Research Article

On Simultaneous Approximation of Modified Baskakov-Durrmeyer Operators

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We discuss properties of modified Baskakov-Durrmeyer-Stancu (BDS) operators with parameter $\gamma > 0$. We compute the moments of these modified operators. Also, we establish pointwise convergence, Voronovskaja type asymptotic formula, and an error estimation in terms of second order modification of continuity of the function for the operators $B_{n,\gamma}^{\alpha,\beta}(f, x)$.

1. Introduction

For $x \in [0, \infty)$, $\gamma > 0$, $0 \leq \alpha \leq \beta$, and $f \in C[0, \infty)$, we consider a certain integral type generalized Baskakov operators as

$$\begin{aligned} B_{n,\gamma}^{\alpha,\beta}(f(t), x) &= \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &\quad + p_{n,0,\gamma}(x) f\left(\frac{\alpha}{n+\beta}\right) \\ &= \int_0^{\infty} W_{n,\gamma}(x, t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \end{aligned} \quad (1)$$

where

$$p_{n,k,\gamma}(x) = \frac{\Gamma(n/\gamma + k)}{\Gamma(k+1)\Gamma(n/\gamma)} \cdot \frac{(\gamma x)^k}{(1 + \gamma x)^{(n/\gamma)+k}},$$

$$b_{n,k,\gamma}(t) = \frac{\gamma \Gamma(n/\gamma + k + 1)}{\Gamma(k)\Gamma(n/\gamma + 1)} \cdot \frac{(\gamma t)^{k-1}}{(1 + \gamma t)^{(n/\gamma)+k+1}},$$

$$W_{n,\gamma}(x, t) = \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) b_{n,k,\gamma}(t) + (1 + \gamma x)^{-n/\gamma} \delta(t), \quad (2)$$

$\delta(t)$ being the Dirac delta function.

The operators defined by (1) are the generalization of the integral modification of well-known Baskakov operators having weight function of some beta basis function. As a special case, that is, $\gamma = 1$, the operators (1) reduce to the operators very recently studied in [1, 2]. Inverse results of same type of operators were established in [3]. Also, if $\alpha = \beta = 0$, the operators (1) reduce to the operators recently studied in [4] and if $\alpha = \beta = 0$ and $\gamma = 1$, the operators (1) reduce to the operators studied in [5]. The q -analog of the operators (1) is discussed in [6]. We refer to some of the important papers on the recent development on similar type of the operators [7–9].

The present paper deals with the study of simultaneous approximation for the operators $B_{n,\gamma}^{\alpha,\beta}$.

2. Moments and Recurrence Relations

Lemma 1. *If one defines the central moments, for every $m \in \mathbb{N}$, as*

$$\begin{aligned} \mu_{n,m,\gamma}(x) &= B_{n,\gamma}^{\alpha,\beta}((t-x)^m, x) \\ &= \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \\ &\quad + p_{n,0,\gamma}(x) \left(\frac{\alpha}{n+\beta} - x \right)^m, \end{aligned} \quad (3)$$

then $\mu_{n,0,\gamma}(x) = 1$, $\mu_{n,1,\gamma}(x) = (\alpha - \beta x)/(n + \beta)$, and for $n > \gamma m$; one has the following recurrence relation:

$$\begin{aligned} (n - \gamma m)(n + \beta) \mu_{n,m+1,\gamma}(x) &= nx(1 + \gamma x) \\ &\cdot \{ \mu_{n,m,\gamma}^{(1)}(x) + m \mu_{n,m-1,\gamma}(x) \} \\ &+ \{ mn + n^2 x - (2\gamma m - n)(\alpha - (n + \beta)x) \} \\ &\cdot \mu_{n,m,\gamma}(x) \\ &+ \left\{ m\gamma(n + \beta) \left(\frac{\alpha}{n + \beta} - x \right)^2 - mn \left(\frac{\alpha}{n + \beta} - x \right) \right\} \\ &\cdot \mu_{n,m-1,\gamma}(x). \end{aligned} \quad (4)$$

From the recurrence relation, it can be easily verified that, for all $x \in [0, \infty)$, one has $\mu_{n,m,\gamma}(x) = O(n^{-(m+1)/2})$, where $[\alpha]$ denotes the integral part of α .

Proof. Taking derivative of the above,

$$\begin{aligned} \mu_{n,m,\gamma}^{(1)}(x) &= -m \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m-1} dt - m p_{n,0,\gamma}(x) \\ &\cdot \left(\frac{\alpha}{n+\beta} - x \right)^{m-1} + \sum_{k=1}^{\infty} p_{n,k,\gamma}^{(1)}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + p_{n,0,\gamma}^{(1)}(x) \\ &\cdot \left(\frac{\alpha}{n+\beta} - x \right)^m = -m \mu_{n,m-1,\gamma}(x) + \sum_{k=1}^{\infty} p_{n,k,\gamma}^{(1)}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + p_{n,0,\gamma}^{(1)}(x) \\ &\cdot \left(\frac{\alpha}{n+\beta} - x \right)^m, \end{aligned}$$

$$\begin{aligned} &x(1 + \gamma x) \{ \mu_{n,m,\gamma}^{(1)}(x) + m \mu_{n,m-1,\gamma}(x) \} \\ &= \sum_{k=1}^{\infty} x(1 + \gamma x) p_{n,k,\gamma}^{(1)}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + x(1 + \gamma x) \\ &\cdot p_{n,0,\gamma}^{(1)}(x) \left(\frac{\alpha}{n+\beta} - x \right)^m. \end{aligned} \quad (5)$$

