Research Article

Weighted Pluricomplex Energy II

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We continue our study of the complex Monge-Ampère operator on the weighted pluricomplex energy classes. We give more characterizations of the range of the classes $\mathcal{E}_\chi$ by the complex Monge-Ampère operator. In particular, we prove that a nonnegative Borel measure μ is the Monge-Ampère of a unique function $\varphi \in \mathcal{E}_\chi$ if and only if $\chi(\mathcal{E}_\chi) \subset L^1(d\mu)$. Then we show that if $\mu = (dd^c \varphi)^n$ for some $\varphi \in \mathcal{E}_\chi$ then $\mu = (dd^c u)^n$ for some $u \in \mathcal{E}_\chi$, where f is given boundary data. If moreover the nonnegative Borel measure μ is suitably dominated by the Monge-Ampère capacity, we establish a priori estimates on the capacity of sublevel sets of the solutions. As a consequence, we give a priori bounds of the solution of the Dirichlet problem in the case when the measure has a density in some Orlicz space.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain, that is, a connected, bounded open subset of $\mathbb{C}^n$, such that there exists a negative plurisubharmonic function ρ such that $\{z \in \Omega; \rho(z) < -c\} \Subset \Omega$, for all $c > 0$. Such a function ρ is called an exhaustion function. We let $\text{PSH}(\Omega)$ denote the cone of plurisubharmonic functions (psh for short) on Ω and let $\text{PSH}^-(\Omega)$ denote the subclass of negative functions.

As known (see [1, 2]), the complex Monge-Ampère operator $(dd^c \cdot)^n$ is well defined, as a nonnegative measure, on the set of locally bounded plurisubharmonic functions. Therefore the question of describing the measures which are the Monge-Ampère of bounded psh functions is very important for pluripotential theory, complex dynamic, and complex geometry. This problem has been studied extensively by various authors; see, for example, [2–6] and reference therein. In [7], Cegrell introduced the pluricomplex energy classes $\mathcal{E}_p(\Omega)$ and $\mathcal{F}_p(\Omega)$ ($p \geq 1$) on which the complex Monge-Ampère operator is well defined. He proved that a measure μ is the Monge-Ampère of some function $u \in \mathcal{E}_p(\Omega)$ if and only if it satisfies

$$\int_{\Omega} (-v)^p d\mu \leq \text{Const} \left(\int_{\Omega} (-v)^p (dd^c v)^n \right)^{p/(n+p)}, \quad (1)$$

$$\forall v \in \mathcal{E}_0(\Omega),$$

where $\mathcal{E}_0(\Omega)$ is the cone of all bounded psh functions φ defined on the domain Ω with finite total Monge-Ampère mass and $\lim_{z \rightarrow \zeta} \varphi(z) = 0$, for every $\zeta \in \partial\Omega$. Recently, Åhag et al. in [8] proved that, in the case $p = 1$, inequality (1) is equivalent to $\mathcal{E}_1(\Omega) \subset L^1(d\mu)$. In this note, our first objective is to extend this result by showing that, for all positive number p , inequality (1) is equivalent to $\mathcal{E}_p(\Omega) \subset L^p(d\mu)$. In fact, we prove some more general result. Given a nondecreasing function $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$, we consider the set $\mathcal{E}_\chi(\Omega)$ of plurisubharmonic functions of finite χ -weighted Monge-Ampère energy and, in some sense, has boundary values zero. These are the functions $u \in \text{PSH}(\Omega)$ for which there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ with limit u and

$$\sup_{j \in \mathbb{N}} \int_{\Omega} -\chi(u_j) (dd^c u_j)^n < \infty. \quad (2)$$

Then we have the following characterization of the image of the complex Monge-Ampère acting in the class $\mathcal{E}_\chi(\Omega)$.

Theorem 1. *Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing convex or homogeneous function such that $\chi(-\infty) = -\infty$. The following assertions are equivalent:*

- (1) *there exists a unique function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\mu = (dd^c \varphi)^n$;*
- (2) *$\chi(\mathcal{E}_\chi(\Omega)) \subset L^1(d\mu)$.*

Next, we extend our previous result to families of functions having prescribed boundary data. Let $f \in \text{PSH}(\Omega)$ be a maximal psh function. We define the class $\mathcal{E}_\chi(f)$ to be the class of psh functions u such that there exists a function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that

$$\varphi(z) + f(z) \leq u(z) \leq f(z), \quad \forall z \in \Omega. \quad (3)$$

Some particular cases of the classes $\mathcal{E}_\chi(f)$ have been studied in [6, 7, 9–16].

More precisely, we prove the following result.

Theorem 2. *Let μ be a nonnegative measure in Ω , let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing convex or homogeneous function such that $\chi(-\infty) = -\infty$, and let f be a maximal function. Then, if $\mu = (dd^c u)^n$ for some $u \in \mathcal{E}_\chi(\Omega)$ then there exists a unique function $\varphi \in \mathcal{E}_\chi(f)$ such that $\mu = (dd^c \varphi)^n$.*

Moreover, when the nonnegative measure μ is dominated by the Monge-Ampère capacity, we give an estimate of the growth of solutions of the equation $(dd^c \varphi)^n = \mu$. As in [12], let us consider the function

$$F_\mu(t) := \sup \{ \mu(K), K \Subset \Omega; \text{cap}_\Omega(K) \leq t \}, \quad \forall t \geq 0. \quad (4)$$

Then $F := F_\mu$ is a nondecreasing function on $\mathbb{R}^+$ and satisfies

$$\mu(K) \leq F_\mu(\text{cap}_\mu(K)), \quad \forall \text{Borel subsets } K \subset \Omega. \quad (5)$$

Write $F(x) = F_\varepsilon(x) = x(\varepsilon(-\ln(x/n)))^n$ where $\varepsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing.

