Research Article

Variational Problem Involving Operator Curl in a Multiconnected Domain

Junichi Aramaki

Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama 350-0394, Japan

Correspondence should be addressed to Junichi Aramaki; aramaki@mail.dendai.ac.jp

Received 26 August 2016; Accepted 12 October 2016

Academic Editor: Zui-Cha Deng

Copyright © 2016 Junichi Aramaki. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We shall study the problem of minimizing a functional involving the curl of vector fields in a three-dimensional, bounded multiconnected domain with prescribed tangential component on the boundary. The paper is an extension of $L^2$ minimization problem of the curl of vector fields. We shall prove the existence and the estimate of minimizers of more general functional which contains $L^p$ norm of the curl of vector fields.

1. Introduction

In this paper, we consider the following problem which was proposed by Pan [1, p. 9].

Problem A. Minimize the $L^p$ norm of the curl of vector fields in a given space with tangential trace on the boundary being prescribed.

The problem is related to the mathematical theory of liquid crystal, of superconductivity, and of electromagnetic field. When $p = 2$ and $\Omega$ is a simply connected domain without holes, Bates and Pan [2, 3] showed the existence of minimizer. For the multiconnected domain, the author of [1] obtained the existence of a minimizer of the Problem A in the case $p = 2$.

In the present paper we shall extend the results to more general functional containing Problem A.

More precisely, let $S(x, t)$ be a Carathéodory function on $\Omega \times [0, \infty)$ and $S(x, t) \in C^1((0, \infty))$, and there exist $1 < p < \infty$ and $\lambda, \Lambda > 0$ such that for a.e. $x \in \Omega$ and all $t > 0$:

$$\lambda t^{(p-2)/2} \leq S_t \leq \Lambda t^{(p-2)/2}. \quad (1)$$

Without loss of generality, we may assume that $S(x, 0) = 0$. We furthermore assume the following structure condition:

$$\left( S_t \left( x, |a|^2 \right) a - S_t \left( x, |b|^2 \right) b \right) \cdot (a - b) > 0$$

for any $a, b \in \mathbb{R}^3$ with $a \neq b$. \quad (2)

Under (1) with $S(x, 0) = 0$, we have

$$\frac{2}{p} \lambda t^{p/2} \leq S(x, t) \leq \frac{2}{p} \Lambda t^{p/2}. \quad (3)$$

For example, the function $S(x, t) = \nu(x) t^{p/2}$ where $\nu(x)$ is a measurable function satisfying $0 < \nu_* = \nu(x) \leq \nu^* < \infty$ for a.e. $x \in \Omega$ satisfies (1)-(2).

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^2$ boundary $\partial \Omega$. Let $\mathcal{H}_T$ be a given tangential vector field on $\partial \Omega$. Let $W^{1,p}((\Omega, \mathbb{R}^3))$ be the standard Sobolev space of vector fields. From now, we denote the tangential component of a vector field $u$ by $u_T$: that is, $u_T = u \cdot \nu$, where $\nu$ is the outer normal unit vector to the boundary $\partial \Omega$. For any given tangential vector field on $\partial \Omega$

$$\mathcal{H}_T \in W^{1-1/p,p}\left( \partial \Omega, \mathbb{R}^3 \right), \quad (4)$$
define a space of vector fields
\[ W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T}) = \{ u \in W_{t_1}^1(\Omega, \mathbb{R}^3); u_t = \mathcal{T} \text{ on } \partial \Omega \}. \]  
(5)

Then it is clear that \( W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T}) \) is a closed convex set in \( W_{t_1}^1(\Omega, \mathbb{R}^3) \). We consider the minimization problem
\[ R_T^p(\mathcal{T}) = \inf_{u \in W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T})} \int_{\Omega} S(x, |\text{curl } u|^2) \, dx. \]  
(6)

When \( p = 2 \), \( S(x, t) = t \), and \( \Omega \) is a simply connected domain without holes, the authors of [2, 3] showed that (6) is achieved, and then in the case where \( p = 2, S(x, t) = T \), and \( \Omega \) is bounded multiconnected domain, the author of [1] succeeded to show the existence of a minimizer of (6).

Since we allow \( \Omega \) to be a multiconnected domain in \( \mathbb{R}^3 \), throughout this paper, we assume that the domain \( \Omega \) satisfies the following (O1) and (O2) (cf. Dautray and Lions [4] and Amrouche and Seloula [5]).

(O1) \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with \( C^2 \) boundary \( \partial \Omega \). \( \Omega \) is locally situated on one side of \( \partial \Omega; \partial \Omega \) has a finite number of connected components \( \Gamma_1, \ldots, \Gamma_m (m \geq 0) \) and \( \Gamma_{m+1} \) denoting the boundary of the infinite connected component of \( \mathbb{R}^3 \setminus \Omega \).

(O2) There exist \( n \) manifolds of dimension 2 and of class \( C^2 \) denoted by \( \Sigma_1, \ldots, \Sigma_n \) \((n \geq 0)\) such that \( \Sigma_i \cap \Sigma_j = \emptyset \) \((i \neq j)\) and they are non-tangential to \( \partial \Omega \) and such that \( \Omega \setminus (\bigcup_{i=1}^n \Sigma_i) \) is simply connected and pseudo \( C^{1,1} \).