Using $x(1 + \gamma x) p_{n,k,\gamma}^{(1)}(x) = (k - nx) p_{n,k,\gamma}(x)$, we get

$$\begin{aligned} &x(1 + \gamma x) \{ \mu_{n,m,\gamma}^{(1)}(x) + m \mu_{n,m-1,\gamma}(x) \} \\ &= \sum_{k=1}^{\infty} (k - nx) p_{n,k,\gamma}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + (-nx) p_{n,0,\gamma}(x) \\ &\cdot \left(\frac{\alpha}{n+\beta} - x \right)^m = \sum_{k=1}^{\infty} k p_{n,k,\gamma}(x) \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt - nx \mu_{n,m,\gamma}(x) := I \\ &- nx \mu_{n,m,\gamma}(x). \end{aligned} \quad (6)$$

We can write I as

$$\begin{aligned} I &= \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} \{ (k-1) - (n+2\gamma)t \} b_{n,k,\gamma}(t) \right. \\ &\cdot \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \\ &\cdot \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \left. \right\} + \left\{ (n+2\gamma) \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \right. \\ &\cdot \int_0^{\infty} t b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \left. \right\} := I_1 \\ &+ I_2 \text{ (say)}. \end{aligned} \quad (7)$$

To estimate I_2 using $t = ((n + \beta)/n) \{ ((nt + \alpha)/(n + \beta) - x) - (\alpha/(n + \beta) - x) \}$, we have

$$\begin{aligned} I_2 &= \frac{(n+2\gamma)(n+\beta)}{n} \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \right. \\ &\cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m+1} dt - \left(\frac{\alpha}{n+\beta} \right. \\ &\cdot \left. - x \right) \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \left. \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(n+2\gamma)(n+\beta)}{n} \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \right. \\
&\quad \cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m+1} dt + p_{n,0,\gamma}(x) \\
&\quad \cdot \left(\frac{\alpha}{n+\beta} - x \right)^{m+1} - \left(\frac{\alpha}{n+\beta} - x \right) \\
&\quad \cdot \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right. \\
&\quad \left. \left. + p_{n,0,\gamma}(x) \left(\frac{\alpha}{n+\beta} - x \right)^m \right\} \right\} \\
&= \frac{(n+2\gamma)(n+\beta)}{n} \left\{ \mu_{n,m+1,\gamma}(x) - \left(\frac{\alpha}{n+\beta} - x \right) \right. \\
&\quad \left. \cdot \mu_{n,m,\gamma}(x) \right\}. \tag{8}
\end{aligned}$$

Next, to estimate I_1 using the equality, $\{(k-1) - (n+2\gamma)t\}b_{n,k,\gamma}(t) = t(1+\gamma t)b_{n,k,\gamma}^{(1)}(t)$, we have

$$\begin{aligned}
I_1 &= \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} t b_{n,k,\gamma}^{(1)}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right. \\
&\quad \left. + \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right\} \\
&\quad + \gamma \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} t^2 b_{n,k,\gamma}^{(1)}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \\
&:= J_1 + J_2 \text{ (say)}. \tag{9}
\end{aligned}$$

Again putting $t = ((n+\beta)/n)\{((nt+\alpha)/(n+\beta) - x) - (\alpha/(n+\beta) - x)\}$, we get

$$\begin{aligned}
J_1 &= \frac{n+\beta}{n} \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \right. \\
&\quad \cdot \int_0^{\infty} b_{n,k,\gamma}^{(1)}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m+1} dt + \left(\frac{\alpha}{n+\beta} - x \right) \\
&\quad \cdot \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}^{(1)}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right\} \\
&\quad \left. + \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right\}. \tag{10}
\end{aligned}$$

Now, integrating by parts, we get

$$\begin{aligned}
J_1 &= -(m+1) \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \\
&\quad \cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + m \left(\frac{\alpha}{n+\beta} - x \right) \\
&\quad \cdot \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m-1} dt \\
&\quad + \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \\
&= -(m+1) \\
&\quad \cdot \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt \right. \\
&\quad \left. + p_{n,0,\gamma}(x) \left(\frac{\alpha}{n+\beta} - x \right)^m \right\} + m \left(\frac{\alpha}{n+\beta} - x \right) \\
&\quad \cdot \left\{ \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{m-1} dt \right. \\
&\quad \left. + p_{n,0,\gamma}(x) \left(\frac{\alpha}{n+\beta} - x \right)^{m-1} \right\} + \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \\
&\quad \cdot \int_0^{\infty} b_{n,k,\gamma}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^m dt + p_{n,0,\gamma}(x) \left(\frac{\alpha}{n+\beta} - x \right)^m, \tag{11}
\end{aligned}$$

$$J_1 = -m\mu_{n,m,\gamma}(x) + m \left(\frac{\alpha}{n+\beta} - x \right) \mu_{n,m-1,\gamma}(x).$$

Proceeding in the similar manner, we obtain the estimate J_2 as

$$\begin{aligned}
J_2 &= -\frac{\gamma(n+\beta)(m+2)}{n} \mu_{n,m+1,\gamma}(x) \\
&\quad + 2\gamma \frac{(n+\beta)(m+1)}{n} \left(\frac{\alpha}{n+\beta} - x \right) \mu_{n,m,\gamma}(x) \\
&\quad - \frac{m\gamma(n+\beta)}{n} \left(\frac{\alpha}{n+\beta} - x \right)^2 \mu_{n,m-1,\gamma}(x). \tag{12}
\end{aligned}$$