Such measures dominated by the Monge-Ampère capacity have been extensively studied by Kołodziej in [3–5]. He proved that if $\phi : \partial\Omega \rightarrow \mathbb{R}$ is a continuous function and $\int_0^{+\infty} \varepsilon(t)dt < +\infty$, then μ is the Monge-Ampère of a unique function $\varphi \in \text{PSH}(\Omega)$ with $\varphi|_{\partial\Omega} = \phi$.

When $\int_0^{+\infty} \varepsilon(s)ds = +\infty$, we have the following estimate.

Theorem 3. *Let μ be a positive finite measure. Assume, for all compact subsets $K \subset \Omega$,*

$$\mu(K) \leq F_\varepsilon(\text{cap}_\Omega(K)). \quad (6)$$

Then there exists a unique function $\varphi \in \mathcal{F}(f)$ such that $\mu = (dd^c \varphi)^n$, and

$$\text{cap}_\Omega(\{\varphi < f - s\}) \leq \exp(-nH^{-1}(s)), \quad \forall s > 0. \quad (7)$$

Here H^{-1} is the reciprocal function of $H(x) = e \int_0^x \varepsilon(t)dt + s_0(\mu)$.

In particular if $\int_0^{+\infty} \varepsilon(t)dt < +\infty$ then

$$0 \leq f - \varphi \leq e \int_0^{+\infty} \varepsilon(t)dt + e\varepsilon(0) + \mu(\mu)^{1/n}. \quad (8)$$

The paper is organised as follows. In Section 2, we recall the definitions of the energy classes $\mathcal{E}_\chi(\Omega)$ and some classes of psh functions introduced by Cegrell [7, 13, 14] and we prove Theorem 1. In Section 3, we prove Theorem 2. As a

consequence, we generalize the main result in the paper [9]. In Section 4, we prove Theorem 3. As application, we give a priori bound of the solution of Dirichlet problem in the case when the measure $\mu = g d\lambda$, where g belongs to some Orlicz space $L \log^\alpha L$.

2. Energy Classes with Zero Boundary Data $\mathcal{E}_\chi$

Let us recall some Cegrell's classes (Cf. [7, 13, 14]). The class $\mathcal{E}(\Omega)$ is the set of plurisubharmonic functions u such that, for all $z_0 \in \Omega$, there exists a neighbourhood V_{z_0} of z_0 and $u_j \in \mathcal{E}_0(\Omega)$, a decreasing sequence which converges towards u in V_{z_0} and satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$. Cegrell [13] has shown that the operator $(dd^c \cdot)^n$ is well defined on $\mathcal{E}(\Omega)$, continuous under decreasing limits, and the class $\mathcal{E}(\Omega)$ is stable under taking maximum; that is, if $u \in \mathcal{E}(\Omega)$ and $v \in \text{PSH}^-(\Omega)$ then $\max(u, v) \in \mathcal{E}(\Omega)$. This class is the largest class with these properties (Theorem 4.5 in [13]). The class $\mathcal{E}(\Omega)$ has been further characterized by Błocki [17, 18] and Le Mau et al. in [19].

The class $\mathcal{F}(\Omega)$ is the global version of $\mathcal{E}(\Omega)$: a function u belongs to $\mathcal{F}(\Omega)$ if and only if there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ converging towards u in all of Ω , which satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$. The class $\mathcal{E}(\Omega)$ has been further characterized in [12, 17].

Let $\Omega_j \Subset \Omega$ be an increasing sequence of strictly pseudoconvex domains such that $\Omega = \bigcup_j \Omega_j$. Let $u \in \mathcal{E}(\Omega)$ be a given psh function and put

$$u_{\Omega_j} := \sup \{ \varphi \in \text{PSH}(\Omega); \varphi \leq u \text{ on } \Omega \setminus \Omega_j \}. \quad (9)$$

Then we have $u_{\Omega_j} \in \mathcal{E}(\Omega)$ and u_{Ω_j} is an increasing sequence. Let $\tilde{u} := (\lim_j u_{\Omega_j})^*$. It follows from the properties of $\mathcal{E}(\Omega)$ that $\tilde{u} \in \mathcal{E}(\Omega)$. Note that the definition of $\tilde{u}$ is independent of the choice of the sequence Ω_j and is maximal; that is, $(dd^c \tilde{u})^n = 0$. $\tilde{u}$ is the smallest maximal psh function above u . Define $\mathcal{N}(\Omega) := \{u \in \mathcal{E}(\Omega); \tilde{u} = 0\}$. In fact, this class is the analogous of potentials for subharmonic functions.

