Research Article

Some Fixed Point Theorems in Complex Valued b-Metric Spaces

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Recently, Azam et al. introduced the notion of complex valued metric spaces and proved fixed point theorems under the contraction condition. Rao et al. introduced the notion of complex valued b-metric spaces. In this paper, we obtain some fixed point results for the mappings satisfying rational expressions in complex valued b-metric spaces. Also, an example is given to illustrate our obtained result.

1. Introduction

Banach contraction principle [1] is one of the most important results in fixed point theory. Later, a large number of articles have been devoted to the improvement and generalization of the Banach contraction principle by using different form of contraction condition in various spaces. Bakhtin [2] introduced the notion of b-metric space which is a generalized form of metric spaces. Azam et al. [3] introduced the notion of complex valued metric spaces which is more general than well-known metric spaces and also proved common fixed point theorems for mappings satisfying rational expression. Afterwards, the concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [4]. In the sequel, Mukheimer [5] proved some common fixed point theorems in complex valued b-metric spaces.

The aim of this paper is to prove some fixed point theorems for map with different type of rational expressions in complex valued b-metric spaces. Our results unify, generalize, and complement the comparable results from the current literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows.

\[ z_1 \preceq z_2 \] if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$, $\text{Im}(z_1) \leq \text{Im}(z_2)$.

Thus $z_1 \preceq z_2$ if one of the following holds:

1. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
2. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
3. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
4. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

We will write $z_1 \lesssim z_2$ if $z_1 \not= z_2$ and one of (2), (3), and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

It follows that

1. $0 \preceq z_1 \preceq z_2$ implies $|z_1| < |z_2|$,  
2. $z_1 \preceq z_2$ and $z_2 \preceq z_3$ imply $z_1 \preceq z_3$,  
3. $0 \preceq z_1 \preceq z_2$ implies $|z_1| \leq |z_2|$,  
4. $a, b \in \mathbb{R}$ and $a \leq b$ imply $az \preceq bz$ for all $z \in \mathbb{C}$.

The following definitions and results [4] will be needed in the sequel.

Definition 1 (see [4]). Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{C}$ is called a complex valued b-metric on $X$ if, for all $x, y, z \in X$, the following conditions are satisfied:

1. $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) \( d(x, y) = d(y, x) \);
(iii) \( d(x, y) \leq s(d(x, z) + d(z, y)) \).

The pair \((X, d)\) is called a complex valued b-metric space.

**Example 2** (see [4]). Let \( X = [0, 1] \). Define the mapping \( d : X \times X \to \mathbb{C} \) by \( d(x, y) = |x - y|^2 + i|x - y|^2 \) for all \( x, y \in X \). Then \((X, d)\) is a complex valued b-metric space with \( s = 2 \).

**Definition 3** (see [4]). Let \((X, d)\) be a complex valued b-metric space.

(i) A point \( x \in X \) is called interior point of a set \( A \subseteq X \) whenever there exists \( 0 < r \in \mathbb{C} \) such that \( B(x, r) = \{ y \in X : d(x, y) < r \} \subseteq A \).
(ii) A point \( x \in X \) is called limit point of a set \( A \) whenever, for every \( 0 < r \in \mathbb{C} \), \( B(x, r) \cap (A - \{x\}) \neq \emptyset \).
(iii) A subset \( A \subseteq X \) is called open whenever each element of \( A \) is an interior point of \( A \).
(iv) A subset \( A \subseteq X \) is called closed whenever each element of \( A \) belongs to \( A \).
(v) A subbasis for Hausdorff topology \( r \) on \( X \) is a family \( F = \{B(x, r) : x \in X \text{ and } 0 < r \} \).

**Definition 4** (see [4]). Let \((X, d)\) be a complex valued b-metric space; let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \).