The number \( n \) is called the first Betti number and \( m \) the second Betti number of \( \Omega \). We say that \( \Omega \) is simply connected if \( n = 0 \), and \( \Omega \) has no holes if \( m = 0 \). If we define the spaces
\[ K^p_n(\Omega) = \{ u \in W_{t_1}^1(\Omega, \mathbb{R}^3); \text{curl } u = 0, \text{div } u \text{ is } \partial \Omega \}, \]
\[ K^p_T(\Omega) = \{ u \in W_{t_1}^1(\Omega, \mathbb{R}^3); \text{curl } u = 0, \text{div } u = 0 \text{ on } \partial \Omega \}, \]  
(7)

then it is well known that \( \dim K^p_n(\Omega) = n \) and \( \dim K^p_T(\Omega) = m \). We note that \( K^p_n(\Omega) \) and \( K^p_T(\Omega) \) are contained in \( W_{t_1}^1(\Omega, \mathbb{R}^3) \); moreover, \( K^p_n(\Omega) \) and \( K^p_T(\Omega) \) are closed subspaces of \( W_{t_1}^1(\Omega, \mathbb{R}^3) \). Also it will be shown in Lemma 4 that \( K^p_n(\Omega) \) and \( K^p_T(\Omega) \) are closed subspaces of \( L^p(\Omega, \mathbb{R}^3) \). Thus since \( K^p_T(\Omega) \) is a finite-dimensional closed subspace of \( L^p(\Omega, \mathbb{R}^3) \), \( K^p_T(\Omega) \) has a complement \( L_{p_0}^p \) in \( L^p(\Omega, \mathbb{R}^3) \); that is, \( L^p \) is a closed subspace of \( L^p(\Omega, \mathbb{R}^3) \), \( L^p \cap K^p_T(\Omega) = \{ 0 \} \), and \( L^p(\Omega, \mathbb{R}^3) = L^p \oplus K^p_T(\Omega) \) (the direct sum). Therefore, for any \( w \in L^p(\Omega, \mathbb{R}^3) \), there exist uniquely \( v \in L^p \) and \( u \in K^p_T(\Omega) \) such that \( w = v + u \). We denote the projection \( P : L^p(\Omega, \mathbb{R}^3) \to L^p \) by \( Pw = v \).

Define
\[ H^p(\Omega, \text{curl, div} 0) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{curl } u \in L^p(\Omega, \mathbb{R}^3), \text{div } u = 0 \text{ in } \Omega \}, \]
\[ H_T^p(\Omega, \text{curl, div} 0, \mathcal{T}) = \{ u \in H^p(\Omega, \text{curl, div} 0); u_T = \mathcal{T} \text{ on } \partial \Omega \}. \]  
(8)

Note that if \( u \in L^p(\Omega, \mathbb{R}^3) \) and \( \text{curl } u \in L^p(\Omega, \mathbb{R}^3) \), then the tangent trace \( u_T \) is well defined as an element of \( W_{t_1}^1(\partial \Omega, \mathbb{R}^3) \) (cf. [5, p. 45]), and
\[ H_T^p(\Omega, \text{curl, div} 0, \mathcal{T}) = \{ u \in H^p(\Omega, \text{curl, div} 0); u_T = \mathcal{T} \text{ on } \partial \Omega \}. \]  
(9)

Moreover, we note that if \( \mathcal{T} \in W_{t_1}^1(\partial \Omega, \mathbb{R}^3) \), then
\[ H_T^p(\Omega, \text{curl, div} 0, \mathcal{T}) \subset W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T}) \].  
(10)
(cf. Amrouche and Seloula [6, Theorem 2.3]). We will see, in Lemma 2 of Section 2, that
\[ R_T^p(\mathcal{T}) = \inf_{u \in H_T^p(\Omega, \text{curl, div} 0, \mathcal{T})} \int_{\Omega} S(x, |\text{curl } u|^2) \, dx. \]  
(11)

We are in a position to state the main theorem.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain satisfying (O1) and (O2), and let \( \mathcal{T} \in W_{t_1}^1(\partial \Omega, \mathbb{R}^3) \) be a tangential vector field on \( \partial \Omega \). Then \( R_T^p(\mathcal{T}) \) is achieved, and the minimizer \( A \) of \( R_T^p(\mathcal{T}) \) in \( H_T^p(\Omega, \text{curl, div} 0, \mathcal{T}) \) satisfies the following estimate. There exists a constant \( C = C(\Omega) > 0 \) independent of \( \mathcal{T} \) such that
\[ \| P A \|_{W_{t_1}^1(\Omega)} \leq C \| \mathcal{T} \|_{W_{t_1}^1(\partial \Omega)}. \]  
(12)

### 2. Preliminaries

In this section, we shall give some lemmas as preliminaries.

**Lemma 2.** Let \( \mathcal{T} \in W_{t_1}^1(\partial \Omega, \mathbb{R}^3) \) be a tangential vector field on \( \partial \Omega \). Then one has
\[ R_T^p(\mathcal{T}) = \inf_{u \in H_T^p(\Omega, \text{curl, div} 0, \mathcal{T})} \int_{\Omega} S(x, |\text{curl } u|^2) \, dx. \]  
(13)

**Proof.** Put
\[ \alpha = \inf_{u \in W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T})} \int_{\Omega} S(x, |\text{curl } u|^2) \, dx, \]
\[ \beta = \inf_{u \in H_T^p(\Omega, \text{curl, div} 0, \mathcal{T})} \int_{\Omega} S(x, |\text{curl } u|^2) \, dx. \]  
(14)

Since \( H_T^p(\Omega, \text{curl, div} 0, \mathcal{T}) \subset W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T}) \), it is trivial that \( \alpha \leq \beta \). For any \( u \in W_{t_1}^1(\Omega, \mathbb{R}^3, \mathcal{T}) \), the problem
\[ \Delta \varphi = \text{div } u \text{ in } \Omega, \]
\[ \varphi = 0 \text{ on } \partial \Omega \]  
(15)
3. Proof of the Main Theorem 1

In this section, we give a proof of Theorem 1. The proof consists of some lemmas and propositions. Throughout this section, we assume that $\mathcal{H}_T$ is a given tangential vector field on $\partial\Omega$.

