A Generalization of the Fuglede-Putnam Theorem to Unbounded Operators

Fotios C. Paliogiannis

Department of Mathematics, St. Francis College, 180 Remsen Street, Brooklyn Heights, NY 11201, USA

Correspondence should be addressed to Fotios C. Paliogiannis; fpaliogiannis@sfc.edu

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In this note we prove a generalization of the classical Fuglede-Putnam theorem to unbounded operators. A special case of this generalization is given in [1]. We begin with some preliminary results.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators in $\mathcal{H}$. Let $Op(\mathcal{H})$ denote the set of unbounded densely defined linear operators in $\mathcal{H}$. For $A \in Op(\mathcal{H})$ we denote the domain of $A$ by $\mathcal{D}(A)$. Given $A, B \in Op(\mathcal{H})$, the operator $B$ is called an extension of $A$, denoted by $A \subseteq B$, if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ax = Bx$ for all $x \in \mathcal{D}(A)$. An operator $A \in Op(\mathcal{H})$ is called closed if $A = A^\star$ (the closure of $A$). A closed densely defined operator $A \in Op(\mathcal{H})$ is said to commute with the bounded operator $T \in B(\mathcal{H})$, if $TA \subseteq AT$. This means that for each $x \in \mathcal{D}(A)$, we have $Tx \in \mathcal{D}(A)$ and $TAx = A^\star x$. Let $[A]_T = \{T \in B(\mathcal{H}) : TA \subseteq AT\}$. If $A \in B(\mathcal{H})$ this notion agrees with the usual notion of commutant. One sees $[A]_T$ is a strongly closed subalgebra of $B(\mathcal{H})$, and $T \in [A]_T$ if and only if $T^\star \in [A^*]^I$. Hence, $[A]^I \cap [A^*]^I$ is a von Neumann algebra.

Definition 1. Let $A \in Op(\mathcal{H})$ be closed and $\mathcal{A}$ a von Neumann algebra. If $\mathcal{A}' \subseteq [A]^I$, the operator $A$ is said to be affiliated with $\mathcal{A}$, denoted by $A \mathcal{A}'$.

The algebra $W^*(A) = \{[A]^I \cap [A^*]^I'\}'$ is the smallest von Neumann algebra with which $A$ is affiliated, and is referred to it as the von Neumann algebra generated by $A$.

Definition 2. Let $A \in Op(\mathcal{H})$. A bounding sequence for $A$ is a non-decreasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of projections on $\mathcal{H}$ such that $\bigcup_{n=1}^{\infty} F_n = I$, $F_nA \subseteq AF_n$ and $AF_n \in B(\mathcal{H})$ for all $n \in \mathbb{N}$.

A closed operator $N \in Op(\mathcal{H})$ is normal if $N^*N = NN^*$. This implies that $\mathcal{D}(N) = \mathcal{D}(N^*)$ and $\|Nx\| = \|N^*x\|$ for every $x \in \mathcal{D}(N)$ [2, page 51]. It turns out that the von Neumann algebra $W^*(N)$ is abelian, and $W^*(N) = [N]^I$ [3]. Hence, from Lemma 3, there is a bounding sequence $\{E_n\}$ for $N$ in $W^*(N)$.

Theorem 4 (Fuglede-Putnam). Let $N$ and $M$ be normal operators in a Hilbert space. If $T$ is any bounded operator satisfying $TN \subseteq MT$, then $TN^* \subseteq M^*T$.

The following result from [2, page 97] is essential to our proof of the generalization of the Fuglede-Putnam theorem.
Lemma 5. Let $A_1, A_2 \in \text{Op}(H)$ be self-adjoint operators and let $T \in \mathcal{B}(H)$. Then $TA_1 \subseteq A_2 T$ if and only if $TE_{\lambda} = P_{\lambda}$ for all $\lambda \in \mathbb{R}$, where $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ and $\{P_{\lambda}\}_{\lambda \in \mathbb{R}}$ are the spectral families of $A_1$ and $A_2$, respectively.

Theorem 6. Let $N, M \in \text{Op}(H)$ be normal operators and let $T \in \text{Op}(H)$ be a closed operator such that $\mathcal{D}(N) \subseteq \mathcal{D}(T)$ and $\mathcal{D}(M) \subseteq \mathcal{D}(T^*)$. If $TN \subseteq MT$, then $TN^* \subseteq M^* T$.

Proof. Let $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ and $\{P_{\lambda}\}_{\lambda \in \mathbb{R}}$ be the spectral families of the self-adjoint operators $N^* N$ and $M^* M$, respectively. For $m, n \in \mathbb{N}$, consider the bounding sequences $F_n = E_n - E_{n-1}$ and $G_m = P_m - P_{m-1}$ for $N$ and $M$, respectively. Since $\mathcal{D}(N) \subseteq \mathcal{D}(T)$, it follows $\mathcal{H} = \mathcal{D}(N^*) \subseteq \mathcal{D}(T^*)$. Since $TF_n$ is closed, the closed graph theorem implies $TF_n \in \mathcal{B}(\mathcal{H})$. Similarly, by the hypothesis on the domain of $M$ and the closed graph theorem, we see $T^* G_m \in \mathcal{B}(\mathcal{H})$.

From the hypothesis $TN \subseteq MT$, we have $TNF_n \subseteq MTF_n$. Moreover, since $F_nN \subseteq NF_n$, we also have $TF_nN \subseteq TNF_n$.