(i) If, for every \( \varepsilon \in \mathbb{C} \), with \( 0 < \varepsilon \), there is \( N \in \mathbb{N} \) such that, for all \( n \in \mathbb{N} \), \( d(x_n, x) < \varepsilon \), then \( \{x_n\} \) is said to be convergent, \( \{x_n\} \) converges to \( x \), and \( x \) is the limit point of \( \{x_n\} \). One denotes this by \( \lim_{n \to \infty} x_n = x \) or \( \{x_n\} \to x \) as \( n \to \infty \).
(ii) If, for every \( \varepsilon \in \mathbb{C} \), with \( 0 < \varepsilon \), there is \( N \in \mathbb{N} \) such that, for all \( n > N \), \( d(x_n, x_{m+n}) < \varepsilon \), where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.
(iii) If every Cauchy sequence in \( X \) is convergent, then \((X, d)\) is said to be a complete complex valued b-metric space.

**Lemma 5** (see [4]). Let \((X, d)\) be a complex valued b-metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) converges to \( x \) if and only if \( |d(x_n, x)| \to 0 \) as \( n \to \infty \).

**Lemma 6** (see [4]). Let \((X, d)\) be a complex valued b-metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is a Cauchy sequence if and only if \( |d(x_n, x_{m+n})| \to 0 \) as \( n \to \infty \), where \( m \in \mathbb{N} \).

**3. Main Results**

**Theorem 7.** Let \((X, d)\) be a complete complex valued b-metric space with the coefficient \( s \geq 1 \) and let \( T : X \to X \) be a mapping satisfying

\[
d(Tx, Ty) \leq \frac{\lambda d^2(x, y)}{1 + d(x, y)} + \mu d(y, Ty),
\]

for all \( x, y \in X \), where \( \lambda, \mu \) are nonnegative reals with \( s\lambda + \mu < 1 \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** For any arbitrary point, \( x_0 \in X \). Define sequence \( \{x_n\} \) in \( X \) such that \( x_{2n+1} = Tx_{2n} \) for \( n = 0, 1, 2, 3, \ldots \).

Now, we show that the sequence \( \{x_n\} \) is Cauchy:

\[
d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Tx_{2n+1})
\]

\[
\leq \frac{\lambda d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, Tx_{2n+1})
\]

\[
= \frac{\lambda d^2(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu d(x_{2n+1}, x_{2n+2})
\]

which implies that

\[
|d(x_{2n+1}, x_{2n+2})| \leq \lambda |d(x_{2n}, x_{2n+1})|
\]

\[
+ \mu |d(x_{2n+1}, x_{2n+2})|.
\]

Since \( 1 + d(x_{2n}, x_{2n+1}) \geq |d(x_{2n}, x_{2n+1})| \), we get

\[
|d(x_{2n+1}, x_{2n+2})| \leq \lambda |d(x_{2n}, x_{2n+1})|
\]

\[
+ \mu |d(x_{2n+1}, x_{2n+2})|
\]

and hence

\[
|d(x_{2n+1}, x_{2n+2})| \leq \frac{\lambda}{1 - \mu} |d(x_{2n}, x_{2n+1})|.
\]

Similarly, we obtain

\[
|d(x_{2n+2}, x_{2n+3})| \leq \frac{\lambda}{1 - \mu} |d(x_{2n+1}, x_{2n+2})|.
\]

Since \( s\lambda + \mu < 1 \) and \( s \geq 1 \), we get \( \lambda + \mu < 1 \).

Therefore, with \( \delta = \lambda/(1 - \mu) < 1 \) and for all \( n \geq 0 \) and consequently, we have

\[
|d(x_{2n+1}, x_{2n+2})| \leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})|
\]

\[
\leq \cdots \leq \delta^{2n+1} |d(x_0, x_1)|,
\]

\[
|d(x_{n+1}, x_{n+2})| \leq \delta |d(x_n, x_{n+1})| \leq \delta^2 |d(x_{n-1}, x_n)|
\]