Combining (6)–(12), we get

$$\begin{aligned}
x(1+\gamma x) \{ \mu_{n,m,\gamma}^{(1)}(x) + m\mu_{n,m-1,\gamma}(x) \} &= -m\mu_{n,m,\gamma}(x) \\
&\quad + m \left(\frac{\alpha}{n+\beta} - x \right) \mu_{n,m-1,\gamma}(x) - \frac{\gamma(n+\beta)(m+2)}{n} \\
&\quad \cdot \mu_{n,m+1,\gamma}(x) + 2\gamma \frac{(n+\beta)(m+1)}{n} \left(\frac{\alpha}{n+\beta} - x \right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \mu_{n,m,\gamma}(x) - \frac{m\gamma(n+\beta)}{n} \left(\frac{\alpha}{n+\beta} - x \right)^2 \\
& \cdot \mu_{n,m-1,\gamma}(x) - nx\mu_{n,m,\gamma}(x) \\
& + \frac{(n+2\gamma)(n+\beta)}{n} \left\{ \mu_{n,m+1,\gamma}(x) \right. \\
& \left. - \left(\frac{\alpha}{n+\beta} - x \right) \mu_{n,m,\gamma}(x) \right\}.
\end{aligned} \tag{13}$$

Hence,

$$\begin{aligned}
& (n-\gamma m)(n+\beta)\mu_{n,m+1,\gamma}(x) = nx(1+\gamma x) \left\{ \mu_{n,m,\gamma}^{(1)}(x) \right. \\
& + m\mu_{n,m-1,\gamma}(x) \left. + \{mn+n^2x \right. \\
& - (2\gamma m-n)(\alpha-(n+\beta)x) \} \mu_{n,m,\gamma}(x) \\
& + \left\{ m\gamma(n+\beta) \left(\frac{\alpha}{n+\beta} - x \right)^2 \right. \\
& \left. - mn \left(\frac{\alpha}{n+\beta} - x \right) \right\} \mu_{n,m-1,\gamma}(x).
\end{aligned} \tag{14}$$

This completes the proof of Lemma 1. $\square$

Remark 2 (see [10]). For $m \in \mathbb{N} \cup \{0\}$, if the m th order moment is defined as

$$U_{n,m,\gamma}(x) = \sum_{k=0}^{\infty} p_{n,k,\gamma}(x) \left(\frac{k}{n} - x \right)^m, \tag{15}$$

then $U_{n,0,\gamma}(x) = 1$, $U_{n,1,\gamma}(x) = 0$, and $nU_{n,m+1,\gamma}(x) = x(1+\gamma x)(U_{n,m,\gamma}^{(1)}(x) + mU_{n,m-1,\gamma}(x))$.

Consequently, for all $x \in [0, \infty)$, we have $U_{n,m,\gamma}(x) = O(n^{-(m+1)/2})$.

Remark 3. It is easily verified from Lemma 1 that for each $x \in [0, \infty)$

$$\begin{aligned}
B_{n,\gamma}^{\alpha,\beta}(t^m, x) &= \frac{n^m \Gamma(n/\gamma + m) \Gamma(n/\gamma - m + 1)}{(n+\beta)^m \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} x^m \\
&+ \frac{mn^{m-1} \Gamma(n/\gamma + m - 1) \Gamma(n/\gamma - m + 1)}{(n+\beta)^m \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ n(m-1) \right. \\
&+ \alpha \left(\frac{n}{\gamma} - m + 1 \right) \left. \right\} x^{m-1} \\
&+ \frac{\alpha m(m-1) n^{m-2} \Gamma(n/\gamma + m - 2) \Gamma(n/\gamma - m + 2)}{(n+\beta)^m \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ n(m \right. \\
&- 2) + \frac{\alpha(n/\gamma - m + 2)}{2} \left. \right\} x^{m-2} + O(n^{-2}).
\end{aligned} \tag{16}$$

Lemma 4 (see [10]). The polynomials $Q_{i,j,r,\gamma}(x)$ exist independent of n and k such that

$$\begin{aligned}
& \{x(1+\gamma x)\}^r D^r [p_{n,k,\gamma}(x)] \\
&= \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i (k-nx)^j Q_{i,j,r,\gamma}(x) p_{n,k,\gamma}(x),
\end{aligned} \tag{17}$$

where $D \equiv \frac{d}{dx}$.

Lemma 5. If f is r times differentiable on $[0, \infty)$, such that $f^{(r-1)} = O(t^v)$, $v > 0$ as $t \rightarrow \infty$, then for $r = 1, 2, 3, \dots$ and $n > v + \gamma r$ one has

$$\begin{aligned}
(B_{n,\gamma}^{\alpha,\beta})^{(r)}(f, x) &= \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n+\beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \\
&\cdot \sum_{k=0}^{\infty} p_{n+\gamma r, k, \gamma}(x) \\
&\cdot \int_0^{\infty} b_{n-\gamma r, k+r, \gamma}(t) f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right) dt.
\end{aligned} \tag{18}$$

Proof. First

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(1)}(f, x) \\
&= \sum_{k=1}^{\infty} p_{n,k,\gamma}^{(1)}(x) \int_0^{\infty} b_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
&- n(1+\gamma x)^{-n/\gamma-1} f\left(\frac{\alpha}{n+\beta}\right).
\end{aligned} \tag{19}$$

Now, using the identities

$$\begin{aligned}
p_{n,k,\gamma}^{(1)}(x) &= n \{ p_{n+\gamma, k-1, \gamma}(x) - p_{n+\gamma, k, \gamma}(x) \}, \\
b_{n,k,\gamma}^{(1)}(x) &= (n+\gamma) \{ b_{n+\gamma, k-1, \gamma}(x) - b_{n+\gamma, k, \gamma}(x) \},
\end{aligned} \tag{20}$$

for $k \geq 1$, we have

$$\begin{aligned}
(B_{n,\gamma}^{\alpha,\beta})^{(1)}(f, x) &= \sum_{k=1}^{\infty} n \{ p_{n+\gamma, k-1, \gamma}(x) - p_{n+\gamma, k, \gamma}(x) \} \\
&\cdot \int_0^{\infty} b_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt - n(1+\gamma x)^{-n/\gamma-1} \\
&\cdot f\left(\frac{\alpha}{n+\beta}\right) = np_{n+\gamma, 0, \gamma}(x) \\
&\cdot \int_0^{\infty} b_{n+\gamma, 1, \gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt - n(1+\gamma x)^{-n/\gamma-1} \\
&\cdot f\left(\frac{\alpha}{n+\beta}\right) + n \sum_{k=1}^{\infty} p_{n+\gamma, k, \gamma}(x) \\
&\cdot \int_0^{\infty} \{ b_{n,k+1, \gamma}(t) - b_{n,k, \gamma}(t) \} f\left(\frac{nt+\alpha}{n+\beta}\right) dt,
\end{aligned}$$

