Research Article

An Algorithm for Computing All Berge Equilibria

H. W. Corley¹ and Phantipa Kwain²

¹Center On Stochastic Modelling, Optimization, and Statistics (COSMOS), The University of Texas at Arlington,
 P.O. Box 19017, Arlington, TX 76019-0017, USA

²TransSolutions, LLC., 14600 Trinity Boulevard, Fort Worth, TX 76155, USA

Correspondence should be addressed to H. W. Corley; corley@uta.edu

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An algorithm is presented in this note for determining all Berge equilibria for an n -person game in normal form. This algorithm is based on the notion of disappointment, with the payoff matrix (PM) being transformed into a disappointment matrix (DM). The DM has the property that a pure strategy profile of the PM is a BE if and only if $(0, \dots, 0)$ is the corresponding entry of the DM. Furthermore, any $(0, \dots, 0)$ entry of the DM is also a more restrictive Berge-Vaisman equilibrium if and only if each player's BE payoff is at least as large as the player's maximin security level.

1. Introduction

In a Berge equilibrium (BE) for an n -person game, every $n-1$ player has pure strategies that maximized the remaining player's payoff. The BE was intuitively defined for pure strategies in [1] as a refinement to the Nash equilibrium (NE) [2]. The BE was formalized in [3] as a game-theoretic solution concept modeling mutual support and cooperation, as opposed to the selfishness of the NE. However, the BE was not studied extensively until recently. For example, see [4–9]. Only [6, 7] offer approaches for obtaining BEs, though, and these methods are stated abstractly without examples.

A simple algorithm is presented here for computing all BEs for an n -person game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in normal form, with $N = \{1, \dots, n\}$ the set of players, S_i the finite set of pure strategies for player i , and $u_i(s_1, \dots, s_n)$ the von Neumann-Morgenstern utility of player i for a pure strategy profile $s = (s_1, \dots, s_n) \in \times_{j \in N} S_j = S$. The BE is defined as follows, where an incomplete strategy profile $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ denotes a member of $S_{-i} = \times_{j \in N \setminus \{i\}} S_j$.

Definition 1. The strategy profile s^* is a BE of G if and only if

$$u_i(s^*) = \max_{s_{-i} \in S_{-i}} u_i(s_i^*, s_{-i}) \quad \forall i \in N. \quad (1)$$

The BE s^* is further called a Berge-Vaisman equilibrium (BVE) if and only if the maximin security levels α_i of the players satisfy

$$\alpha_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \leq u_i(s_i^*) \quad \forall i \in N \quad (2)$$

so that no player is guaranteed better payoff than that given by s^* .

In Section 2 we define the disappointment matrix (DM) corresponding to the payoff matrix (PM) for G and prove that a pure strategy s^* is a BE if and only if $(0, \dots, 0)$ is the corresponding entry of the DM. In Section 3 the approach of Section 2 is formalized as an algorithm to obtain all BEs and BVEs for G . In Section 4 a computational example is presented for $n = 3$.

2. The Disappointment Matrix

The disappointment function is defined as a transformation of a player's payoffs to losses.

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(1)  $1 \leftarrow i, 1 \leftarrow j$ 
(2) While  $i \leq n$  do
(3)   While  $j \leq m_i$  do
(4)     Compute  $d_i(s_i^j, s_{-i})$  for all  $s_{-i} \in S_{-i}$  from (3).
(5)    $d_i(s_i^j, s_{-i}) \leftarrow u_i(s_i^j, s_{-i})$  for all  $s_{-i} \in S_{-i}$ 
(6)   End While
(7) End While
(8)  $\text{BE} = \bigcup \{s \in S : d_i(s) = 0, i = 1, \dots, n\}$ .
(9)  $\text{BVE} = \{s \in \text{BE} : \alpha_i \leq u_i(s), i = 1, \dots, n\}$ .

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ALGORITHM 1

Definition 2. For the game G , the disappointment incurred by any player i choosing s_i and the other $n - 1$ players choosing s_{-i} is defined as

$$d_i(s) = d_i(s_i, s_{-i}) = \max_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i}) - u_i(s_i, s_{-i}) \quad (3)$$

$\forall i \in N.$

The disappointment incurred by player i for a strategy profile $s = (s_i, s_{-i})$ is thus the difference between the best payoff that player i could obtain by choosing s_i and the actual payoff that player i would obtain for the strategy profile (s_i, s_{-i}) . This definition immediately gives the following result.

Theorem 3. The pure strategy profile s^* is a BE for the game G if and only if the disappointment $d_i(s^*) = 0$ for all $i \in N$. Moreover, a BE s^* is also a BVE if and only if $\alpha_i \leq u_i(s^*)$ for all $i \in N$.

Proof. The strategy profile s^* is a BE if and only if it satisfies (1) of Definition 1. However, (1) holds if and only if $d_i(s^*) = 0$ in (3) for all $i \in N$. To complete the proof, the BE s^* is also a BVE if and only if it satisfies (2). $\square$

The DM of a game is now defined.

Definition 4. The disappointment matrix of G is the matrix obtained from its PM by replacing $(u_1(s), \dots, u_n(s))$ by $(d_1(s), \dots, d_n(s))$ for all $s \in S$.

3. Algorithm

Consider the game G in which each player $i \in N$ has m_i pure strategies. Denote the j th strategy of player i by s_i^j , $j = 1, \dots, m_i$. Algorithm 1 describes the algorithm *BECOMP* for obtaining all BEs and BVEs for G from its DM. The computational complexity of *BECOMP* is $O(n \prod_{i=1}^n m_i)$ because of the unavoidable enumeration in Step 4 of all strategy profiles (s_i^j, s_{-i}) , $j = 1, \dots, m_i$, for each player i .

4. Example

For $n = 3$ we use the notation of *BECOMP* with $m_1 = m_2 = m_3 = 2$ and consider the game G with a PM of Table 1. The DM for G is obtained from the PM via *BECOMP* and

TABLE 1: PM for G .

Player 1	Player 3			
	Player 2		Player 2	
	s_2^1	s_2^2	s_2^1	s_2^2
s_1^1	(3, 1, 2)	(3, 4, 0)	(6, 3, 0)	(3, 5, 1)
s_1^2	(1, 4, 5)	(2, 2, 3)	(2, 4, 4)	(-1, 2, 3)

TABLE 2: DM for G .

Player 1	Player 3			
	Player 2		Player 2	
	s_2^1	s_2^2	s_2^1	s_2^2
s_1^1	(3, 3, 3)	(3, 1, 5)	(0, 1, 4)	(3, 0, 3)
s_1^2	(1, 0, 0)	(0, 3, 2)	(0, 0, 0)	(3, 3, 1)

shown in Table 2. The unique BE for G is $s^* = (s_1^2, s_2^1, s_3^2)$ with associated payoffs $(u_1(s^*), \dots, u_n(s^*)) = (2, 4, 4)$. However, s^* is not a BVE since $\alpha_1 = 3 > u_1(s^*) = 2$. Player 1 is guaranteed better payoff by choosing s_1^1 instead of s_1^2 .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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