\[
\leq \cdots \leq \delta^{n+1} |d(x_0, x_1)|.
\]
Thus for any \( m > n, m, n \in \mathbb{N} \) and since \( s \delta = s \lambda / (1 - \mu) < 1 \), we get
\[
|d(x_m, x_n)| \\
\leq s |d(x_m, x_{m-1})| + s |d(x_{m-1}, x_n)| \\
\leq s^2 |d(x_m, x_{m-1})| + s^2 |d(x_{m-1}, x_{m-2})| + s^2 |d(x_{m-2}, x_m)| \\
|d(x_{m+1}, x_{m+2})| + \cdots + s^{m-n-1} |d(x_{m-n-1}, x_{m-1})| \\
+ s^{m-n} |d(x_{m-1}, x_m)|.
\]
By using (8), we get
\[
|d(x_m, x_n)| \\
\leq s^{m-n} |d(x_0, x_1)| + s^2 \delta^{m-n+1} |d(x_0, x_1)| \\
+ s^3 \delta^{m-n+2} |d(x_0, x_1)| + \cdots + s^{m-n-1} \delta^{m-n-2} |d(x_0, x_1)| \\
+ s^{m-n} \delta^{m-n-1} |d(x_0, x_1)| = \sum_{i=1}^{m-n} s^i \delta^{i+n-1} |d(x_0, x_1)|.
\]
Therefore,
\[
|d(x_m, x_n)| \leq \sum_{i=1}^{m-n} s^i \delta^{i+n-1} |d(x_0, x_1)| \\
= \sum_{i=n}^{m-1} s^i \delta^i |d(x_0, x_1)| \\
\leq \sum_{i=n}^{\infty} (s \delta)^i |d(x_0, x_1)| \\
= \frac{(s \delta)^n}{1 - s \delta} |d(x_0, x_1)|.
\]
and hence
\[
|d(x_n, x_m)| \leq \frac{(s \delta)^n}{1 - s \delta} |d(x_0, x_1)| \to 0 \quad \text{as} \ m,n \to \infty.
\]
Thus, \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists some \( u \in X \) such that \( x_n \to u \) as \( n \to \infty \). Suppose this is not possible; then there exists \( z \in X \) such that
\[
|d(u, Tu)| = |z| > 0.
\]
Now,
\[
z = d(u, Tu) \leq sd(u, x_{2n+2}) + sd(x_{2n+2}, Tu) \\
= sd(u, x_{2n+2}) + sd(TX_{2n+1}, Tu) \\
\leq sd(u, x_{2n+2}) + sd\left(\frac{s\lambda d^2(x_{2n+1}, u)}{1 + d(x_{2n+1}, u)} + s\mu d(u, Tu)\right).
\]
which implies that
\[
|z| = |d(u, Tu)| \leq s|d(u, x_{2n+2})| \\
+ s\lambda d^2(x_{2n+1}, u) + s\mu d(u, Tu) \\
\leq \lambda d^2(u, u^*) + \mu d(u, Tu^*).
\]
Taking the limit of (15) as \( n \to \infty \), we obtain that \( |z| = |d(u, Tu)| \leq 0 \), a contradiction with (13).
So \( |z| = 0 \). Hence \( Tu = u \).
Now, we show that \( T \) has a unique fixed point in \( X \). To show this, assume that \( u^* \) is another fixed point of \( T \). Then,
\[
d(u, u^*) = d(Tu, Tu^*) \leq \frac{\lambda d^2(u, u^*)}{1 + d(u, u^*)} + \mu d(u^*, Tu^*).
\]
So
\[
|d(u, u^*)| \leq \frac{\lambda}{1 + d(u, u^*)} |d(u, u^*)| + \mu |d(u^*, Tu^*)|,
\]
and hence
\[
|d(u, u^*)| < \lambda |d(u, u^*)| + \mu |d(u^*, u^*)| = \lambda |d(u, u^*)|, \quad \text{a contradiction.}
\]
So \( u = u^* \), which proves the uniqueness of fixed point in \( X \). This completes the proof.

**Corollary 8.** Let \((X, d)\) be a complete complex valued \( b \)-metric space with the coefficient \( s \geq 1 \) and let \( T : X \to X \) be a mapping satisfying (for some fixed \( n \))
\[
d(T^n x, T^n y) \leq \frac{\lambda d^2(x, y)}{1 + d(x, y)} + \mu d(y, T^n y),
\]
for all \( x, y \in X \), where \( \lambda, \mu \) are nonnegative reals with \( s \lambda + \mu < 1 \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** From Theorem 7, we obtain \( u \in X \) such that
\[
T^n u = u.
\]
The uniqueness follows from
\[
d(Tu, u) = d(TT^n u, T^n u) = d(T^n Tu, T^n u) \leq \frac{\lambda d^2(Tu, u)}{1 + d(Tu, u)} + \mu d(u, T^n u).
\]
By taking modulus (22), we have
\[ |d(Tu, u)| \leq \lambda |d(Tu, u)| + \mu |d(u, u)|, \] since
\[ |1 + d(Tu, u)| > |d(Tu, u)|. \] Therefore,
\[ |d(Tu, u)| < \lambda |d(Tu, u)|, \] a contradiction. \[ \square \]