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(1)}(f, x) = n(1+\gamma x)^{-n/\gamma-1} \\
& \cdot \int_0^\infty (n+\gamma)(1+\gamma t)^{-n/\gamma-2} f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
& + n \sum_{k=1}^\infty p_{n+\gamma,k,\gamma}(x) \\
& \cdot \int_0^\infty \left(-\frac{1}{n} b_{n-\gamma,k+1,\gamma}^{(1)}(t)\right) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
& - n(1+\gamma x)^{-n/\gamma-1} f\left(\frac{\alpha}{n+\beta}\right). \quad (21)
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(1)}(f, x) = n(1+\gamma x)^{-n/\gamma-1} f\left(\frac{\alpha}{n+\beta}\right) \\
& + \frac{n^2}{n+\beta} (1+\gamma x)^{-n/\gamma-1} \\
& \cdot \int_0^\infty (1+\gamma t)^{-n/\gamma-1} f^{(1)}\left(\frac{nt+\alpha}{n+\beta}\right) dt + \frac{n}{n+\beta} \\
& \cdot \sum_{k=1}^\infty p_{n+\gamma,k,\gamma}(x) \int_0^\infty b_{n-\gamma,k+1,\gamma}(t) f^{(1)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \quad (22) \\
& - n(1+\gamma x)^{-n/\gamma-1} f\left(\frac{\alpha}{n+\beta}\right), \\
& (B_{n,\gamma}^{\alpha,\beta})^{(1)}(f, x) = \frac{n}{n+\beta} \sum_{k=0}^\infty p_{n+\gamma,k,\gamma}(x) \\
& \cdot \int_0^\infty b_{n-\gamma,k+1,\gamma}(t) f^{(1)}\left(\frac{nt+\alpha}{n+\beta}\right) dt.
\end{aligned}$$

Thus the result is true for $r = 1$. We prove the result by induction method. Suppose that the result is true for $r = i$; then

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(i)}(f, x) = \frac{n^i \Gamma(n/\gamma + i) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \\
& \cdot \sum_{k=0}^\infty p_{n+\gamma,k,\gamma}(x) \int_0^\infty b_{n-\gamma,k+i,\gamma}(t) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt. \quad (23)
\end{aligned}$$

Thus, using the identities (20), we have

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(i+1)}(f, x) \\
& = \frac{n^i \Gamma(n/\gamma + i) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ \sum_{k=1}^\infty \left(\frac{n}{\gamma} + i\right) \right. \\
& \cdot \{p_{n+\gamma(i+1),k-1,\gamma}(x) - p_{n+\gamma(i+1),k,\gamma}(x)\} \int_0^\infty b_{n-\gamma,k+i,\gamma}(t)
\end{aligned}$$

$$\begin{aligned}
& \cdot f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt - \left(\frac{n}{\gamma} + i\right) (1+\gamma x)^{-n/\gamma-i-1} \\
& \cdot \int_0^\infty b_{n-\gamma,i,\gamma}(t) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \Big\} \\
& = \frac{n^i \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} p_{n+\gamma(i+1),0,\gamma}(x) \\
& \cdot \int_0^\infty b_{n-\gamma,i+1,\gamma}(t) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
& - \frac{n^i \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} p_{n+\gamma(i+1),0,\gamma}(x) \\
& \cdot \int_0^\infty b_{n-\gamma,i,\gamma}(t) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
& + \frac{n^i \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ \sum_{k=1}^\infty p_{n+\gamma(i+1),k,\gamma}(x) \right. \\
& \cdot \int_0^\infty \{b_{n-\gamma,k+i+1,\gamma}(t) - b_{n-\gamma,k+i,\gamma}(t)\} \\
& \cdot f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \Big\} \\
& = \frac{n^i \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} p_{n+\gamma(i+1),0,\gamma}(x) \\
& \cdot \int_0^\infty \left(-\frac{1}{n/\gamma - i} b_{n-\gamma(i-1),1+i,\gamma}^{(1)}(t)\right) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
& + \frac{n^i \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^i \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \sum_{k=1}^\infty p_{n+\gamma(i+1),k,\gamma}(x) \\
& \cdot \int_0^\infty \left(-\frac{1}{n/\gamma - i} b_{n-\gamma(i-1),k+i+1,\gamma}^{(1)}(t)\right) f^{(i)}\left(\frac{nt+\alpha}{n+\beta}\right) dt. \quad (24)
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
& (B_{n,\gamma}^{\alpha,\beta})^{(i+1)}(f, x) = \frac{n^{i+1} \Gamma(n/\gamma + i + 1) \Gamma(n/\gamma - i + 1)}{(n+\beta)^{i+1} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \\
& \cdot \sum_{k=0}^\infty p_{n+\gamma(i+1),k,\gamma}(x) \\
& \cdot \int_0^\infty b_{n-\gamma(i-1),k+i+1,\gamma}(t) f^{(i+1)}\left(\frac{nt+\alpha}{n+\beta}\right) dt. \quad (25)
\end{aligned}$$

This completes the proof of Lemma 5. $\square$

3. Direct Theorems

This section deals with the direct results; we establish here pointwise approximation, asymptotic formula, and error estimation in simultaneous approximation.

