Research Article
On Stability of Vector Nonlinear Integrodifferential Equations

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Let \( \Omega \) be a bounded domain in a real Euclidean space. We consider the equation

\[
\frac{\partial u(t, x)}{\partial t} = C(x)u(t, x) + \int_{\Omega} K(x, s)u(t, s)ds + [F(u)](t, x) \quad (t > 0; \ x \in \Omega),
\]

where \( C(\cdot) \) and \( K(\cdot, \cdot) \) are matrix-valued functions and \( F(\cdot) \) is a nonlinear mapping. Conditions for the exponential stability of the steady state are established. Our approach is based on a norm estimate for operator commutators.

1. Introduction and Statement of the Main Result

Throughout this paper, \( \mathbb{C}^n \) is the complex \( n \)-dimensional Euclidean space with a scalar product \( \langle \cdot, \cdot \rangle_n \) and norm \( \| \cdot \|_n = \sqrt{\langle \cdot, \cdot \rangle_n} \); \( \mathbb{C}^{n \times n} \) is the set of \( n \times n \)-matrices; \( I \) is the unit operator in corresponding space; \( \Omega \) is a bounded domain with a smooth boundary in a real Euclidean space; \( L^2 (\mathbb{C}^n, \Omega) = L^2 \) is the Hilbert space of functions defined on \( \Omega \) with values in \( \mathbb{C}^n \), the scalar product

\[
\langle v, w \rangle_{L^2} = \int_{\Omega} \langle v(x), w(x) \rangle_n \ dx \quad (v, w \in L^2),
\]

and the norm \( \| \cdot \|_{L^2} = \sqrt{\langle \cdot, \cdot \rangle_{L^2}}. \)

Our main object in this paper is the equation

\[
\frac{\partial u(t, x)}{\partial t} = C(x)u(t, x) + \int_{\Omega} K(x, s)u(t, s)ds + [F(u)](t, x) \quad (t > 0; \ x \in \Omega), \tag{2}
\]

where \( C(\cdot) \) and \( K(\cdot, \cdot) \) are matrix-valued functions defined on \( \Omega \) and \( \Omega \times \Omega \), respectively, with values in \( \mathbb{C}^{n \times n} \), and \( F(\cdot) : L^2 \rightarrow L^2 \) satisfy conditions pointed out below, and \( u(\cdot, \cdot) \) is unknown.

Traditionally, (2) is called the Barbashin type integrodifferential equation or simply the Barbashin equation. It plays an essential role in numerous applications, in particular, in kinetic theory [1], transport theory [2], continuous mechanics [3], control theory [4], radiation theory [5, 6], and the dynamics of populations [7]. Regarding other applications, see [8]. The classical results on the Barbashin equation are represented in the well-known book [9]. The recent results about various aspects of the theory of the Barbashin equation can be found, for instance, in [10–14] and the references given therein. In particular, in [11], the author investigates the solvability conditions for the Cauchy problem for a Barbashin equation in the space of bounded continuous functions and in the space of continuous vector-valued functions with the values in an ideal Banach space. The stability and boundedness of solutions to a linear scalar nonautonomous Barbashin equation have been investigated in [15].

The literature on the asymptotic properties of integrodifferential equations is rather rich (cf. [16–22] and the references given therein), but the stability of nonlinear vector integrodifferential equations is almost not investigated. It is at an early stage of development.

A solution of (2) is a function \( u(t, \cdot) : [0, \infty) \rightarrow L^2 \) having a measurable derivative bounded on each finite interval.

It is assumed that under consideration \( F \) provides the existence and uniqueness of solutions (e.g., it is Lipschitz continuous). The zero solution of (2) is said to be exponentially stable, if there are constants \( m_0 \geq 1, \delta_0 > 0, \) and \( \alpha > 0, \) such that \( \|u(t)\|_{L^2} \leq m_0\|u(0)\|_{L^2}e^{\alpha t} \) \((t \geq 0), \) provided \( \|u(0)\|_{L^2} \leq \delta_0. \) It is globally exponentially stable if \( \delta_0 = \infty. \)

Suppose that, for a positive \( r \leq \infty, \)

\[
\|F(h)\|_{L^2} \leq q \|h\|_{L^2}; \quad (h \in L^2; \ \|h\|_{L^2} \leq r). \tag{3}
\]
For example, for an integer \( p > 1 \), let \([F(h)](x) = (Th(x))^p\). Here,

\[
Th(x) = \int_{\Omega} b(x, s) h(s) ds
\]

with a matrix kernel \( b(x, s) \) satisfying

\[
J_p = \left( \int_{\Omega} \left( \int_{\Omega} \|b(x, s)\|^p ds \right)^{\frac{1}{p}} dx \right)^{\frac{1}{2}} < \infty.
\]

Then, by the Schwarz inequality,

\[
\|Th(x)\|^2 \leq \int_{\Omega} \|b(x, s)\|^2 ds \|h\|^2_{L^2}.
\]

Thus,

\[
\int_{\Omega} \|F(h)(x)\|^2 dx \leq \int_{\Omega} \left( \int_{\Omega} \|b(x, s)\|^2 ds \right)^p dx \|h\|^2_{L^2}.
\]

Hence, for any \( r < \infty \), we have condition (3) with \( q = r^{p-1}J_p \).