**Theorem 9.** Let \((X, d)\) be a complete complex valued \(b\)-metric space with the coefficient \(s \geq 1\) and let \(T: X \to X\) be a mapping satisfying
\[ d(Tx, Ty) \leq \lambda d(x, y) + \frac{\mu d(x, Tx)d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)}, \] for all \(x, y \in X\) such that \(x \neq y\), \(d(x, Ty) + d(y, Tx) + d(x, y) \neq 0\), where \(\lambda, \mu\) are nonnegative reals with \(\lambda + \mu < 1\) or \(d(Tx, Ty) = 0\) if \(d(x, Ty) + d(y, Tx) + d(x, y) = 0\). Then \(T\) has a unique fixed point in \(X\).

**Proof.** For any arbitrary point, \(x_0 \in X\). Define sequence \(\{x_n\}\) in \(X\) such that
\[ x_{2n+1} = Tx_{2n} \quad \text{for} \quad n = (0, 1, 2, 3, \ldots). \] Now, we show that the sequence \(\{x_n\}\) is Cauchy:
\[ d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}) \leq \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, Tx_{2n})d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, Tx_{2n}) + d(x_{2n+1}, Tx_{2n+1}) + d(x_{2n}, x_{2n+1})} \]
\[ = \lambda d(x_{2n}, x_{2n+1}) + \frac{\mu d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})} \] which implies that
\[ |d(x_{2n+1}, x_{2n+2})| \leq \lambda |d(x_{2n}, x_{2n+1})| + \frac{\mu |d(x_{2n+1}, x_{2n+2})|}{|d(x_{2n}, x_{2n+1})|} + |d(x_{2n}, x_{2n+1})| \times |d(x_{2n}, x_{2n+1})|, \] since
\[ |d(x_{2n+1}, x_{2n+2})| \leq |d(x_{2n+1}, x_{2n})| + |d(x_{2n}, x_{2n+1})|. \] Therefore,
\[ |d(x_{2n+1}, x_{2n+2})| \leq \lambda |d(x_{2n+1}, x_{2n})| + \mu |d(x_{2n}, x_{2n+1})| \]
\[ = (\lambda + \mu) |d(x_{2n}, x_{2n+1})|. \] Similarly, we obtain
\[ |d(x_{2n+2}, x_{2n+3})| \leq (\lambda + \mu) |d(x_{2n+1}, x_{2n+2})|. \] Since \(\lambda + \mu < 1\) and \(s \geq 1\), we get \(\lambda + \mu < 1\). Therefore, with \(\delta = \lambda + \mu < 1\) and for all \(n \geq 0\) and consequently, we have
\[ |d(x_{2n+1}, x_{2n+2})| \leq \delta |d(x_{2n}, x_{2n+1})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \leq \ldots \leq \delta^{n+1} |d(x_0, x_1)|, \]
\[ |d(x_{2n+2}, x_{2n+3})| \leq \delta |d(x_{2n+1}, x_{2n+2})| \leq \delta^2 |d(x_{2n-1}, x_{2n})| \leq \ldots \leq \delta^{n+1} |d(x_0, x_1)|. \] Thus, for any \(m > n, m, n \in \mathbb{N}\), we have
\[ |d(x_m, x_n)| \leq s |d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| \]
\[ \leq s \delta |d(x_0, x_1)| + s^2 \delta^2 |d(x_{-1}, x_0)| \]
\[ \leq \ldots \leq s^{m-n-1} \delta |d(x_0, x_1)| + s^{m-n} |d(x_{-1}, x_0)|. \]
By using (33), we get

\[ |d(x_n, x_m)| \]
\[ \leq s \delta^n |d(x_0, x_1)| + s^2 \delta^{n+1} |d(x_0, x_1)| + \ldots + s^{m-n-1} \delta^{m-2} |d(x_0, x_1)| + s^{m-n} \delta^{m-1} |d(x_0, x_1)| 
\]
\[ = \sum_{i=1}^{m-n} s^i \delta^{i+n-1} |d(x_0, x_1)|. \]  
(35)