We denote $C_\mu[0, \infty) = \{f \in C[0, \infty) : |f(t)| \leq Mt^\mu \text{ for some } M > 0, \mu > 0\}$, and the norm $\|\cdot\|_\mu$ on the class $C_\mu[0, \infty)$ is defined as $\|f\|_\mu = \sup_{0 \leq t < \infty} |f(t)|t^{-\mu}$. It can be easily verified that the operators $B_{n,\gamma}^{\alpha,\beta}(f, x)$ are well defined for $f \in C_\mu[0, \infty)$.

Theorem 6. Let $f \in C_\mu[0, \infty)$ and let $f^{(r)}$ exist at a point $x \in (0, \infty)$. Then one has

$$\lim_{n \rightarrow \infty} (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f, x) = f^{(r)}(x). \quad (26)$$

Proof. By Taylor's expansion of f , we have

$$f(t) = \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (t-x)^i + \epsilon(t, x) (t-x)^r, \quad (27)$$

where $\epsilon(t, x) \rightarrow 0$ as $t \rightarrow x$. Hence,

$$\begin{aligned} (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f, x) &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} (B_{n,\gamma}^{\alpha,\beta})^{(r)}((t-x)^i, x) \\ &\quad + (B_{n,\gamma}^{\alpha,\beta})^{(r)}(\epsilon(t, x) (t-x)^r, x) \\ &:= R_1 + R_2. \end{aligned} \quad (28)$$

First, to estimate R_1 , using binomial expansion of $((nt + \alpha)/(n + \beta) - x)^i$ and Remark 3, we have

$$\begin{aligned} R_1 &= \sum_{i=0}^r \frac{f^{(i)}(x)}{i!} \sum_{j=0}^i \binom{i}{j} (-x)^{i-j} (B_{n,\gamma}^{\alpha,\beta})^{(r)}(t^j, x) \\ &= \frac{f^{(r)}(x)}{r!} \left\{ \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n + \beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} r! \right\} \\ &= f^{(r)}(x) \left\{ \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n + \beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \right\} \\ &\longrightarrow f^{(r)}(x) \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (29)$$

Next, applying Lemma 4, we obtain

$$\begin{aligned} R_2 &= \int_0^\infty W_{n,\gamma}^{(r)}(t, x) \epsilon(t, x) \left(\frac{nt + \alpha}{n + \beta} - x \right)^r dt, \\ |R_2| &\leq \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \frac{|Q_{i,j,r,\gamma}(x)|}{\{x(1 + \gamma x)\}^r} \sum_{k=1}^\infty |k - nx|^j p_{n,k,\gamma}(x) \\ &\quad \cdot \int_0^\infty b_{n,k,\gamma}(t) |\epsilon(t, x)| \left| \frac{nt + \alpha}{n + \beta} - x \right|^r dt \\ &\quad + \frac{\Gamma(n/\gamma + r + 2)}{\Gamma(n/\gamma)} (1 + \gamma x)^{-n/\gamma - r} |\epsilon(0, x)| \\ &\quad \cdot \left| \frac{\alpha}{n + \beta} - x \right|^r. \end{aligned} \quad (30)$$

The second term in the above expression tends to zero as $n \rightarrow \infty$. Since $\epsilon(t, x) \rightarrow 0$ as $t \rightarrow x$ for given $\epsilon > 0$, there exists a $\delta \in (0, 1)$ such that $|\epsilon(t, x)| < \epsilon$ whenever $0 < |t - x| < \delta$. If $\tau > \max\{\mu, r\}$, where τ is any integer, then we can find a constant $M_3 > 0$, such that $|\epsilon(t, x)((nt + \alpha)/(n + \beta) - x)^\tau| \leq M_3 |(nt + \alpha)/(n + \beta) - x|^\tau$ for $|t - x| \geq \delta$. Therefore,

$$\begin{aligned} |R_2| &\leq M_3 \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \sum_{k=0}^\infty |k - nx|^j p_{n,k,\gamma}(x) \\ &\quad \cdot \left\{ \epsilon \int_{|t-x| < \delta} b_{n,k,\gamma}(x) \left| \frac{nt + \alpha}{n + \beta} - x \right|^r dt \right. \\ &\quad \left. + \int_{|t-x| \geq \delta} b_{n,k,\gamma}(t) \left| \frac{nt + \alpha}{n + \beta} - x \right|^\tau dt \right\} =: R_3 + R_4. \end{aligned} \quad (31)$$

Applying the Cauchy-Schwarz inequality for integration and summation, respectively, we obtain

$$\begin{aligned} |R_3| &\leq \epsilon M_3 \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \left\{ \sum_{k=1}^\infty (k - nx)^{2j} p_{n,k,\gamma}(x) \right\}^{1/2} \\ &\quad \cdot \left\{ \sum_{k=1}^\infty p_{n,k,\gamma}(x) \int_0^\infty b_{n,k,\gamma}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^{2r} dt \right\}^{1/2}. \end{aligned} \quad (32)$$

Using Remark 2 and Lemma 1, we get $R_3 \leq \epsilon O(n^{r/2}) O(n^{-r/2}) = \epsilon \cdot O(1)$.

