iv) The elastic beam model tends to overestimate the resonant frequencies of CNTs as compared to its shell counterpart, due to not incorporating circumferential mode number into the model, especially for shorter CNTs.
- (v) The results generated provide valuable information concerning new noncoaxial modes affecting the properties of MWCNTs.

Appendix

Partial operators appeared in (14a), (14b), and (14c):

$$\begin{aligned}
 l_{11}^{(i)} &= \frac{Eh}{1-\nu^2} \frac{\partial^2}{\partial x^2} + \frac{1}{R_i^2} \frac{Eh}{2(1+\nu)} \frac{\partial^2}{\partial \theta^2}, \\
 l_{12}^{(i)} &= \frac{1}{R_i} \left(\frac{Eh\nu}{1-\nu^2} + \frac{Eh}{2(1+\nu)} \right) \frac{\partial^2}{\partial x \partial \theta}, \\
 l_{13}^{(i)} &= \frac{1}{R_i} \frac{E\nu h}{1-\nu^2} \frac{\partial}{\partial x}, \\
 l_{21}^{(i)} &= l_{12}^{(i)}, \\
 l_{22}^{(i)} &= \left(\frac{Eh}{2(1+\nu)} + \frac{D}{R_i^2} (1-\nu) \right) \frac{\partial^2}{\partial x^2} \\
 &\quad + \frac{1}{R_i^2} \left(\frac{Eh}{1-\nu^2} + \frac{D}{R_i^2} \right) \frac{\partial^2}{\partial \theta^2}, \\
 l_{23}^{(i)} &= -\frac{D}{R_i^2} \frac{\partial^3}{\partial x^2 \partial \theta} - \frac{\nu D}{R_i^4} \frac{\partial^3}{\partial \theta^3} + \frac{1}{R_i^2} \frac{Eh}{1-\nu^2} \frac{\partial}{\partial \theta}, \\
 l_{31}^{(i)} &= -l_{13}^{(i)}, \\
 l_{32}^{(i)} &= \frac{D}{R_i^2} (2-\nu) \frac{\partial^3}{\partial x^2 \partial \theta} + \frac{D}{R_i^4} \frac{\partial^3}{\partial \theta^3} - \frac{1}{R_i^2} \frac{Eh}{1-\nu^2} \frac{\partial}{\partial \theta}, \\
 l_{33}^{(i)} &= -D \frac{\partial^4}{\partial x^4} - \frac{2D}{R_i^2} \frac{\partial^4}{\partial x^2 \partial \theta^2} - \frac{D}{R_i^4} \frac{\partial^4}{\partial \theta^4} - \frac{1}{R_i^2} \frac{Eh}{1-\nu^2}.
 \end{aligned} \tag{A.1}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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