Definition 4. Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be a nondecreasing function. We let $\mathcal{E}_\chi(\Omega)$ denote the set of all functions $u \in \text{PSH}(\Omega)$ for which there exists a sequence $u_j \in \mathcal{E}_0(\Omega)$ decreasing to u in Ω and satisfying

$$\sup_{j \in \mathbb{N}} \int_\Omega -\chi(u_j) (dd^c u_j)^n < \infty. \quad (10)$$

It was proved in [15, 20] that if $\chi \not\equiv 0$ then

$$\mathcal{E}_\chi(\Omega) \subset \mathcal{E}(\Omega). \quad (11)$$

In particular, for any function $u \in \mathcal{E}_\chi(\Omega)$, the complex Monge-Ampère operator $(dd^c u)^n$ is well defined as nonnegative measure. Furthermore, if $\chi(-t) < 0$ for all $t > 0$, then

$$\mathcal{E}_\chi(\Omega) = \left\{ u \in \mathcal{N}(\Omega); \int_\Omega -\chi(u) (dd^c u)^n < +\infty \right\}. \quad (12)$$

The class $\mathcal{E}_\chi(\Omega)$ has been characterized by the speed of decrease of the capacity of sublevel sets [11, 12].

Recall that the Monge-Ampère capacity has been introduced and studied by Bedford and Taylor in [1]. Given $K \subset \Omega$, a compact subset, its Monge-Ampère capacity relatively to Ω is defined by

$$\text{cap}_\Omega(K) := \sup \left\{ \int_K (dd^c u)^n; u \in \text{PSH}(\Omega), -1 \leq u \leq 0 \right\}. \quad (13)$$

The following estimates (cf [12]) will be useful later on. For any $\varphi \in \mathcal{E}_0$

$$\begin{aligned} t^n \text{cap}_\Omega(\varphi < -s - t) \\ \leq \int_{(\varphi < -s)} (dd^c \varphi)^n \leq s^n \text{cap}_\Omega(\varphi < -s), \quad \forall s, t > 0. \end{aligned} \quad (14)$$

Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be a nondecreasing function. Without loss of generality, from now on, we assume that $\chi(0) = 0$. We define the class $\widehat{\mathcal{E}}_\chi(\Omega)$

$$\begin{aligned} \widehat{\mathcal{E}}_\chi(\Omega) := \left\{ \varphi \in \text{PSH}^-(\Omega); \right. \\ \left. \int_{t_\chi}^{+\infty} t^n \chi'(t) \text{cap}_\Omega(\varphi < -t) dt < +\infty \right\}. \end{aligned} \quad (15)$$

Proposition 5. One has $\widehat{\mathcal{E}}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$, while

$$\mathcal{E}_\chi(\Omega) \subset \widehat{\mathcal{E}}_{\widehat{\chi}}(\Omega), \quad \text{where } \widehat{\chi}(t) = \chi\left(\frac{t}{2}\right). \quad (16)$$

Moreover, if $\chi :]-\infty, -t_\chi[\rightarrow \mathbb{R}^-$ is convex, then

$$\mathcal{E}_\chi(\Omega) = \widehat{\mathcal{E}}_\chi(\Omega). \quad (17)$$

Here t_χ denote the real number satisfying $\chi(t) < 0$, for all $t < -t_\chi$, and $\chi(t) = 0$, for all $t \geq -t_\chi$.

Proof. Compare with [12, 20]. $\square$

Theorem 6. Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing convex function such that $\chi(-\infty) = -\infty$. The following conditions are equivalent:

- (1) there exists a unique function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\mu = (dd^c \varphi)^n$;
- (2) $\chi(\mathcal{E}_\chi(\Omega)) \subset L^1(\Omega, d\mu)$;
- (3) there exists a constant $C > 0$ such that

$$\int_\Omega -\chi(u) d\mu \leq C, \quad \forall u \in \widehat{\mathcal{E}}_0(\Omega); \quad (18)$$

- (4) there exists a constant $A > 0$ such that

$$\begin{aligned} \int_\Omega -\chi(u) d\mu \\ \leq C_2 \max \left(1, \left(\int_0^\infty s^n \chi'(-s) \text{cap}_\Omega(u < -s) ds \right)^{1/n} \right), \quad (19) \\ \forall u \in \mathcal{E}_0(\Omega); \end{aligned}$$

- (5) there exists a locally bounded function $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\limsup_{t \rightarrow +\infty} F(t)/t < 1$ and

$$\int_\Omega -\chi(u) d\mu \leq F(C_\chi(u)), \quad \forall u \in \mathcal{E}_0(\Omega). \quad (20)$$

Here $\widehat{\mathcal{E}}_0(\Omega)$ denotes the class $\widehat{\mathcal{E}}_0(\Omega) := \{u \in \mathcal{E}_0(\Omega); \int_0^\infty s^n \chi'(-s) \text{cap}_\Omega(u < -s) ds \leq 1\}$ and $C_\chi(u) := \int_0^\infty s^n \chi'(-s) \text{cap}_\Omega(u < -s) ds$.

The equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are proved in [11] (Theorem 5.1) and the implication (5) $\Rightarrow$ (1) is proved in [12] (Theorem 5.2). For the sake of completeness we include a complete proof.