**Lemma 5.** Let $A \in \mathcal{H}^0_T(\Omega, \text{curl}, \text{div}0, \mathcal{H}_T)$. Then the minimization problem

$$y = \inf_{u \in \mathcal{K}_T(\Omega)} \|A - u\|_{L^p(\Omega)}$$

(21)

has a unique minimizer.

**Proof.** From Lemma 4, we know that $\mathcal{K}_T(\Omega)$ is a closed subspace of $L^p(\Omega, \mathbb{R}^3)$. Thus it is well known that (21) has a minimizer. For the uniqueness of the minimizer, it suffices to show that the unit sphere $B = \{u \in L^p(\Omega, \mathbb{R}^3); \|u\|_{L^p(\Omega)} = 1\}$ does not contain any line segment $\{u + tv; 0 \leq \lambda \leq 1\}$ for $u, v \in B$ and $u \neq v$. (cf. Fujita et al. [9, p. 306 and the remark]). However, this is clear because the functional

$$f(u) = \int_\Omega |u|^p \, dx$$

(22)

is strictly convex.

For $A \in \mathcal{H}^0_T(\Omega, \text{curl}, \text{div}0, \mathcal{H}_T)$, let $u \in \mathcal{K}_T^p(\Omega)$ be a unique minimizer of (21) and define $B = A - u$. Then since for any $z \in \mathcal{K}_T^p(\Omega)$ and $t \in \mathbb{R}$, $\|B + tz\|_{L^p(\Omega)}^p \leq \|B\|_{L^p(\Omega)}^p + 2t \cdot z \cdot \mathcal{F}B$.

If we define a space

$$B(\Omega, \mathcal{H}_T) = \left\{ B \in L^p(\Omega, \mathbb{R}^3) : \text{curl } B \in L^p(\Omega, \mathbb{R}^3), \text{div } B = 0 \text{ in } \Omega, B_T = \mathcal{H}_T \text{ on } \partial\Omega \right\},$$

(24)

then we see that $B \in B(\Omega, \mathcal{H}_T)$. Then we have the following.

**Lemma 6.** One can see that

$$\mathcal{H}^p_T(\Omega, \text{curl}, \text{div}0, \mathcal{H}_T) = B(\Omega, \mathcal{H}_T) \oplus \mathcal{K}_T^p(\Omega)$$

(25)

(the direct sum).

**Proof.** For any $A \in \mathcal{H}^p_T(\Omega, \text{curl}, \text{div}0, \mathcal{H}_T)$, as the above we can write

$$A = B + u, \quad \text{where } B \in B(\Omega, \mathcal{H}_T), \ u \in \mathcal{K}_T^p(\Omega).$$

(26)

We show the uniqueness of the above decomposition. If we can write

$$A = B_1 + u_1 = B_2 + u_2,$$

(27)
where \( B_1, B_2 \in B(\Omega, H_{\mathcal{T}}) \), \( u_1 \) and \( u_2 \in K_{p}(\Omega) \), then \( B_1 - B_2 = u_2 - u_1 \in K_{p}(\Omega) \). Therefore we have
\[
\int_{\Omega} |B_1|^{p-2} B_1 \cdot (B_1 - B_2) \, dx = 0,
\]
\[
\int_{\Omega} |B_2|^{p-2} B_2 \cdot (B_1 - B_2) \, dx = 0.
\]
(28)

Hence
\[
\int_{\Omega} (|B_1|^{p-2} B_1 - |B_2|^{p-2} B_2) \cdot (B_1 - B_2) \, dx = 0.
\]
(29)

Here we use the following inequality. There exists a constant \( c > 0 \) such that
\[
(|a|^{p-2} a - |b|^{p-2} b) \cdot (a - b) = \begin{cases} c |a - b|^p & \text{if } p \geq 2, \\ c (|a| + |b|)^{p-2} |a - b|^2 & \text{if } 1 < p < 2 \end{cases}
\]
(30)

for all \( a, b \in \mathbb{R}^3 \). For the proof of this inequality, see DiBenedetto [10, Lemma 4.4] for \( p \geq 2 \), and see Miranda et al. [11, (7C)]. Applying (30) with \( a = B_1, b = B_2 \) to (29), we have
\[
\int_{\Omega} |B_1 - B_2|^p \, dx = 0 \quad \text{for } p \geq 2,
\]
\[
\int_{\Omega} (|B_1| + |B_2|)^{p-2} |B_1 - B_2|^2 \, dx = 0 \quad \text{for } 1 < p < 2.
\]
(31)

From these equalities, we have \( B_1 = B_2 \), so \( u_1 = u_2 \). \( \square \)

Now we state a refinement of Fatou’s lemma (cf. Evans [12, pp. 11-12]).