The following notations are introduced: for a linear operator \( A \), \( A^* \) is the adjoint operator, \( \|A\| \) is the operator norm, and \( \sigma(A) \) is the spectrum. For \( n \times n \)-matrix \( C \), put

\[
g(C) = \left[ N_2(C) - \sum_{k=1}^{n} |\lambda_k(C)|^2 \right]^{1/2},
\]

where \( \lambda_k(C), k = 1, \ldots, n \), are the eigenvalues of \( C \), counted with their multiplicities; \( N_2(C) = (\text{Trace } C^*)^{1/2} \) is the Frobenius (Hilbert-Schmidt) norm of \( C \). The following relations are checked in [23, Section 2.1]:

\[
g(C) = N_2(C) - |\text{Trace } C^*|,
\]

where 

\[
g \left( e^{\gamma C} + zI \right) = g(C) \quad (\gamma \in \mathbb{R}, \ z \in \mathbb{C}),
\]

\[
g^2(C) \leq \frac{N_2(C - C^*)}{2}.
\]

If \( C \) is a normal matrix, \( CC^* = C^*C \), then \( g(C) = 0 \).

Furthermore, denote

\[
\xi = \frac{1}{2} \sup_{(x, s) \in \Omega} \int_{\Omega} \int_{\Omega} ((K(x, s) + K^*(s, x)) h(s), h(x))_n ds dx,
\]

\[
h(x)_n ds dx,
\]

\[
\gamma = \left( \int_{\Omega} \int_{\Omega} \|C(x) K(x, s) - K(x, s) C(s)\|^2 ds dx \right)^{1/2}
\]

and assume that

\[
\alpha_0 = \xi + \sup_x \Re \sigma(C(x)) = \xi + \sup_k \Re \lambda_k(C(x)) < 0.
\]

In addition, with the notation \( g_0 = \sup_x g(C(x)) \), put

\[
\chi = \sum_{j,k=0}^{n-1} \frac{g^j_k (j+k)!}{|\alpha_0|^{j+k+1} (j+k)^{3/2}},
\]

\[
p(t) = \sum_{k=0}^{\infty} \frac{t^k g_k}{|\alpha_0|^{3/2}},
\]

This integral is simply calculated. If \( C(x) \) is a normal matrix for all \( x \), then

\[
g(C(x)) = 0,
\]

\[
p(t) \equiv 1,
\]

\[
\chi = \frac{1}{|\alpha_0|},
\]

\[
\gamma = 0,
\]

\[
\xi = \frac{1}{2} \frac{1}{|\alpha_0|^2}.
\]

Now, we are in a position to formulate our main result.

**Theorem 1.** Let conditions (3), (11), and

\[
\gamma \xi_0 + \chi \gamma < 1
\]

hold. Then, the zero solution to (2) is exponentially stable. If, in addition, \( r = \infty \) in (3), then the zero solution is globally exponentially stable.

This theorem is proved in the next 3 sections. It gives us "good" results when \( \gamma \) is "small," that is, if matrices \( K(x, s) \) and \( C(x) \) almost commute" and \( \sup_x \|C(x) - C(s)\|_n \) is "small." If (2) is scalar, then \( g_0 = 0 \),

\[
\chi = \frac{1}{|\alpha_0|},
\]

\[
\gamma = 0,
\]

\[
|\alpha_0| = \frac{1}{2}.
\]

So, in the scalar case, condition (14) takes the form

\[
q < |\alpha_0|.
\]

This condition is similar to the stability test derived in [24] for scalar integrodifferential equations.

**2. Auxiliary Results**

Let \( \mathcal{H} \) be a Hilbert space with a scalar product \( (\cdot, \cdot)_{\mathcal{H}} \) and the norm \( \| \cdot \|_{\mathcal{H}} = \sqrt{(\cdot, \cdot)_{\mathcal{H}}} \); \( B(\mathcal{H}) \) denotes the set of bounded linear operators in \( \mathcal{H} \) and \( [A_1, A_2] = A_1 A_2 - A_2 A_1 \) is the commutator of \( A_1, A_2 \in B(\mathcal{H}) \).
Lemma 2. Let $A, B \in B(H)$ and $C = [A, B]$. Then,
\[ [e^{itA}, B] = \int_0^t e^{isA} Ce^{s-A} ds. \] (17)

Proof. Put $J(t) = \int_0^t e^{isA} Ce^{s-A} ds$. Then, $(d/dt)(J(t)e^{-tA}) = e^{tA} Ce^{-tA}$. On the other hand,
\[ \frac{d}{dt} \left( [e^{itA}, B] \right) = \frac{d}{dt} \left( e^{itA} Be^{-itA} - B \right) = e^{itA} Ce^{-itA}. \] (18)

So, $[e^{itA}, B] = J$, as claimed.