Again using the Cauchy-Schwarz inequality and Lemma 1, we get

$$\begin{aligned} |R_4| &\leq M_4 \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \sum_{k=1}^\infty |k - nx|^j p_{n,k,\gamma}(x) \\ &\quad \cdot \int_{|t-x| \geq \delta} b_{n,k,\gamma}(t) \left| \frac{nt + \alpha}{n + \beta} - x \right|^\tau dt \leq M_4 \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \\ &\quad \cdot \sum_{k=1}^\infty |k - nx|^j p_{n,k,\gamma}(x) \left\{ \int_{|t-x| \geq \delta} b_{n,k,\gamma}(t) dt \right\}^{1/2} \\ &\quad \cdot \left\{ \int_{|t-x| \geq \delta} b_{n,k,\gamma}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^{2\tau} dt \right\}^{1/2} \\ &\leq M_4 \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \left\{ \sum_{k=1}^\infty (k - nx)^{2j} p_{n,k,\gamma}(x) \right\}^{1/2} \\ &\quad \cdot \left\{ \sum_{k=1}^\infty p_{n,k,\gamma}(x) \int_0^\infty b_{n,k,\gamma}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^{2\tau} dt \right\}^{1/2} \\ &= \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i O(n^{j/2}) O(n^{-\tau/2}) = O(n^{(r-\tau)/2}) = o(1). \end{aligned} \quad (33)$$

Collecting the estimation of $R_1 - R_4$, we get the required result. $\square$

Theorem 7. Let $f \in C_\mu[0, \infty)$. If $f^{(r+2)}$ exists at a point $x \in (0, \infty)$, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left\{ \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (f, x) - f^{(r)}(x) \right\} \\ &= r(\gamma(r-1) - \beta) f^{(r)}(x) \\ &+ \{ r\gamma(1+2x) + \alpha - \beta x \} f^{(r+1)}(x) \\ &+ x(1+\gamma x) f^{(r+2)}(x). \end{aligned} \quad (34)$$

Proof. Using Taylor's expansion of f , we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \epsilon(t, x) (t-x)^{r+2}, \quad (35)$$

where $\epsilon(t, x) \rightarrow 0$ as $t \rightarrow x$ and $\epsilon(t, x) = O((t-x)^\mu)$, $t \rightarrow \infty$ for $\mu > 0$.

Applying Lemma 1, we have

$$\begin{aligned} & n \left\{ \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (f, x) - f^{(r)}(x) \right\} \\ &= n \left\{ \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} ((t-x)^i, x) - f^{(r)}(x) \right\} \\ &+ n \left\{ \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (\epsilon(t, x) (t-x)^{r+2}, x) \right\} \\ &:= E_1 + E_2. \end{aligned} \quad (36)$$

First, we have

$$\begin{aligned} E_1 &= n \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^i \binom{i}{j} (-x)^{i-j} \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^j, x) \\ &- n f^{(r)}(x) = \frac{f^{(r)}(x)}{r!} n \left\{ \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^r, x) - r! \right\} \\ &+ \frac{f^{(r+1)}(x)}{(r+1)!} n \left\{ (r+1) (-x) \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^r, x) \right. \\ &+ \left. \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^{r+1}, x) \right\} + \frac{f^{(r+2)}(x)}{(r+2)!} \\ &\cdot n \left\{ \frac{(r+2)(r+1)}{2} x^2 \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^r, x) + (r+2) \right. \\ &\cdot (-x) \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^{r+1}, x) + \left. \left(B_{n,\gamma}^{\alpha,\beta} \right)^{(r)} (t^{r+2}, x) \right\} \\ &= f^{(r)}(x) n \left\{ \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n+\beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} - 1 \right\} \\ &+ \frac{f^{(r+1)}(x)}{(r+1)!} n \left\{ (r+1) (-x) \right. \\ &\cdot \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n+\beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} r! \\ &+ \frac{n^{r+1} \Gamma(n/\gamma + r + 1) \Gamma(n/\gamma - r)}{(n+\beta)^{r+1} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} (r+1)! x \end{aligned}$$

$$\begin{aligned} &+ \frac{(r+1) n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r)}{(n+\beta)^{r+1} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ nr + \alpha \left(\frac{n}{\gamma} \right. \right. \\ &- r \left. \left. \right\} r! \right\} + \frac{f^{(r+2)}(x)}{(r+2)!} n \left\{ \frac{(r+1)(r+2)}{2} x^2 \right. \\ &\cdot \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n+\beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} r! - x(r+2) \\ &\cdot \left\{ \frac{n^{r+1} \Gamma(n/\gamma + r + 1) \Gamma(n/\gamma - r)}{(n+\beta)^{r+1} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} (r+1)! x \right. \\ &+ \frac{(r+1) n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r)}{(n+\beta)^{r+1} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ nr \right. \\ &+ \alpha \left(\frac{n}{\gamma} - r \right) \left. \right\} r! \left. \right\} \\ &+ \frac{n^{r+2} \Gamma(n/\gamma + r + 2) \Gamma(n/\gamma - r - 1) (r+2)!}{(n+\beta)^{r+2} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} x^2 \\ &+ \frac{(r+2) n^{r+1} \Gamma(n/\gamma + r + 1) \Gamma(n/\gamma - r - 1)}{(n+\beta)^{r+2} \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ n(r \right. \\ &+ 1) + \alpha \left(\frac{n}{\gamma} - r - 1 \right) \left. \right\} (r+1)! x \\ &+ \frac{\alpha(r+1)(r+2) n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r)}{(n+\beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \left\{ nr \right. \\ &+ \left. \frac{\alpha(n/\gamma - r)}{2} \right\} r! \left. \right\}. \end{aligned} \quad (37)$$

Now, the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$, and $f^{(r+2)}(x)$ in the above expression tend to $r(\gamma(r-1) - \beta)$, $r\gamma(1+2x) + \alpha - \beta x$, and $x(1+\gamma x)$, respectively, which follows by using induction hypothesis on r and taking the limit as $n \rightarrow \infty$. Hence, in order to prove (34), it is sufficient to show that $E_2 \rightarrow 0$ as $n \rightarrow \infty$, which follows along the lines of the proof of Theorem 6 and by using Remark 2 and Lemmas 1 and 4. $\square$

Remark 8. Particular case $\alpha = \beta = 0$ was discussed in Theorem 4.1 in [4], which says that the coefficient of $f^{(r+1)}(x)$ converges to $r(1+2\gamma x)$ but it converges to $r\gamma(1+2x)$ and we get this by putting $\alpha = \beta = 0$ in the above theorem.