**Lemma 7.** Assume that \( 1 < p < \infty \). Let \( B_j \to B \) weakly in \( L^p(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \). Then one has
\[
\lim_{j \to \infty} \int_{\Omega} \left( |B_j|^p - |B_j|^{p-2} B_j - |B|^{p-2} B \right) \, dx = \int_{\Omega} |B|^p \, dx.
\]
(32)

If furthermore
\[
\lim_{j \to \infty} \int_{\Omega} |B_j|^p \, dx = \int_{\Omega} |B|^p \, dx,
\]
(33)

then
\[
|B_j|^{p-2} B_j \to |B|^{p-2} B \quad \text{strongly in } L^{p'}(\Omega, \mathbb{R}^3),
\]
(34)

where \( p' \) denotes the conjugate exponent of \( p \); that is, \( (1/p) + (1/\mathcal{C}) = 1 \). In particular, if \( B_j \to B \) strongly in \( L^p(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \), then (34) holds.

**Proof.** We use an elementary estimate. Let \( 1 \leq q < \infty \). Then, for any fixed \( \epsilon > 0 \), there exists a constant \( C = C(\epsilon, q) > 0 \) such that
\[
|a + b|^q - |a|^q \leq \epsilon |a|^q + C |b|^q
\]
(35)

for any \( a, b \in \mathbb{R}^3 \) (cf. [12, (1.13)]). Define
\[
g_j^\epsilon = \left\{ \begin{array}{ll}
|B_j|^{p-2} B_j & - |B|^{p-2} B_j - |B|^{p-2} B \vspace{1ex} \\
|B_j|^{p-2} B_j & - |B|^{p-2} B_j - |B|^{p-2} B
\end{array} \right. + \epsilon |B_j|^{p-2} B_j - |B|^{p-2} B
\]
(36)

where \( [a]^+ = \max\{a, 0\} \) for \( a \in \mathbb{R} \). Then we have
\[
g_j^\epsilon \leq \left| |B_j|^{p-2} B_j - |B|^{p-2} B_j - |B|^{p-2} B \right| + \epsilon \left| |B_j|^{p-2} B_j - |B|^{p-2} B_j - |B|^{p-2} B \right| + \epsilon |B_j|^{p-2} B_j - |B|^{p-2} B
\]
(37)

If we apply (35) with \( a = |B_j|^{p-2} B_j - |B|^{p-2} B, b = |B|^{p-2} B \) and \( q = p' \), we have
\[
g_j^\epsilon \leq (C + 1) \left| |B|^{p-2} B \right|^{p'} = (C + 1) |B|^{p'}.
\]
(38)

We note that the right-hand side is integrable. By the hypothesis, we can see that \( g_j^\epsilon \to 0 \) a.e. in \( \Omega \). Therefore by the Lebesgue dominated theorem, we have
\[
\lim_{j \to \infty} \int_{\Omega} g_j^\epsilon \, dx = 0.
\]
(39)

Therefore we have
\[
\limsup_{j \to \infty} \int_{\Omega} \left( |B_j|^{p-2} B_j - |B|^{p-2} B_j - |B|^{p-2} B \right) \, dx \leq \epsilon \limsup_{j \to \infty} \int_{\Omega} \left| |B_j|^{p-2} B_j - |B|^{p-2} B \right| \, dx
\]
(40)

\[
+ \epsilon |B_j|^{p-2} B \right) \, dx = \epsilon |B_j|^{p-2} B \right) \, dx = \epsilon |B_j|^{p-2} B \right) \, dx.
\]
Since \( B_j \to B \) weakly in \( L^p(\Omega, \mathbb{R}^3) \), \( \|B_j\|_{L^p(\Omega)} \) is bounded. Since \( \varepsilon \) is arbitrary, we have
\[
\lim_{j \to \infty} \int_\Omega \left( |B_j|^p - |B_j|^{p-2} B_j - |B|^{p-2} B \right)^{p'} \, dx = \int_\Omega |B|^p \, dx. \tag{41}
\]
If furthermore
\[
\lim_{j \to \infty} \int_\Omega |B_j|^{p} \, dx = \int_\Omega |B|^p \, dx, \tag{42}
\]
then we have
\[
\lim_{j \to \infty} \int_\Omega \left( |B_j|^{p-2} B_j - |B|^{p-2} B \right)^{p'} \, dx = 0. \tag{43}
\]

**Lemma 8.** \( B(\Omega, \mathcal{H}_T) \) is a weakly closed set in \( \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \).

**Proof.** Let \( B_j \in B(\Omega, \mathcal{H}_T) \), \( B_j \to B \) weakly in \( \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \). Then we have \( \text{curl } B_j \in L^p(\Omega, \mathbb{R}^3) \), \( \text{div } B = 0 \) in \( \Omega \), \( B_j \to B \) in \( \mathcal{H}_T \) on \( \partial \Omega \), and
\[
\int_\Omega |B_j|^{p-2} B_j \cdot z \, dx = 0 \quad \forall z \in \mathbb{R}^3. \tag{44}
\]
Passing to a subsequence, we may assume that \( B_j \to B \) strongly in \( L^p(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \). Thus from Lemma 7, we have \( |B_j|^{p-2} B_j \to |B|^{p-2} B \) in \( L^p(\Omega, \mathbb{R}^3) \). Therefore we have
\[
\int_\Omega |B|^{p-2} B \cdot z \, dx = 0 \quad \forall z \in \mathbb{R}^3. \tag{45}
\]
This implies that \( B \in B(\Omega, \mathcal{H}_T) \). \( \square \)

**Lemma 9.** There exists a constant \( c(\Omega) > 0 \) such that for all \( B \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \) satisfying \( \text{div } B = 0 \) in \( \Omega \) and
\[
\int_\Omega |B|^{p-2} B \cdot z \, dx = 0 \quad \forall z \in \mathbb{R}^3, \tag{46}
\]
one has
\[
\|B\|_{W^{1,p}(\Omega)} \leq c(\Omega) \left( \|\text{curl } B\|_{L^p(\Omega)} + \|B_T\|_{W^{1,p}(\partial \Omega)} \right). \tag{47}
\]