Lemma 3. Under condition (19), one has
\[ \lambda (A) := \sup \Re \sigma (A) < 0. \] (19)

Then, the Lyapunov equation
\[ WA + (WA)^* = -2I \] (20)
has a unique solution $W \in \mathcal{B}(\mathcal{H})$ and it can be represented as
\[ W = 2 \int_0^\infty e^{st} e^{-st} dt \] (21)
(cf. [25]). Denote $\Lambda_B = (1/2) \sup \sigma (B + B^*)$,
\[ \zeta (A) = 2 \int_0^\infty \| e^{st} \| \int_0^t \| e^{s-t} \| ds dt, \]
\[ \psi (W, B) = \begin{cases} \Lambda_B \| W \| & \text{if } \Lambda_B > 0, \\ \Lambda_B \lambda_W & \text{if } \Lambda_B \leq 0, \end{cases} \] (22)
where $\lambda_W = \inf \sigma (W)$.

Lemma 4. Let conditions (19) and (29) with $r = \infty$ hold. Then, any solution of (28) satisfies the inequality
\[ \| u(t) \| \leq \left( \frac{\| W \|}{\lambda_W} \right)^{1/2} \| u(0) \| e^{-\nu t}, \quad t \geq 0, \] (30)
where $\nu = 1 - \psi (W, B) - \zeta (A) \| C \| - \| q \| \| W \|.$

Proof. For brevity, we write $[F u(t)] = F u(t)$. Multiplying (28) by $W$ and doing the scalar product, we get
\[ \left( W u'(t), u(t) \right) = \left( W (A + B) u(t), u(t) \right) \]
\[ + (WF u(t), u(t)). \] (31)

Since $(d/dt)(W u(t), u(t)) = (W u'(t), u(t)) + (u(t), W u'(t))$, due to (20) and Lemma 3, it can be written that
\[ \frac{d}{dt} (W u(t), u(t)) = 2 \Re (W (A + B) u(t), u(t)) \]
\[ + 2 \Re (WF u(t), u(t)) \]
\[ \leq 2 (1 + \psi(W,B) + \zeta(A)\| C \|) (u(t), u(t)) + 2 \Re (WF u(t), u(t)). \] (32)

Taking into account the fact that due to (29)
\[ (WF u(t), u(t)) \leq \| W \| \| F u(t) \| \| u(t) \| \]
\[ \leq \| W \| \| q \| \| u(t) \|^2, \] (33)
we get
\[ \frac{d}{dt} (Wu(t), u(t)) \leq -2 \nu (u(t), u(t)). \quad (34) \]
From this inequality, we have \((Wu(t), u(t)) \leq (Wu(0), u(0)) e^{-2\nu t} \). Hence,
\[ \lambda_W (u(t), u(t)) \leq \|W\| (u(0), u(0)) e^{-2\nu t}, \quad (35) \]
as claimed.

**Lemma 5.** Let conditions (29) and \( \nu < 0 \) hold. Then, the zero solution to (28) is exponentially stable. If \( r = \infty \) in (29), then the zero solution to (28) is globally exponentially stable.

**Proof.** If \( r = \infty \), then the required result is due to the previous lemma. If \( r < \infty \), then, taking \( \|u(0)\| < r(\lambda_W/\|W\|)^{1/2} \) due to the previous lemma, \( \|u(t)\| < r \) as claimed.

**4. Proof of Theorem 1**

Take
\[ A h(x) = (C(x) + \xi I) h(x), \]
\[ B h(x) = \int_{\Omega} K(x,s) h(s) ds - \xi h(x) \]
(\( h \in L^2 \)).

Then, \( A(B) \leq 0 \) and
\[ [A, B] h(x) = \int_{\Omega} [C(x) K(x,s) - K(x,s) C(s)] h(s) ds. \]
So \( \|[A, B]\| \leq \nu \). Due to [23, Example 1.7.3],
\[ \|e^{A(t)}\|_2 \leq e^{\alpha(C(x))} \sum_{k=0}^{n-1} \frac{\nu^k}{(k!)^{1/2}} (t \geq 0), \quad (38) \]
where \( \alpha(C(x)) = \operatorname{Re} \sigma(C(x)) \). Hence,
\[ \|e^{A(t)}\|_2 \leq \sup_x \|e^{A(C(x)) + \xi I}\|_2 \leq e^{\alpha(C)} p(t) \quad (t \geq 0), \quad (39) \]
since \( g(C(x) + \xi I) = g(C(x)) \). Consequently, \( \zeta(A) \leq \zeta_0 \). In addition,
\[ \|W\|_2 \leq 2 \int_0^\infty \|e^{A(t)}\|_2^2 dt \leq 2 \int_0^\infty e^{2\nu t} p^2(t) dt = \chi. \quad (40) \]
Now, the required result is due to Lemma 5.

**Competing Interests**

The author declares that there are no competing interests regarding the publication of this paper.

**References**