Definition 9. The m th order modulus of continuity $\omega_m(f, \delta, [a, b])$ for a function continuous on $[a, b]$ is defined by

$$\begin{aligned} & \omega_m(f, \delta, [a, b]) \\ &= \sup \{ |\Delta_h^m f(x)| : |h| \leq \delta; x, x+h \in [a, b] \}. \end{aligned} \quad (38)$$

For $m = 1$, $\omega_m(f, \delta)$ is usual modulus of continuity.

Theorem 10. Let $f \in C_\mu[0, \infty)$ for some $\mu > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then, for n sufficiently large, one has

$$\left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} \quad (39)$$

$$\leq M_1 \omega_2(f^{(r)}, n^{-1/2}, [a_1, b_1]) + M_2 n^{-1} \|f\|_\mu,$$

where $M_1 = M_1(r)$ and $M_2 = M_2(r, f)$.

Proof. Let us assume that $0 < a < a_1 < b_1 < b < \infty$. For sufficiently small $\eta > 0$, we define the function $f_{\eta,2}$ corresponding to $f \in C_\mu[a, b]$ and $t \in [a_1, b_1]$ as follows:

$$f_{\eta,2}(t) = \eta^{-2} \iint_{-\eta/2}^{\eta/2} (f(t) - \Delta_h^2 f(t)) dt_1 dt_2, \quad (40)$$

where $h = (t_1 + t_2)/2$ and Δ_h^2 is the second order forward difference operator with step length h . For $f \in C[a, b]$, the functions $f_{\eta,2}$ are known as the Steklov mean of order 2, which satisfy the following properties [11]:

- (a) $f_{\eta,2}$ has continuous derivatives up to order 2 over $[a_1, b_1]$.
- (b) $\|f_{\eta,2}^{(r)}\|_{C[a_1, b_1]} \leq \widehat{M}_1 \eta^{-r} \omega_2(f, \eta, [a, b])$, $r = 1, 2$.
- (c) $\|f - f_{\eta,2}\|_{C[a_1, b_1]} \leq \widehat{M}_2 \omega_2(f, \eta, [a, b])$.
- (d) $\|f_{\eta,2}\|_{C[a_1, b_1]} \leq \widehat{M}_3 \|f\|_\mu$,

where $\widehat{M}_i$, $i = 1, 2, 3$, are certain constants which are different in each occurrence and are independent of f and η .

We can write by linearity properties of $B_{n,\gamma}^{\alpha,\beta}$,

$$\begin{aligned} & \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} \\ & \leq \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f - f_{\eta,2}, \cdot) \right\|_{C[a_1, b_1]} \\ & \quad + \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(f_{\eta,2}, \cdot) - f_{\eta,2}^{(r)} \right\|_{C[a_1, b_1]} \\ & \quad + \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a_1, b_1]} := P_1 + P_2 + P_3. \end{aligned} \quad (41)$$

Since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}(t)$, by property (c) of the function $f_{\eta,2}$, we get

$$P_3 \leq \widehat{M}_4 \omega_2(f^{(r)}, \eta, [a, b]). \quad (42)$$

Next, on an application of Theorem 7, it follows that

$$P_2 \leq \widehat{M}_5 n^{-1} \sum_{i=r}^{r+2} \|f_{\eta,2}^{(i)}\|_{C[a, b]}. \quad (43)$$

Using the interpolation property due to Goldberg and Meir [12], for each $j = r, r+1, r+2$, it follows that

$$\|f_{\eta,2}^{(j)}\|_{C[a, b]} \leq \widehat{M}_6 \left\{ \|f_{\eta,2}\|_{C[a, b]} + \|f_{\eta,2}^{(r+2)}\|_{C[a, b]} \right\}. \quad (44)$$

Therefore, by applying properties (c) and (d) of the function $f_{\eta,2}$, we obtain

$$P_2 \leq \widehat{M}_7 \cdot n^{-1} \{ \|f\|_\mu + \delta^{-2} \omega_2(f^{(r)}, \mu, [a, b]) \}. \quad (45)$$

Finally we will estimate P_1 , choosing a^*, b^* satisfying the conditions $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Suppose $\hbar(t)$ denotes the characteristic function of the interval $[a^*, b^*]$. Then

$$\begin{aligned} P_1 & \leq \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(\hbar(t)(f(t) - f_{\eta,2}(t)), \cdot) \right\|_{C[a_1, b_1]} \\ & \quad + \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}((1 - \hbar(t))(f(t) - f_{\eta,2}(t)), \cdot) \right\|_{C[a_1, b_1]} \\ & := P_4 + P_5. \end{aligned} \quad (46)$$

By Lemma 5, we have

$$\begin{aligned} & (B_{n,\gamma}^{\alpha,\beta})^{(r)}(\hbar(t)(f(t) - f_{\eta,2}(t)), x) \\ & = \frac{n^r \Gamma(n/\gamma + r) \Gamma(n/\gamma - r + 1)}{(n + \beta)^r \Gamma(n/\gamma + 1) \Gamma(n/\gamma)} \sum_{k=0}^{\infty} P_{n+\gamma r, k, \gamma}(x) \\ & \quad \cdot \int_0^{\infty} b_{n-\gamma r, k+\gamma, \gamma}(t) \hbar(t) \\ & \quad \cdot \left(f^{(r)}\left(\frac{nt + \alpha}{n + \beta}\right) - f_{\eta,2}^{(r)}\left(\frac{nt + \alpha}{n + \beta}\right) \right) dt. \end{aligned} \quad (47)$$

Hence,

$$\begin{aligned} & \left\| (B_{n,\gamma}^{\alpha,\beta})^{(r)}(\hbar(t)(f(t) - f_{\eta,2}(t)), \cdot) \right\|_{C[a_1, b_1]} \\ & \leq \widehat{M}_8 \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a^*, b^*]}. \end{aligned} \quad (48)$$

Now, for $x \in [a_1, b_1]$ and $t \in [0, \infty) \setminus [a^*, b^*]$, we choose a $\delta > 0$ satisfying $|nt + \alpha|/(n + \beta) - x| \geq \delta$.