Proof. We start by the implication (1) $\Rightarrow$ (2). Let $u, \varphi \in \mathcal{E}_\chi(\Omega)$. It follows from Proposition 5 that $u + \varphi \in \mathcal{E}_\chi(W)$. Hence

$$\int_\Omega -\chi(u) (dd^c \varphi)^n \leq \int_\Omega -\chi(u + \varphi) (dd^c(u + \varphi))^n < \infty. \quad (21)$$

Now, for the implication (2) $\Rightarrow$ (3), assume that (3) is not satisfied. Then for each $j \in \mathbb{N}$ we can find a function $u_j \in \widehat{\mathcal{E}}_0(\Omega)$ such that

$$\int_\Omega -\chi(u_j) d\mu \geq 2^{3j}. \quad (22)$$

Consider the function

$$u := \sum_{j=1}^\infty \frac{1}{2^{2j}} u_j. \quad (23)$$

Observe that

$$(u < -s) \subset \bigcup_1^\infty (u_j < -2^j s). \quad (24)$$

Hence

$$\text{cap}_\Omega(u < -s) \leq \sum_j^\infty \text{cap}_\Omega(u_j < -2^j s). \quad (25)$$

Now, since the weight χ is convex or homogeneous and using the estimates (14), we get

$$\begin{aligned} \int_0^\infty s^n \chi'(-s) \text{cap}_\Omega(u < -s) ds \\ \leq \int_0^\infty s^n \chi'(-s) \sum_j^\infty \text{cap}_\Omega(u_j < -2^j s) ds \\ \leq \sum_j^\infty \frac{1}{2^{nj-n}} \int_0^\infty (2^{j-1} s)^n \chi'(-s) \text{cap}_\Omega(u_j < -2^j s) ds \\ \leq \sum_j^\infty \frac{1}{2^{nj-n}} \int_0^\infty (2^{j-1} s)^n \chi'(-s) \text{cap}_\Omega(u_j < -s) ds \\ \leq 2^n \sum_j^\infty \frac{1}{2^{nj}} < \infty. \end{aligned} \quad (26)$$

Hence $u \in \mathcal{E}_\chi(\Omega)$. On the other hand, from (22) we have

$$\int_{\Omega} -\chi(u) d\mu \geq \frac{1}{2^{2j}} \int_{\Omega} -\chi(u_j) d\mu \geq 2^j, \quad \forall j \in \mathbb{N}, \quad (27)$$

which yields a contradiction.

Now, we prove that (3) $\Rightarrow$ (4). Let $\psi \in \mathcal{E}_0(\Omega)$, denote $E_\chi(\psi) := \int_{\Omega} -\chi(\psi)(dd^c\psi)^n$. If $\psi \in \widetilde{\mathcal{E}}_0(\Omega)$, that is, $C_\chi(\psi) \leq 1$, then

$$\int_{\Omega} -\chi(\psi) d\mu \leq 2^n = C. \quad (28)$$

If $C_\chi(\psi) > 1$ the function $\tilde{\psi}$ defined by

$$\tilde{\psi} := \frac{\psi}{1 + C_\chi(\psi)^{1/n}} \in \widetilde{\mathcal{E}}_0(\Omega). \quad (29)$$

Indeed, from the monotonicity of χ , we have

$$\begin{aligned} & \int_0^\infty \chi'(s) s^n \text{cap}_\Omega \left(\frac{\psi}{1 + C_\chi(\psi)^{1/n}} < -s \right) ds \\ &= \frac{1}{C_\chi(\psi)} \int_0^\infty \chi'(s) (s C_\chi(\psi)^{1/n})^n \\ & \quad \cdot \text{cap}_\Omega (\psi < -s - s C_\chi(\psi)^{1/n}) ds \\ &\leq \frac{1}{C_\chi(\psi)} \int_0^\infty \chi'(s) s^n \text{cap}_\Omega (\psi < -s) ds = 1. \end{aligned} \quad (30)$$

It follows from (18) and the convexity of χ that

$$\begin{aligned} \int_{\Omega} -\chi(\psi) d\mu &\leq 2C_\chi(\psi)^{1/n} \int_{\Omega} -\chi \left(\frac{\psi}{1 + C_\chi(\psi)^{1/n}} \right) d\mu \\ &\leq AC_\chi(\psi)^{1/n}. \end{aligned} \quad (31)$$

Hence we get (19).

For the implication (4) $\Rightarrow$ (5), we consider $F(t) = A \max(1, t^{1/n})$.

(5) $\Rightarrow$ (1). It follows from [12] (Theorem 4.5) that the class $\mathcal{E}_\chi(\Omega)$ characterizes pluripolar sets in the sense that if P is a locally pluripolar subset of Ω then $P \subset \{v = -\infty\}$, for some $v \in \mathcal{E}_\chi(\Omega)$. Then assumption (20) on μ implies that it vanishes on pluripolar sets. It follows from [13] that there exists a function $u \in \mathcal{E}_0(\Omega)$ and $f \in L_{\text{loc}}^1((dd^c u)^n)$ such that $\mu = f(dd^c u)^n$.

Consider $\mu_j := \min(f, j)(dd^c u)^n$. This is a finite measure which is bounded from above by the complex Monge-Ampère measure of a bounded function. It follows therefore from [3] that there exist $\varphi_j \in \mathcal{E}_0(\Omega)$ such that

$$(dd^c \varphi_j)^n = \min(f, j)(dd^c u)^n. \quad (32)$$

The comparison principle shows that φ_j is a decreasing sequence. Set $\varphi = \lim_{j \rightarrow \infty} \varphi_j$. It follows from (20) that

$$\int_{\Omega} -\chi(\varphi_j)(dd^c \varphi_j)^n \leq F \left(\int_0^\infty \chi'(s) s^n \text{cap}_\Omega (\varphi_j < -s) ds \right). \quad (33)$$

Hence

$$\sup_j \int_0^\infty \chi'(s) s^n \text{cap}_\Omega (\varphi_j < -s) ds < \infty. \quad (34)$$

This implies that

$$\int_0^{+\infty} t^n \chi'(-t) \text{cap}_\Omega (\{\varphi < -t\}) dt < +\infty. \quad (35)$$

Then $\varphi \not\equiv -\infty$ and therefore $\varphi \in \mathcal{E}_\chi(\Omega)$.