Therefore,

\[ |d(x_n, x_m)| \leq \sum_{i=1}^{m-n} s^i \delta^{i+n-1} |d(x_0, x_1)| \]
\[ = \sum_{i=1}^{m-n} s^i \delta^i |d(x_0, x_1)| \]
\[ \leq \sum_{i=1}^{\infty} (s \delta)^i |d(x_0, x_1)| \]
\[ = \frac{(s \delta)^n}{1 - s \delta} |d(x_0, x_1)| \]  
(36)

and hence

\[ |d(x_n, x_m)| \leq \frac{(s \delta)^n}{1 - s \delta} |d(x_0, x_1)| \to 0 \text{ as } m, n \to \infty. \]  
(37)

Thus, \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists some \( u \in X \) such that \( x_n \to u \) as \( n \to \infty \). Suppose this is not possible; then there exists \( z \in X \) such that

\[ |d(u, Tu)| = |z| > 0. \]  
(38)

Now,

\[ z = d(u, Tu) \]
\[ \leq sd(u, x_{2n+2}) + sd(x_{2n+2}, Tu) \]
\[ = sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Tu) \]
\[ \leq sd(u, x_{2n+2}) + s \lambda d(x_{2n+1}, u) \]
\[ + \frac{s \mu d(x_{2n+1}, Tu)}{d(x_{2n+1}, Tu) + d(u, Tx_{2n+1}) + d(x_{2n+1}, u)} |d(u, Tu)| \]
\[ + \frac{s \mu d(x_{2n+1}, x_{2n+2}) d(u, Tu)}{d(x_{2n+1}, Tu) + d(u, x_{2n+2}) + d(x_{2n+1}, u)} \]  
(39)

which implies that

\[ |z| \]
\[ = |d(u, Tu)| \leq s |d(u, x_{2n+2})| + s \lambda |d(x_{2n+1}, u)| \]
\[ + \frac{s \mu |d(x_{2n+1}, x_{2n+2})|}{d(x_{2n+1}, Tu) + d(u, x_{2n+2}) + d(x_{2n+1}, u)} \]
\[ \times |d(u, Tu)|. \]  
(40)

Taking the limit of (40) as \( n \to \infty \), we obtain that \( |z| = |d(u, Tu)| \leq 0 \), a contradiction with (38).

So \( |z| = 0 \). Hence \( Tu = u \).

Now, we show that \( T \) has a unique fixed point in \( X \). To show this, assume that \( u^* \) is another fixed point of \( T \).

\[ d(u, u^*) = d(Tu, u^*) \]
\[ \leq \lambda d(u, u^*) \]
\[ + \frac{\mu d(u, Tu)}{d(u, Tu^*) + d(u^*, Tu) + d(u, u^*)} \]  
(41)

so that

\[ |d(u, u^*)| \leq \lambda |d(u, u^*)| \]
\[ + \frac{\mu |d(u, Tu)|}{|d(u, Tu^*)| + |d(u^*, Tu)| + |d(u, u^*)|} \]  
(42)

\[ < \lambda |d(u, u^*)|, \text{ a contradiction.} \]

So \( u = u^* \), which proves the uniqueness of fixed point in \( X \).

Now, we consider the following case: \( d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+2}) + d(x_{2n+2}, Tx_{2n+3}) = 0 \) (for any \( n \)) implies \( d(Tx_{2n}, Tx_{2n+1}) = 0 \), so that \( x_{2n} = Tx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2} \). Thus we have \( x_{2n+1} = Tx_{2n} = x_{2n} \), so there exist \( K_1 \) and \( l_1 \) such that \( K_1 = Tl_1 = l_1 \). Using foregoing arguments, one can also show that there exist \( K_2 \) and \( l_2 \) such that \( K_2 = Tl_2 = l_2 \). As \( d(l_1, Tl_2) + d(l_2, Tl_1) + d(l_1, l_2) = 0 \) (due to definition) implies \( d(Tl_1, Tl_2) = 0, K_1 = Tl_1 = l_1 = K_2 \), which in turn yields that \( K_1 = Tl_1 = T K_1 \). Similarly, one can also have \( K_2 = T K_2 \). As \( K_2 = K_2 \) implies \( TK_1 = K_1 \), therefore \( K_1 = K_2 \) is fixed point of \( T \).