Therefore, by Lemma 4 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I & \equiv (B_{n,\gamma}^{\alpha,\beta})^{(r)}((1 - \hbar(t))(f(t) - f_{\eta,2}(t)), x), \\ |I| & \leq \sum_{\substack{2i+j \leq r \\ i, j \geq 0}} n^i \frac{|Q_{i,j,r,\gamma}(x)|}{\{x(1 + \gamma x)\}^r} \sum_{k=1}^{\infty} P_{n,k,\gamma}(x) |k - nx|^j \\ & \quad \cdot \int_0^{\infty} b_{n,k,\gamma}(t) (1 - \hbar(t)) \\ & \quad \cdot \left| f\left(\frac{nt + \alpha}{n + \beta}\right) - f_{\eta,2}\left(\frac{nt + \alpha}{n + \beta}\right) \right| dt \\ & \quad + \frac{\Gamma(n/\gamma + r)}{\Gamma(n/\gamma)} (1 + \gamma x)^{-n/\gamma - r} (1 - \hbar(0)) \\ & \quad \cdot \left| f\left(\frac{\alpha}{n + \beta}\right) - f_{\eta,2}\left(\frac{\alpha}{n + \beta}\right) \right|. \end{aligned} \quad (49)$$

For sufficiently large n , the second term tends to zero. Thus,

$$\begin{aligned}
 |I| &\leq \widehat{M}_9 \|f\|_\mu \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) |k - nx|^j \\
 &\cdot \int_{|t-x| \geq \delta} b_{n,k,\gamma}(t) dt \leq \widehat{M}_9 \|f\|_\mu \delta^{-2m} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \\
 &\cdot \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) |k - nx|^j \left(\int_0^\infty b_{n,k,\gamma}(t) dt \right)^{1/2} \\
 &\cdot \left(\int_0^\infty b_{n,k,\gamma}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^{4m} dt \right)^{1/2} \quad (50) \\
 &\leq \widehat{M}_9 \|f\|_\mu \delta^{-2m} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \\
 &\cdot \left(\sum_{k=1}^{\infty} p_{n,k,\gamma}(x) |k - nx|^{2j} \right)^{1/2} \left(\sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \right. \\
 &\cdot \left. \int_0^\infty b_{n,k,\gamma}(t) \left(\frac{nt + \alpha}{n + \beta} - x \right)^{4m} dt \right)^{1/2}.
 \end{aligned}$$

Hence, by using Remark 2 and Lemma 1, we have

$$|I| \leq \widehat{M}_{10} \|f\|_\mu \delta^{-2m} O\left(n^{(i+(j/2)-m)}\right) \leq \widehat{M}_{11} n^{-q} \|f\|_\mu, \quad (51)$$

where $q = m - (r/2)$. Now choosing $m > 0$ satisfying $q \geq 1$, we obtain $I \leq \widehat{M}_{11} n^{-1} \|f\|_\mu$. Therefore, by property (c) of the function $f_{\eta,2}(t)$, we get

$$\begin{aligned}
 P_1 &\leq \widehat{M}_8 \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a^*,b^*]} + \widehat{M}_{11} n^{-1} \|f\|_\mu \\
 &\leq \widehat{M}_{12} \omega_2(f^{(r)}, \eta, [a, b]) + \widehat{M}_{11} n^{-1} \|f\|_\mu. \quad (52)
 \end{aligned}$$

Choosing $\eta = n^{-1/2}$, the theorem follows. $\square$

Remark 11. In the last decade the applications of q -calculus in approximation theory are one of the main areas of research. In 2008, Gupta [13] introduced q -Durrmeyer operators whose approximation properties were studied in [14]. More work in this direction can be seen in [15–17].

A Durrmeyer type q -analogue of the $B_{n,\gamma}^{\alpha,\beta}(f, x)$ is introduced as follows:

$$\begin{aligned}
 B_{n,\gamma,q}^{\alpha,\beta}(f, x) &= \sum_{k=1}^{\infty} p_{n,k,\gamma}^q(x) \int_0^{\infty/A} q^{-k} b_{n,k,\gamma}^q(t) f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) d_q t \\
 &+ p_{n,0,\gamma}^q(x) f\left(\frac{\alpha}{[n]_q + \beta}\right), \quad (53)
 \end{aligned}$$

where

$$\begin{aligned}
 p_{n,k,\gamma}^q(x) &= q^{k^2/2} \frac{\Gamma_q(n/\gamma + k)}{\Gamma_q(k+1) \Gamma_q(n/\gamma)} \\
 &\cdot \frac{(q\gamma x)^k}{(1 + q\gamma x)_q^{(n/\gamma)+k}}, \\
 b_{n,k,\gamma}^q(x) &= \gamma q^{k^2/2} \frac{\Gamma_q(n/\gamma + k + 1)}{\Gamma_q(k) \Gamma_q(n/\gamma + 1)} \\
 &\cdot \frac{(\gamma t)^{k-1}}{(1 + \gamma t)_q^{(n/\gamma)+k+1}}, \quad (54)
 \end{aligned}$$

$$\int_0^{\infty/A} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0.$$

Notations used in (53) can be found in [18]. For the operators (53), one can study their approximation properties based on q -integrals.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this research article.

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