We conclude now by continuity of the complex Monge-Ampère operator along decreasing sequences that $(dd^c \varphi)^n = \mu$. The uniqueness of φ follows from the comparison principle. $\square$

3. The Weighted Energy Class with Boundary Values

Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be a nondecreasing function and let $f \in \mathcal{M}(\Omega)$ be a maximal psh function. We define the class $\mathcal{E}_\chi(f)$ (resp., $\mathcal{N}(f)$, $\mathcal{F}(f)$, $\mathcal{N}^a(f)$, and $\mathcal{F}^a(f)$) to be the class of psh functions u such that there exists a function $\varphi \in \mathcal{E}_\chi(\Omega)$ (resp., $\mathcal{N}$, $\mathcal{F}$, $\mathcal{N}^a$, $\mathcal{F}^a$) such that

$$\varphi(z) + f(z) \leq u(z) \leq f(z), \quad \forall z \in \Omega. \quad (36)$$

Later on, we will use repeatedly the following well known comparison principle from [1] as well as its generalizations to the class $\mathcal{N}(f)$ (cf. [10, 14]).

Theorem 7 (see [1, 10, 14]). *Let $f \in \mathcal{E}(\Omega)$ be a maximal function and $u, v \in \mathcal{N}(f)$ be such that $(dd^c u)^n$ vanishes on all pluripolar sets in Ω . Then*

$$\int_{(u < v)} (dd^c v)^n \leq \int_{(u < v)} (dd^c u)^n. \quad (37)$$

Furthermore if $(dd^c u)^n = (dd^c v)^n$ then $u = v$.

The following lemma, which gives an estimate of the size of sublevel set in terms of the mass of Monge-Ampère measure, will be useful shortly.

Lemma 8. *Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be a nondecreasing function such that $\chi(t) < 0$, for all $t < 0$, and $f \in \mathcal{E}$ a maximal function. Then for all $\varphi \in \mathcal{E}_\chi(f)$*

$$\begin{aligned} & t^n \text{cap}_\Omega (\varphi < -s - t + f) \\ & \leq \int_{(\varphi < -s + f)} (dd^c \varphi)^n, \quad \forall s > 0, t > 0. \end{aligned} \quad (38)$$

Proof. Fix $s, t > 0$. Let $K \subset \{\varphi < f - s - t\}$ be a compact subset. Then

$$\begin{aligned} \text{cap}_\Omega(K) &= \int_{\Omega} (dd^c u_K^*)^n = \int_{(\varphi < f - s - t)} (dd^c u_K^*)^n \\ &= \int_{(\varphi < f - s + t u_K^*)} (dd^c u_K^*)^n \leq \frac{1}{t^n} \int_{\{\varphi < v\}} (dd^c v)^n, \end{aligned} \quad (39)$$

where u_K^* is the relative extremal function of the compact K and $v := f - s + tu_K^*$. It follows from Theorem 7 that

$$\begin{aligned} \frac{1}{t^n} \int_{\{\varphi < v\}} (dd^c v)^n &= \frac{1}{t^n} \int_{\{\varphi < \max(\varphi, v)\}} (dd^c \max(\varphi, v))^n \\ &\leq \frac{1}{t^n} \int_{\{\varphi < \max(\varphi, v)\}} (dd^c \varphi)^n \\ &= \frac{1}{t^n} \int_{\{\varphi < f - s + tu_K^*\}} (dd^c \varphi)^n \\ &\leq \frac{1}{t^n} \int_{\{\varphi < f - s\}} (dd^c \varphi)^n. \end{aligned} \quad (40)$$

Taking the supremum over all K 's yields the first inequality. $\square$

Proposition 9. Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing function. Then one has

$$\mathcal{E}_\chi(f) \subset \left\{ u \in \text{PSH}(\Omega); u \leq f, \int_0^{+\infty} s^n \chi'(-s) \text{cap}_\Omega(u < f - 2s) ds < +\infty \right\}. \quad (41)$$

In particular, if $\chi \not\equiv 0$, then $\text{cap}_\Omega(u < f - s) < +\infty$ for all $s > 0$ and $u \in \mathcal{E}_\chi(f)$.

Proof. Let $u \in \mathcal{E}_\chi(f)$. Then there exists a function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\varphi + f \leq u$. Therefore $(u < f - s) \subset (\varphi < -s)$. It follows from Lemma 8

$$\begin{aligned} \int_0^{+\infty} s^n \chi'(-s) \text{cap}_\Omega(u < f - 2s) ds \\ &\leq \int_0^{+\infty} s^n \chi'(-s) \text{cap}_\Omega(\varphi < -2s) ds \\ &\leq \int_0^{+\infty} \chi'(-s) \int_{\{\varphi < -s\}} (dd^c \varphi)^n ds \\ &= \int_\Omega -\chi(\varphi) (dd^c \varphi)^n < \infty. \end{aligned} \quad (42)$$

$\square$

Theorem 10. Let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing function which satisfies $\chi(-\infty) = -\infty$ and $f \in \mathcal{E}$ a maximal function. Then if there exists a decreasing sequence $u_j \in \mathcal{E}_0(f)$ such that

$$\sup_j \int_\Omega -\chi(u_j - f) (dd^c u_j)^n < \infty \quad (43)$$

then $u := \lim_{j \rightarrow \infty} u_j \in \mathcal{E}_\chi(f)$ and $\chi(u - f) \in L^1((dd^c u)^n)$.