**Proof.** If the conclusion (47) is false, there exists a sequence \( \{B_j\} \subset \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \) satisfying \( \text{div } B_j = 0 \) in \( \Omega \) and
\[
\int_\Omega |B_j|^{p-2} B_j \cdot z \, dx = 0 \quad \forall z \in \mathbb{R}^3, \tag{48}
\]
such that \( \|B_j\|_{W^{1,p}(\Omega)} \to 1 \), \( \|\text{curl } B_j\|_{L^p(\Omega)} \to 0 \), \( \|B_j\|_{W^{1,p}(\partial \Omega)} \to 0 \) as \( j \to \infty \). After passing to a subsequence, we may assume that \( B_j \to B_0 \) weakly in \( \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \), strongly in \( L^p(\Omega, \mathbb{R}^3) \), and a.e. in \( \Omega \). Therefore we have \( \text{div } B_0 = 0 \), \( \text{curl } B_0 = 0 \) in \( \Omega \) and \( B_{0,T} = 0 \) on \( \partial \Omega \), so \( B_0 \in \mathcal{C}_T^p(\Omega) \). From Lemma 7,
\[
\int_\Omega |B_0|^p \, dx = \int_\Omega |B_0|^{p-2} B_0 \cdot B_0 \, dx = \lim_{j \to \infty} \int_\Omega |B_j|^{p-2} B_j \cdot B_0 \, dx = 0. \tag{49}
\]
Thus we have \( B_0 = 0 \). Hence \( B_j \to 0 \) strongly in \( L^p(\Omega, \mathbb{R}^3) \).
From (19), we see that
\[
\|B_j\|_{W^{1,p}(\Omega)} \leq c_2(\Omega)
\]
\[
\cdot \left( \|B_j\|_{L^p(\Omega)} + \|\text{curl } B_j\|_{L^p(\Omega)} + \|B_j\|_{W^{1,p}(\partial \Omega)} \right) \to 0 \tag{50}
\]
as \( j \to \infty \). This contradicts \( \|B_j\|_{W^{1,p}(\Omega)} = 1 \). \( \square \)

**Proposition 10.** Let \( \mathcal{H}_T \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3) \). Then the minimization problem
\[
\inf_{B \in \mathcal{B}(\Omega, \mathcal{H}_T)} \int_\Omega S(x, |\text{curl } B|^2) \, dx \tag{51}
\]
is achieved and
\[
R^p_T(\mathcal{H}_T) = \inf_{B \in \mathcal{B}(\Omega, \mathcal{H}_T)} \int_\Omega S(x, |\text{curl } B|^2) \, dx \tag{52}
\]

**Proof.** By Lemma 2, we can see that
\[
R^p_T(\mathcal{H}_T) = \inf_{A \in H^1_T(\Omega, \text{curl } 0, \mathcal{H}_T)} \int_\Omega S(x, |A|^2) \, dx. \tag{53}
\]
Since \( B(\Omega, \mathcal{H}_T) \subset H^1_T(\Omega, \text{curl } 0, \mathcal{H}_T) \), it is clear that
\[
R^p_T(\mathcal{H}_T) \leq \inf_{B \in \mathcal{B}(\Omega, \mathcal{H}_T)} \int_\Omega S(x, |\text{curl } B|^2) \, dx. \tag{54}
\]
On the other hand, for any \( A \in H^1_T(\Omega, \text{curl } 0, \mathcal{H}_T) \), we can write \( A = B + u \), where \( B \in B(\Omega, \mathcal{H}_T) \), and \( u \in \mathcal{C}_T^p(\Omega) \). Hence we have
\[
\int_\Omega S(x, |\text{curl } A|^2) \, dx = \int_\Omega S(x, |\text{curl } B|^2) \, dx \geq \inf_{B \in \mathcal{B}(\Omega, \mathcal{H}_T)} \int_\Omega S(x, |\text{curl } B|^2) \, dx. \tag{55}
\]
Thus (52) holds. We show that the right-hand side of (52) has a minimizer. Let \( \{B_j\} \subset B(\Omega, \mathcal{H}_T) \) be a minimizing sequence. Then
\[
\int_\Omega S(x, |\text{curl } B_j|^2) \, dx = R^p_T(\mathcal{H}_T) + o(1) \tag{56}
\]
as \( j \to \infty \).
By (1), we have
\[
\frac{2}{p} \int_\Omega |\text{curl } B_j|^p \, dx \leq \int_\Omega S(x, |\text{curl } B_j|^2) \, dx = R^p_T(\mathcal{H}_T) + o(1). \tag{57}
\]
Thus, by Lemma 9, \( \{B_j\} \) is bounded in \( W^{1,p}(\Omega, \mathbb{R}^3) \). Passing to a subsequence, we may assume that \( B_j \to B_0 \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \), strongly in \( L^p(\Omega, \mathbb{R}^3) \), and a.e. in \( \Omega \). Therefore we have \( \text{div } B_0 = 0, \ B_0 = \mathcal{H}_\tau \) on \( \partial \Omega \). Since

\[
\int_\Omega |B_j|^{p-2} B_j \cdot z \, dx = 0 \quad \forall z \in \mathcal{H}_\tau^p(\Omega), \tag{58}
\]

it follows from Lemma 7 that

\[
\int_\Omega |B_0|^{p-2} B_0 \cdot z \, dx = 0 \quad \forall z \in \mathcal{H}_\tau^p(\Omega). \tag{59}
\]