We now prove that \( T \) has unique fixed point. For this, assume that \( K_1^* \) in \( X \) is another fixed point of \( T \). Then we have \( TK_1^* = K_1^* \). As \( d(K_1, TK_1^*) + d(K_1^*, TK_1) + d(K_1, K_1^*) = 0 \), therefore \( d(K_1, K_1^*) = d(TK_1, TK_1^*) = 0 \).

This implies that \( K_1^* = K_1 \). This completes the proof of the theorem.

**Corollary 10.** Let \( (X, d) \) be a complete complex valued b-metric space with the coefficient \( s \geq 1 \) and let \( T : X \to X \) be a mapping satisfying (for some fixed \( n \))

\[ d(T^n x, T^n y) \]
\[ \leq \lambda d(x, y) + \frac{\mu d(x, T^n y)}{d(x, T^n y) + d(y, T^n x) + d(x, y)}, \]  
(43)

for all \( x, y \in X \) such that \( x \neq y, d(x, Ty) + d(y, Tx) + d(x, y) \neq 0 \), where \( \lambda, \mu \) are nonnegative reals with \( \lambda + \mu s < 1 \) or
To verify that \((X, d)\) is a complete complex valued b-metric space with \(s = 2\), it is enough to verify the triangular inequality condition:

\[
d(x, y) = \frac{2}{3} \left[ |x - y|^2 + i |x - y|^2 \right]
\]

\[
= \frac{2}{3} \left[ |x - z + z - y|^2 + i |x - z + z - y|^2 \right]
\]

\[
\leq \frac{2}{3} \left[ (|x - z|^2 + |z - y|^2 + 2|x - z||z - y|) + i (|x - z|^2 + |z - y|^2 + 2|x - z||z - y|) \right]
\]

\[
\leq 2 \left( \frac{2}{3} \left[ |x - z|^2 + |z - y|^2 + |x - z|^2 + |z - y|^2 \right] \right)
\]

\[
= 2 \left[ \frac{2}{3} \left[ |x - y|^2 + i |x - z|^2 \right] + \frac{2}{3} \left[ |z - y|^2 + i |z - x|^2 \right] \right]
\]

\[
= 2 \left[ d(x, z) + d(z, y) \right].
\]

That is,

\[
d(x, y) \leq s \left[ d(x, z) + d(z, y) \right].
\]

Therefore, \(s = 2\). Define \(T : X \to X\) as \(T(x, y) = (x/2, y/2)\) for all \(x, y \in X\). Then

\[
d(Tx, Ty) = d(x/2, y/2) = \frac{2}{3} \left[ |x - y|^2 + i |x - y|^2 \right]
\]

\[
= \frac{2}{3} \left[ \left( \frac{x}{2} - \frac{y}{2} \right)^2 + i \left( \frac{x}{2} - \frac{y}{2} \right)^2 \right]
\]

\[
= \frac{1}{6} \left[ |x - y|^2 + i |x - y|^2 \right] = \frac{1}{6} d(x, y),
\]

\[
d(Tx, Ty)
\]

\[
\leq \frac{3}{8} d(x, y) + \frac{(1/5) d(x, x/2) d(y, y/2)}{d(x, y/2) + d(y, x/2) + d(x, y)}.
\]

Here

\[
\lambda + s \mu = \frac{3}{8} + \frac{1}{5} = \frac{31}{40} < 1.
\]

It is easily and clearly verified that the map \(T\) satisfies contractive condition (26) of Theorem 9 with the coefficients \(s = 2, \lambda = 3/8, \mu = 1/5\). Observe that the point \(0 \in X\) remains fixed under \(T\) and is indeed unique.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
References


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