Conversely, if $u \in \mathcal{E}_\chi(f)$ and $\chi(u - f) \in L^1((dd^c u)^n)$ then there exists sequence $u_j \in \mathcal{E}_0(f)$ decreasing towards u such that

$$\sup_j \int_\Omega -\chi(u_j - f) (dd^c u_j)^n < \infty. \quad (44)$$

Proof. Assume that the sequence $u_j \in \mathcal{E}_0(f) \cap C(\Omega)$ (if necessary, we approximate u_j by a continuous sequence $u_j^k \in \mathcal{E}_0(f)$). For a fixed $j \in \mathbb{N}$, let φ_j denote the function defined by

$$\varphi_j(z) := \sup \{ v \in \text{PSH}(\Omega); v + f \leq u_j \}, \quad \forall z \in \Omega. \quad (45)$$

We claim that

- (1) $\varphi_j \in \mathcal{E}_0$;
- (2) $(dd^c \varphi_j)^n \leq (dd^c u_j)^n$;
- (3) $(dd^c \varphi_j)^n = 0$ on the subset $(\varphi_j + f < u_j)$.

Then, it follows from statements (1), (2), and (3) that for each $j \in \mathbb{N}$

$$\begin{aligned} \int_\Omega -\chi(\varphi_j)^n (dd^c \varphi_j)^n &= \int_{(\varphi_j + f = u_j)} -\chi(\varphi_j)^n (dd^c \varphi_j)^n \\ &\leq \int_\Omega -\chi(u_j - f)^n (dd^c u_j)^n. \end{aligned} \quad (46)$$

Therefore

$$\begin{aligned} \sup_j \int_\Omega -\chi(\varphi_j)^n (dd^c \varphi_j)^n \\ &\leq \sup_j \int_\Omega -\chi(u_j - f)^n (dd^c u_j)^n < +\infty. \end{aligned} \quad (47)$$

Hence, the function $\varphi := \lim_{j \rightarrow \infty} \varphi_j$ satisfies $\varphi + f \leq u \leq f$.

For the converse implication, fix $u \in \mathcal{E}_\chi(f)$. Then there exists a function $\varphi \in \mathcal{E}_\chi$ such that $\varphi + f \leq u \leq f$. Let $\varphi_j \in \mathcal{E}_0 \cap C(\Omega)$ be a decreasing sequence with limit function φ . Then for each $j \in \mathbb{N}$, consider the function $u_j := \max(\varphi_j + f, u) \in \mathcal{E}_\chi(f)$. The sequence u_j decreases towards u and

$$\begin{aligned} \int_\Omega -\chi(u_j - f)^n (dd^c u_j)^n \\ &\leq C \int_\Omega -\chi(u - f)^n (dd^c u)^n < \infty, \end{aligned} \quad (48)$$

where C is a constant which depends only on u and the proof of the theorem is completed. $\square$

Theorem 11. Let μ be a nonnegative measure in Ω , let $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ be an increasing convex (or homogeneous) function such that $\chi(-\infty) = -\infty$, and let $f \in \mathcal{E}(\Omega)$ be a maximal function. Then there exists a unique function $\varphi \in \mathcal{E}_\chi(f)$ such that $\mu = (dd^c \varphi)^n$ if and only if μ satisfies one of the conditions of Theorem 6.

Proof. Assume that $\mu = (dd^c v)^n$ for some $v \in \mathcal{E}_\chi$. Let $(\Omega_j)_j$ be a fundamental sequence of strictly pseudoconvex subsets of Ω . Choose a sequence $f_j \in \text{PSH}(\Omega) \cap C(\overline{\Omega})$ decreasing towards f on Ω and f_j is maximal on Ω_{j+1} . It follows from [13] that there exist a function $g \in \mathcal{E}_0$ and a function $\theta \in L^1_{\text{loc}}(dd^c g)^n$ such that

$$\mu = \theta (dd^c g)^n. \quad (49)$$

Consider the measure $\mu_j = \chi_{\Omega_j} \min(\theta, j)(dd^c g)^n$, where χ_{Ω_j} denotes the characteristic function of the set Ω_j . Now, solving the Dirichlet problem in the strictly pseudoconvex domain Ω_j , we state that there exist functions $u_j, v_j \in \text{PSH}(\Omega_j) \cap C(\overline{\Omega_j})$ such that

$$\begin{aligned} (dd^c u_j)^n &= (dd^c v_j)^n = \mu_j, & v_j &= 0, & u_j &= f_j \\ & & & & & \text{on } \partial\Omega_j. \end{aligned} \quad (50)$$

By the comparison principle, we have u_j and v_j are decreasing sequences and

$$v + f \leq v_j + f_j \leq u_j \leq f_j \quad \text{on } \Omega_j. \quad (51)$$

Letting $j \rightarrow +\infty$ we get that $u := \lim_{j \rightarrow \infty} u_j \in \mathcal{E}_\chi(f)$. The continuity of the complex Monge-Ampère operator under monotonic sequences yields that $(dd^c u)^n = \mu$. Uniqueness of u follows from the comparison principle. $\square$