Therefore \( B_0 \in B(\Omega, \mathcal{H}_\tau) \). It suffices to prove that

\[
\int_\Omega S\left(x, |\text{curl } B_0|^2\right) \, dx
\leq \liminf_{j \to \infty} \int_\Omega S\left(x, |\text{curl } B_j|^2\right) \, dx.
\]

In fact, we can choose a subsequence \( \{\text{curl } B_{j_k}\} \) of \( \{\text{curl } B_j\} \) so that

\[
\lim_{k \to \infty} \int_\Omega S\left(x, |\text{curl } B_{j_k}|^2\right) \, dx = \liminf_{j \to \infty} \int_\Omega S\left(x, |\text{curl } B_j|^2\right) \, dx.
\]

Since \( \text{curl } B_{j_k} \to \text{curl } B_0 \) weakly in \( L^p(\Omega, \mathbb{R}^3) \), it follows from the Mazur theorem that there exist \( g_k \in L^p(\Omega, \mathbb{R}^3) \) such that \( g_k \) is convex hull of \( \{\text{curl } B_{j_k}; k \geq l\} \) and \( g_k \to \text{curl } B_0 \) strongly in \( L^p(\Omega, \mathbb{R}^3) \). Hence we can choose a subsequence \( \{g_{m_l}\} \) of \( \{g_k\} \) so that \( g_{m_l} \to \text{curl } B_0 \) strongly in \( L^p(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \). By the Fatou lemma, we have

\[
\int_\Omega S\left(x, |\text{curl } B_0|^2\right) \, dx \leq \liminf_{m \to \infty} \int_\Omega S\left(x, |g_{m_l}|^2\right) \, dx.
\]

Since \( S(x, t^2) \) is a convex function with respect to \( t \), we have

\[
\int_\Omega S\left(x, |g_{m_l}|^2\right) \, dx
\leq \sup \left\{ \int_\Omega S\left(x, |\text{curl } B_{j_k}|^2\right) \, dx; k \geq l_m \right\}.
\]

Therefore we have

\[
\int_\Omega S\left(x, |\text{curl } B_0|^2\right) \, dx \leq \liminf_{m \to \infty} \int_\Omega S\left(x, |g_{m_l}|^2\right) \, dx
\leq \lim_{m \to \infty} \sup \left\{ \int_\Omega S\left(x, |\text{curl } B_{j_k}|^2\right) \, dx; k \geq l_m \right\}
\leq \lim_{k \to \infty} \int_\Omega S\left(x, |\text{curl } B_{j_k}|^2\right) \, dx
\leq \liminf_{j \to \infty} \int_\Omega S\left(x, |\text{curl } B_j|^2\right) \, dx.
\]

This completes the proof.

**Lemma 11.** Let \( A \in H^1_0(\Omega, \text{curl}, \text{div } 0, \mathcal{H}_\tau) \) be a minimizer of \( R^1(\mathcal{H}_\tau) \). Then \( A \) is a weak solution of the following system:

\[
\text{curl } \left[ S_t \left(x, |\text{curl } A|^2\right) \text{curl } A \right] = 0, \quad \text{div } A = 0 \text{ in } \Omega,
\]

\[
A = \mathcal{H}_\tau \text{ on } \partial \Omega. \tag{65}
\]

**Proof.** If \( A \in H^1_0(\Omega, \text{curl}, \text{div } 0, \mathcal{H}_\tau) \) is a minimizer of \( R^1(\mathcal{H}_\tau) \), then we can see that, for any \( w \in H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \), we have

\[
\frac{d}{dt} \int_\Omega \left\{ S \left(x, |\text{curl } A + t \text{ curl } w|^2\right) \right\} \, dx = 0. \tag{66}
\]

Thus we have

\[
\int_\Omega S_t \left(x, |\text{curl } A|^2\right) \text{curl } A \cdot \text{curl } w \, dx = 0 \tag{67}
\]

for all \( w \in H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \). We claim that

\[
\text{curl } \left[ H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \right] = \text{curl } \left[ W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \right].
\]

In fact, since it is clear that \( H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \subset W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \), we have

\[
\text{curl } \left[ H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \right] \subset \text{curl } \left[ W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \right]. \tag{68}
\]

Conversely let \( u \in W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \). Choose \( \phi \) to be a solution of

\[
\Delta \phi = \text{div } u \quad \text{in } \Omega,
\]

\[
\phi = 0 \quad \text{on } \partial \Omega. \tag{70}
\]

By the elliptic regularity theorem, we see that \( \phi \in W^{2,p}(\Omega) \). Define \( v = u - \nabla \phi \). Then \( \text{curl } v = \text{curl } u \in L^p(\Omega, \mathbb{R}^3) \), \( \text{div } v = \text{div } u - \Delta \phi = 0 \) in \( \Omega \), and \( v_T = u_T - (\nabla \phi)_T = u_T = 0 \) on \( \partial \Omega \). Therefore \( v \in H^1_0(\Omega, \text{curl}, \text{div } 0, 0) \) and \( \text{curl } u = \text{curl } v \in \text{curl}[H^1_0(\Omega, \text{curl}, \text{div } 0, 0)] \).