Hence,

$$ (TF_n)N \subseteq M(TF_n), \quad \forall n \in \mathbb{N}. \quad (1) $$

Since $TF_n$ is bounded, the Fuglede-Putnam theorem implies

$$ (TF_n)N^* \subseteq M^*(TF_n), \quad \forall n \in \mathbb{N}. \quad (2) $$

From (1), (2), we have

$$ (TF_n)N^* N \subseteq M^*(TF_n)N \subseteq M^* M(TF_n). \quad (3) $$

That is,

$$ (TF_n)N^* N \subseteq M^* M(TF_n), \quad \forall n \in \mathbb{N}. \quad (3) $$

Consequently, from Lemma 5,

$$ (TF_n)E_{\lambda} = P_{\lambda}(TF_n), \quad \forall \lambda \in \mathbb{R}. \quad (4) $$

Therefore

$$ (TF_n)F_k = G_m(TF_n), \quad \forall k, n, m \in \mathbb{N}. \quad (5) $$

Taking adjoints in (5) we have

$$ [G_m(TF_n)]^* = [(TF_n)F_k]^* = F_k(TF_n)^* \supseteq F_k F_n T^*. \quad (6) $$

But

$$ [G_m(TF_n)]^* = (TF_n)^* G_m \supseteq F_n T^* G_m. \quad (7) $$

As $F_n T^* G_m \in \mathcal{B}(\mathcal{H})$, we get

$$ F_k F_n T^* \supseteq F_n T^* G_m. \quad (8) $$

Furthermore, since $F_k$ and $F_n$ commute,

$$ F_k F_n T^* \subseteq F_n T^* G_m; \quad (9) $$

that is, for every $x \in \mathcal{D}(T^*)$, we have $G_m x \in \mathcal{D}(T^*)$ and

$$ F_k F_n T^* x = F_n T^* G_m x. \quad (10) $$

Let $x \in \mathcal{D}(T^*)$ and fix $k, m < n$. Then since $F_n \rightarrow I$ (strongly) as $n \rightarrow \infty$, it follows

$$ F_k T^* \subseteq T^* G_m, \quad \forall k, m \in \mathbb{N}. \quad (11) $$

Taking adjoints in (11) and using the closeness of $T$,

$$ (F_k T^*)^* \supseteq (T^* G_m)^* \supseteq G_m T^{**} = G_m T. \quad (12) $$

But $(F_k T^*)^* = T^{**} F_k = T F_k$. Hence,

$$ G_m T \subseteq T F_k, \quad \forall k, m \in \mathbb{N}. \quad (13) $$

Multiplying (2) by $F_n$, we get

$$ (TF_n)N^* F_n \subseteq M^*(TF_n)F_n = M^* TF_n. \quad (14) $$

Since $(TF_n)(N^* F_n) = T^* F_n N$ and $(TF_n)(N^* F_n) \in \mathcal{B}(\mathcal{H})$, we obtain

$$ TN^* F_n = M^* TF_n \quad \forall n \in \mathbb{N}. \quad (15) $$

Now let $x \in \mathcal{D}(TN^*)$; that is, $x \in \mathcal{D}(N^*)$ and $N^* x \in \mathcal{D}(T)$. Fix $m > k$, and let $m \rightarrow \infty$. Then using (13) and the fact $G_m \rightarrow I$ (strongly), we have

$$ TF_k x = G_m T x \rightarrow T x. \quad (16) $$

Moreover, from (14), the fact $F_n N^* \subseteq N^* F_n$, and (13), we have

$$ M^* TF_k x = TN^* F_k x = TF_k N^* x = G_m T N^* x \rightarrow T N^* x. \quad (17) $$

Since $M^*$ is closed, it follows $x \in \mathcal{D}(M^*T)$ and $M^* T x = T N^* x$. Therefore, $TN^* \subseteq M^* T$.

As a special case for $M = N$, we obtain the following generalization of Fuglede's theorem [5].

Corollary 7. Let $N \in \text{Op}(\mathcal{H})$ be normal and let $T \in \text{Op}(\mathcal{H})$ be a closed operator such that $\mathcal{D}(N) \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*)$. If $TN \subseteq NT$, then $TN^* \subseteq N^* T$.

Corollary 8. Let $N_1, N_2 \in \text{Op}(\mathcal{H})$ be normal operators. If $\mathcal{D}(N_1) \subseteq \mathcal{D}(N_2)$, then $N_2 N_1 \subseteq N_1 N_2 \Leftrightarrow N_2^* N_1^* \subseteq N_1^* N_2$.

Corollary 9. Let $N_1, N_2 \in \text{Op}(\mathcal{H})$ be normal operators. If $\mathcal{D}(N_i) \subseteq \mathcal{D}(N)$, for $i = 1, 2$, then $N_1 N_2 \subseteq N_2 N_1 \Leftrightarrow N_1^* N_2^* \subseteq N_2^* N_1^*$.

Remark 10. Recently in the article “An All-Unbounded-Operator Version of the Fuglede-Putnam Theorem,” Complex Analysis and Operator Theory (2012) [6: 1269–1273], a similar result was offered, but its proof is incorrect. In fact, on the last page of this paper [page 1273] the proof is wrong; note that from the equality $F_{b_n}(M) A N^* F_{b_n}(N) x = P_{b_n}(M) M^* A P_{b_n}(N) x$, the fact $P_{b_n}(M) \rightarrow I$ (strongly) gives $AN^* F_{b_n}(N) x = M^* A P_{b_n}(N) x$; however, (dealing with unbounded operators, as is the case here) the fact (alone) that $P_{b_n}(N) \rightarrow I$ (strongly) does not give the equality $AN^* x = M^* A x$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.
References


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