Corollary 12. *Let μ be nonnegative measure in Ω with total finite mass $\mu(\Omega) < \infty$ and let f be a maximal function. Then there exists a uniquely determined function $\varphi \in \mathcal{F}^a(f)$ such that $(dd^c \varphi)^n = \mu$ if and only if μ vanishes on pluripolar subsets.*

Proof. It follows from [13] that there exist a function $\psi \in \mathcal{E}_0$ and a function $\theta \in L^1_{\text{loc}}(dd^c \psi)^n$ such that

$$\mu = \theta (dd^c \psi)^n. \quad (52)$$

By [3], there exists a unique $h_j \in \mathcal{E}_0$ such that $(dd^c h_j)^n = \min(\theta, j)(dd^c \psi)^n$. The comparison principle yields that h_j is a decreasing sequence. Let denote by $h := \lim_{j \rightarrow \infty} h_j$. It follows from Lemma 8 that $h \not\equiv -\infty$. Therefore $h \in \mathcal{F}^a$. By the continuity of the complex Monge-Ampère operator under decreasing sequences, we have $(dd^c h)^n = \mu$. Now, since

$$\mathcal{F}^a = \bigcup_{\substack{\chi \text{ convex}; \chi(0) \neq 0 \\ \chi(-\infty) = -\infty}} \mathcal{E}_\chi, \quad (53)$$

then there exists a convex function $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ with $\chi(0) \neq 0$ and $\chi(-\infty) = -\infty$ such that $h \in \mathcal{E}_\chi$. By Theorem 11, we can find a unique function $\varphi \in \mathcal{E}_\chi(f) \subset \mathcal{F}^a(f)$ such that $(dd^c \varphi)^n = \mu$. $\square$

4. Measures Dominated by Capacity

Throughout this section, μ denotes a fixed nonnegative measure of finite total mass $\mu(\Omega) < +\infty$. We want to solve the Dirichlet problem

$$(dd^c \varphi)^n = \mu, \quad \text{with } \varphi \in \mathcal{F}^a(f), \quad \varphi|_{\partial\Omega} = f, \quad (54)$$

and measure how far the distance between the solution φ and the given boundary data f is from being bounded, by assuming that μ is suitable dominated by the Monge-Ampère capacity.

Measures dominated by the Monge-Ampère capacity have been extensively studied by Kołodziej in [3–5]. The main

result of his study, achieved in [4], can be formulated as follows. Fix $\varepsilon : \mathbb{R} \rightarrow [0, +\infty[$ a continuous decreasing function and set $F_\varepsilon(x) := x[\varepsilon(-\ln x/n)]^n$. If for all compact subsets $K \subset \Omega$

$$\mu(K) \leq F_\varepsilon(\text{cap}_\Omega(K)), \quad \text{where } \int_0^{+\infty} \varepsilon(t) dt < +\infty, \quad (55)$$

and $l : \partial\Omega \rightarrow \mathbb{R}$ is a continuous function, then $\mu = (dd^c \varphi)^n$ for some continuous function $\varphi \in \text{PSH}(\Omega)$ with $\varphi|_{\partial\Omega} = l$.

The condition $\int_0^{+\infty} \varepsilon(t) dt < +\infty$ means that ε decreases fast enough towards zero at infinity. This gives a quantitative estimate on how fast $\varepsilon(-\ln \text{Cap}_\Omega(K)/n)$, hence $\mu(K)$ decreases towards zero as $\text{cap}_\Omega(K) \rightarrow 0$.

When $\int_0^{+\infty} \varepsilon(t) dt = +\infty$, it is still possible to show that $\mu = (dd^c \varphi)^n$ for some function $\varphi \in \mathcal{F}(\Omega)$, but φ will generally be unbounded. We now measure how far it is from being so.

Theorem 13. *Let μ be a nonnegative finite measure. Assume for all compact subsets $K \subset \Omega$*

$$\mu(K) \leq F_\varepsilon(\text{cap}_\Omega(K)). \quad (56)$$

Then there exists a unique function $\varphi \in \mathcal{F}^a(f)$ such that $\mu = (dd^c \varphi)^n$, and

$$\text{cap}_\Omega(\{\varphi < f - s\}) \leq \exp(-nH^{-1}(s)), \quad \forall s > 0. \quad (57)$$

Here H^{-1} is the reciprocal function of $H(x) = e \int_0^x \varepsilon(t) dt + e\varepsilon(0) + \mu(\Omega)^{1/n}$.

The proof is almost the same as that of Theorem 5.1 in [12], except that we use Corollary 12 for the existence of the solution and Lemma 8 to estimate the capacity of sublevel set.

Observe that if $\int_0^\infty \varepsilon(t) dt < \infty$ then H is bounded by $e \int_0^\infty \varepsilon(t) dt + e\varepsilon(0) + \mu(\Omega)^{1/n}$. Hence $H^{-\infty}(t) = +\infty, \forall t \geq e \int_0^\infty \varepsilon(t) dt + e\varepsilon(0) + \mu(\Omega)^{1/n}$. Therefore

$$0 \leq f - \varphi \leq e \int_0^\infty \varepsilon(t) dt + e\varepsilon(0) + \mu(\Omega)^{1/n}. \quad (58)$$

Now, we consider the case when $\mu = fd\lambda$ is absolutely continuous with respect to Lebesgue measure.