Hence (67) holds for any \( w \in W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \). Since \( D(\Omega, \mathbb{R}^3) \subset W^{1,p}_1(\Omega, \mathbb{R}^3, 0) \), it follows from (67) that \( A \) is a weak solution of (65). \( \square \)

**Remark 12.** The system (65) is so-called the \( p \)-curl system. When \( \Omega \) is a bounded, simply connected domain in \( \mathbb{R}^3 \) without holes and with \( C^{3,\alpha} \) boundary for some \( \alpha \in (0, 1) \), if \( \mathcal{H}_\tau = 0 \), then [8] showed that the weak solution \( A \) of system (65) satisfies the fact that \( A \in C^{3,\beta}(\overline{\Omega}, \mathbb{R}^3) \) for some \( \beta \in (0, 1) \) and there exists a constant \( C \) depending only on \( p, \Omega \) such that \( \|A\|_{C^{3,\beta}(\overline{\Omega})} \leq C \).

**Lemma 13.** Let \( B_0 \in B(\Omega, \mathcal{H}_\tau) \) be a minimizer of (52). Then any minimizer \( A \in H^1_0(\Omega, \text{curl}, \text{div } 0, \mathcal{H}_\tau) \) of (17) must have the form \( A = B_0 + u \) where \( u \in \mathcal{H}_\tau^p(\Omega) \). In particular, the minimizer of (52) is unique. \( \square \)
Proof. Since for any \( u \in K_p(\Omega) \), we see that
\[
B_0 + u \in H^p(\Omega, \text{curl}, \text{div} 0, \mathcal{H}_T),
\]
\[
\int_\Omega |\text{curl}(B_0 + u)^p| \, dx = \int_\Omega |\text{curl} B_0|^p \, dx = R_T^p(\mathcal{H}_T).
\]
Thus \( B_0 + u \) is a minimizer of (17). On the other hand, for any minimizer \( A \in H^p(\Omega, \text{curl}, \text{div} 0, \mathcal{H}_T) \) of (17), define \( w = A - B_0 \). Then \( w \in H^p(\Omega, \text{curl}, \text{div} 0, 0) \). From (67), we have
\[
\int_\Omega S_1(\cdot, |\text{curl} A|^2) \text{curl} A \cdot \text{curl} w \, dx = \int_\Omega S_1(\cdot, |\text{curl} B_0|^2) \text{curl} B_0 \cdot \text{curl} w \, dx = 0.
\]
Therefore,
\[
\int_\Omega \left( S_1(\cdot, |\text{curl} A|^2) \text{curl} A - S_1(\cdot, |\text{curl} B_0|^2) \text{curl} B_0 \right) \cdot (\text{curl} A - \text{curl} B_0) \, dx = 0.
\]
By the structure condition (2), we have \( \text{curl}(A - B_0) = 0 \) in \( \Omega \), so \( A - B_0 \in K_p(\Omega) \).

If \( B \in B(\Omega, \mathcal{H}_T) \subseteq H^p(\Omega, \text{curl}, \text{div} 0, \mathcal{H}_T) \) is a minimizer of (52), we can write \( B = B_0 + u \), where \( u \in K_p(\Omega) \). If follows from Lemma 6 that we see that \( u = 0 \). Thus the minimizer of (52) in \( B(\Omega, \mathcal{H}_T) \) is unique. \( \square \)

For \( \mathcal{H}_T \in W^{1, -1/p', p}(\partial \Omega, \mathbb{R}^3) \), let \( A = A(\mathcal{H}_T) \in H^p(\Omega, \text{curl}, \text{div} 0, \mathcal{H}_T) \) be a minimizer of (17). Then there exist uniquely \( B_0 = B_0(\mathcal{H}_T) \in B(\Omega, \mathcal{H}_T) \) which is a minimizer of (52) and \( u(\mathcal{H}_T) \in K_p(\Omega) \) such that
\[
A(\mathcal{H}_T) = B_0(\mathcal{H}_T) + u(\mathcal{H}_T).
\]
We note that \( PA(\mathcal{H}_T) = B_0(\mathcal{H}_T) \).

In order to show the estimate in Theorem 1, it suffices to prove the following proposition.

**Proposition 14.** There exists a constant \( c = c(\Omega) \) independent of \( \mathcal{H}_T \) such that
\[
\|B_0(\mathcal{H}_T)\|_{W^{1, -1/p}(\Omega)} \leq c \|A(\mathcal{H}_T)\|_{W^{1, -1/p}(\partial \Omega)}.
\]