Let $\mathbb{G} \subset \mathbb{C}^n$ denote a generic subspace of $\mathbb{C}^n$ that is a real subspace such that $\mathbb{G} + J\mathbb{G} = \mathbb{C}^n$, where J is the usual complex structure on $\mathbb{C}^n$ (cf. [21] for more details). $\mathbb{G}$ will be endowed with the induced Euclidean structure and the corresponding Lebesgue measure which will be denoted by $\lambda_{\mathbb{G}}$.

Let $\alpha > 0$ be a positive real number. According to [22, 23], the Orlicz space $L \log^{n+\alpha} L(d\lambda_{\mathbb{G}})$ consists of $\lambda_{\mathbb{G}}$ -measurable functions g defined on $\Omega \cap \mathbb{G}$ such that

$$\int_{\Omega \cap \Gamma} \frac{|f|}{\lambda} \log^{n+\alpha} \left(1 + \frac{|f|}{\lambda} \right) d\lambda_{\mathbb{G}} < \infty, \quad \text{for some } \lambda > 0. \quad (59)$$

On the space $L \log^{n+\alpha} L(d\mu)$, we define the norm

$$\begin{aligned} \|f\|_{L \log^{n+\alpha} L} &:= \inf \left\{ \lambda > 0; \int_{\Omega} \frac{|f|}{\lambda} \log^{n+\alpha} \left(e + \frac{|f|}{\lambda} \right) d\lambda_{\mathbb{G}} < 1 \right\}. \end{aligned} \quad (60)$$

The dual space to $L\log^{n+\alpha}L$ is the exponential class $\text{Exp}L^{1/n+\alpha}$; that is, the vector space

$$\text{Exp}L^{1/n+\alpha} := \left\{ f : \Omega \longrightarrow \overline{\mathbb{R}}; \exists \lambda > 0 : \int_{\Omega} \exp \left(\left(\frac{|f|}{\lambda} \right)^{1/n+\alpha} \right) - 1 d\lambda_{\mathbb{G}} < \infty \right\} \quad (61)$$

equipped with the norm

$$\|f\|_{\text{Exp}L^{1/n+\alpha}} := \inf \left\{ \lambda > 0; \int_{\Omega} \left(\exp \left(\left(\frac{|f|}{\lambda} \right)^{1/n+\alpha} \right) - 1 \right) d\lambda_{\mathbb{G}} < 1 \right\}. \quad (62)$$

Then we have the following Hölder inequality:

$$\left| \int_{\Omega} f g d\lambda_{\mathbb{G}} \right| \leq C_{n,\alpha} \|f\|_{L\log^{n+\alpha}L} \|g\|_{\text{Exp}L^{1/n+\alpha}}, \quad (63)$$

for $f \in L\log^{n+\alpha}L$ and $g \in \text{Exp}L^{1/n+\alpha}$, where $C_{n,\alpha} > 0$ is a positive constant depending only on n and α . By a simple computation, we have

$$\|1\|_{\text{Exp}L^{1/n+\alpha}(K)} = \frac{1}{\log^{n+\alpha} (1 + 1/\lambda_{\mathbb{G}}(K))}. \quad (64)$$

Corollary 14. Let $\mu = \llcorner_{\Omega \cap \mathbb{G}} g \lambda_{\mathbb{G}}$ be a measure with nonnegative density $g \in L\log^{n+\alpha}L(\Omega \cap \mathbb{G})$. Then there exists a unique bounded function $\varphi \in \mathcal{F}^a(f) \cap L^{\infty}(\Omega)$ such that $(dd^c \varphi)^n = \mu$ and

$$0 \leq f - \varphi \leq C \|g\|_{L\log^{n+\alpha}L}^{1/n}, \quad (65)$$

where $C > 0$ only depends on n, α, Ω , and $\mathbb{G}$.

Proof. We claim that there exists a constant $C > 0$ such that

$$\mu(K) \leq \left(C \|g\|_{L\log^{n+\alpha}L}^{1/n} \right)^n \text{cap}_{\Omega}^{(\alpha+n)/n}(K), \quad (66)$$

$$\forall \text{compact } K \subset \Omega.$$

Indeed, Hölder's inequality and inequality (66) yield

$$\mu(K) \leq \|g\|_{L\log^{n+\alpha}L} \frac{1}{\log^{n+\alpha} (1 + 1/\lambda_{\mathbb{G}}(K))}, \quad (67)$$

$$\forall \text{compact } K \subset \Omega.$$

By [21] we have

$$\lambda_{\mathbb{G}}(K) \leq C \exp \left(- \frac{1}{\text{cap}_{\Omega}^{1/n}(K)} \right), \quad \forall \text{compact } K \subset \Omega, \quad (68)$$

where $C > 0$ is a constant which depends only on Ω and $\mathbb{G}$.

Inequality (66) follows by combining (67) and (68). Then we apply Theorem 13 with

$$\varepsilon(x) = C \|g\|_{L\log^{n+\alpha}L} C \|g\|_{L\log^{n+\alpha}L}^{1/n} e^{-\alpha x/n}, \quad (69)$$

which yields

$$0 \leq f - \varphi \leq e \int_0^x \varepsilon(t) dt + e \varepsilon(0) + \mu(\Omega)^{1/n} \leq C \|g\|_{L\log^{n+\alpha}L}^{1/n}. \quad (70)$$

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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