**Proof.** Assume that the conclusion is false. Then there exists a sequence \( \{\mathcal{H}_T\} \subset W^{1, -1/p}(\partial \Omega, \mathbb{R}^3) \) such that \( \|B_0(\mathcal{H}_{T,j})\|_{W^{1, -1/p}(\partial \Omega)} = 1 \) and
\[
\|A(\mathcal{H}_{T,j})\|_{W^{1, -1/p}(\partial \Omega)} \to 0 \text{ as } j \to \infty.
\]
For brevity of notation, we write \( B_j = B_0(\mathcal{H}_{T,j}) \). Passing to a subsequence, we may assume that \( B_j \to B \) weakly in \( W^{1, 2}(\Omega, \mathbb{R}^3) \), strongly in \( L^p(\Omega, \mathbb{R}^3) \), and a.e. in \( \Omega \). Thus \( \text{curl} B \in L^p(\Omega, \mathbb{R}^3) \), \( \text{div} B = 0 \) in \( \Omega \), and \( B_T = 0 \) on \( \partial \Omega \). Since \( B_j \) satisfies
\[
\int_\Omega |B_j|^{p-2} B_j \cdot z \, dx = 0 \quad \forall z \in K_p(\Omega)
\]
and \( B_j \to B \) strongly in \( L^p(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \), it follows from Lemma 7 that
\[
\int_\Omega |B_j|^{p-2} B_j \cdot z \, dx = 0 \quad \forall z \in K_p(\Omega).
\]
Hence we have \( B \in B(\Omega, 0) \). On the other hand, \( B_j \) is a weak solution of
\[
\text{curl} \left[ S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \right] = 0 \quad \text{in} \ \Omega,
\]
\[
B_{j,T} = H_{j,T} \text{ on} \ \partial \Omega.
\]
Since \( S_1(x, |\text{curl} B_j|^2) \text{curl} B_j \in L^p(\Omega, \mathbb{R}^3) \) and \( \|S_1(x, |\text{curl} B_j|^2) \text{curl} B_j\|_{\partial \Omega} \to 0 \), we see that \( S_1(x, |\text{curl} B_j|^2) \text{curl} B_j \in W^{1, -1/p, p}(\partial \Omega, \mathbb{R}^3) \). Since \( \text{curl} \times \mathcal{H}_{j,T} \in W^{1, -1/p}(\partial \Omega, \mathbb{R}^3) \) \( \mathcal{H}_{j,T} \), it follows from the Green formula that
\[
0 = \int_\Omega \text{curl} \left[ S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \right] \cdot B_j \, dx
\]
\[
= \int_\Omega S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \cdot \text{curl} B_j \, dx + \int_{\partial \Omega} \left< \mathcal{H}_{j,T}, \text{curl} \left[ S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \right] \right> \, dS,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality bracket of the spaces \( W^{1, -1/p, p}(\partial \Omega, \mathbb{R}^3) \) and \( W^{1, -1/p', p'}(\partial \Omega, \mathbb{R}^3) \). Here we have
\[
\int_{\partial \Omega} \left< \mathcal{H}_{j,T}, \text{curl} \left[ S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \right] \right> \, dS
\]
\[
\leq \|\mathcal{H}_{j,T}\|_{W^{1, -1/p}(\partial \Omega)} \|S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j\|_{L^{p'}(\Omega)}
\]
\[
\leq \|\mathcal{H}_{j,T}\|_{W^{1, -1/p}(\partial \Omega)} \left( \int_\Omega \langle A, |\text{curl} B_j|^{p-1} \rangle^{1/p'} \, dx \right)^{1/p'}
\]
\[
\leq \Lambda \|\mathcal{H}_{j,T}\|_{W^{1, -1/p}(\partial \Omega)} \|\text{curl} B_j\|_{L^{p'/p'}(\partial \Omega)}.
\]
Since \( \text{curl} B_j \to \text{curl} B \) weakly in \( L^p(\Omega, \mathbb{R}^3) \), we see that \( \|\text{curl} B_j\|_{L^p(\Omega)} \) is bounded. Since \( \|\mathcal{H}_{j,T}\|_{W^{1, -1/p}(\partial \Omega)} \to 0 \), we have
\[
\int_{\partial \Omega} \left< \mathcal{H}_{j,T}, S_1(\cdot, |\text{curl} B_j|^2) \text{curl} B_j \right> \, dS \to 0.
\]
as $j \to \infty$. Since $S(x, t^2) t^2$ is equivalent to $S(x, t)$, using (80), we have

$$
\int_{\Omega} S_j(\nabla \times B) |\nabla \times B|^2 \, dx \\
\leq \liminf_{j \to \infty} \int_{\Omega} S_j(\nabla \times B) |\nabla \times B|^2 \, dx \\
= \liminf_{j \to \infty} \left[ \int_{\Omega} S_j(\nabla \times B) |\nabla \times B|^2 \, dx \\
+ \int_{\partial \Omega} \left( \nabla \times \nabla \times B, S_j(\nabla \times B) \right) \, dS \right] \quad (83)
$$

$$
\leq \limsup_{j \to \infty} \int_{\Omega} S_j(\nabla \times B) |\nabla \times B|^2 \, dx \\
+ \int_{\partial \Omega} \left( \nabla \times \nabla \times B, S_j(\nabla \times B) \right) \, dS \\
= 0.
$$

Since $S_j(\nabla \times B)|\nabla \times B|^2 \geq \lambda |\nabla \times B|^p$, we see that $\nabla \times B = 0$, so $B \in \mathcal{L}_p^p(\Omega)$. From (78) with $z = B$, we have

$$
0 = \int_{\Omega} |B|^p - 2 \nabla \times B \cdot B \, dx = \int_{\Omega} |B|^p \, dx. \quad (84)
$$

Therefore $B = 0$ in $\Omega$, so $B_j \to 0$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$ and strongly in $L^p(\Omega, \mathbb{R}^3)$. From (80), we can see that $\|\nabla \times B\|_{L^p(\Omega)} \to 0$. By (19),

$$
\|B\|_{W^{1,p}(\Omega)} \leq \zeta(\Omega) \\
\cdot \left( \|B\|_{L^p(\Omega)} + \|\nabla \times B\|_{L^p(\Omega)} + \|\nabla \times \nabla \times B\|_{W^{1,p}(\Omega)} \right) \to 0 \\
\text{as } j \to \infty. \text{ This contradicts } \|B\|_{W^{1,p}(\Omega)} = 1. \quad \square
$$

**Proof of Theorem 1.** The proof of Theorem 1 follows from Lemma 2 and Propositions 10 and 14. \quad \square

**Remark 15.** Instead of minimizing $S(t, |\nabla u|^2)$, it is also interesting to minimize $S(x, |\nabla u|^2)$. This problem is related to the mathematical theory of liquid crystals. For $p = 2$ and $S(x, t) = t$, see Aramaki [13].

**Competing Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


