Research Article
On Self-Centeredness of Product of Graphs

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A graph G is said to be a self-centered graph if the eccentricity of every vertex of the graph is the same. In other words, a graph is a self-centered graph if radius and diameter of the graph are equal. In this paper, self-centeredness of strong product, co-normal product, and lexicographic product of graphs is studied in detail. The necessary and sufficient conditions for these products of graphs to be a self-centered graph are also discussed. The distance between any two vertices in the co-normal product of a finite number of graphs is also computed analytically.

1. Introduction

The concept of self-centered graphs is widely used in applications, for example, the facility location problem. The facility location problem is to locate facilities in a locality (network) so that these facilities can be used efficiently. All graphs in this paper are simple and connected graphs. The distance between two vertices u and v in a graph G, denoted by d_{G}(u,v) (or simply d(u,v)), is the minimum length of u-v path in the graph. The eccentricity of a vertex v in G, denoted by ecc_{G}(v), is defined as the distance between v and a vertex farthest from v; that is, ecc_{G}(v) = max{d_{G}(v,u): u \in V(G)}. The radius rad(G) and diameter diam(G) of the graph G are, respectively, the minimum and maximum eccentricity of the vertices of graph G; that is, rad(G) = min{ecc(v): v \in V(G)} and diam(G) = max{ecc(v): v \in V(G)}. The center C(G) of graph G is the induced subgraph of G on the set of all vertices with minimum eccentricity. A graph G is said to be a self-centered graph if the eccentricity of every vertex is the same; that is, C(G) = G or rad(G) = diam(G). If the eccentricity of every vertex is equal to d, then G is called d-self-centered graph.

For any kind of graph product G of the graphs G_{1}, G_{2}, \ldots , G_{n}, the vertex set is taken as V(G) = \{(x_{1},x_{2},\ldots ,x_{n}): x_{i} \in V(G_{i})\}. Because of their adjacency rules, product names are different. Let x = (x_{1},\ldots ,x_{n}) and y = (y_{1},\ldots ,y_{n}) be two vertices in V(G). Then the product is called

(i) Cartesian product, denoted by G = G_{1}\square G_{2}\square \cdots \square G_{n}, where x \sim y if and only if x_{i}y_{i} \in E(G_{i}) for exactly one index i, 1 \leq i \leq n, and x_{j} = y_{j} for each index j \neq i,

(ii) strong product, denoted by G = G_{1} \boxtimes \cdots \boxtimes G_{n}, where x \sim y if and only if x_{i}y_{i} \in E(G_{i}) or x_{i} = y_{i}, for every i, 1 \leq i \leq n,

(iii) lexicographic product, denoted by G = G_{1} \circ \cdots \circ G_{n}, where x \sim y if and only if, for some j \in \{1,2,\ldots ,n\}, x_{j}y_{j} \in E(G_{j}) and x_{i} = y_{i} for each 1 \leq i < j,

(iv) co-normal product, denoted by G = G_{1} \star G_{2} \star \cdots \star G_{n}, where x \sim y if and only if x_{i} \sim y_{i} for some i \in \{1,2,\ldots ,n\}.

Self-centered graphs have been broadly studied and surveyed in [1–3]. In [4], the authors described several algorithms to construct self-centered graphs. Stanic [5] proved that the Cartesian product of two self-centered graphs is a self-centered graph. Inductively, one can prove that Cartesian product of n-self-centered graphs is also a self-centered graph.

In this paper, we find conditions for self-centeredness of strong product, co-normal product, and lexicographic product of graphs.

2. Main Results

In this section, we will discuss the self-centeredness of different types of product graphs. As mentioned before, all graphs
considered here are simple and connected. The following result is given by Stanić [5].

**Theorem 1.** If $G_1$ and $G_2$ are $m$- and $n$-self-centered graphs, respectively, then $G_1 \Box G_2$ is $(m+n)$-self-centered graph. Reciprocally, if $G_1 \Box G_2$ is self-centered, then both graphs $G_1$ and $G_2$ are self-centered.

By method of induction, one can extend the above theorem and get the result given below.

**Theorem 2.** Let $G = G_1 \Box G_2 \Box \cdots \Box G_n$ be the Cartesian product of graphs $G_1, G_2, \ldots, G_n$. If every $G_i$ is $d_i$-self-centered graph, then $G$ is $m$-self-centered graph, where $m = \sum_{i=1}^n d_i$, $1 \leq i \leq n$. Conversely, if $G$ is a self-centered graph, then every $G_i$ is a self-centered graph.

Next we will discuss self-centeredness of strong product of graphs.

**Theorem 3.** Let $G = G_1 \boxtimes \cdots \Box G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$. Then $G$ is $d$-self-centered graph if and only if, for some $k \in \{1, \ldots, n\}$, $G_k$ is $d$-self-centered graph and $\text{diam}(G_k) \leq d$ for every $i$, $1 \leq i \leq n$.

**Proof.** For any two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the distance between $x$ and $y$ is given in [6]:

$$d(x, y) = \max_{1 \leq i < n} \{d_{G_i}(x_i, y_i)\}.$$  \hfill (1)

Now, the eccentricity of any vertex $x$ of $G$ is given by

$$\text{ecc}(x) = \max \{d(x, y) : y \in V(G)\}.$$  \hfill (2)

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

First, let $G_k$ be $d$-self-centered graph for some $k \in \{1, 2, \ldots, n\}$ and $\text{diam}(G_k) \leq d$ for all $i$, $1 \leq i \leq n$. Since $G_k$ is $d$-self-centered, $\text{ecc}(x_k) = d$ and there exists some $y_k$ in $G_k$ such that $d(x_k, y_k) = d$. As $\text{diam}(G_k) \leq d$ for all $i$, $1 \leq i \leq n$, the distance between any two vertices in any $G_i$ cannot exceed $d$. Hence, $\text{ecc}(x) = d$ for all $x \in V(G)$ and thus $G$ is $d$-self-centered graph.

Conversely, let $G$ be a $d$-self-centered graph. If, for some $l \in \{1, \ldots, n\}$, $\text{diam}(G_l) = d_l > d$, then there exist vertices $x_l$ and $y_l$ in $G_l$ such that $d(x_l, y_l) = d_l$. Now for $x = (x_1, \ldots, x_l, x_{l+1}, \ldots, x_n)$ and $y = (y_1, \ldots, y_l, y_{l+1}, \ldots, y_n)$ in $V(G)$, $d(x, y) \geq d(x_l, y_l) = d_l > d$ and so $\text{ecc}(x) \geq d_l > d$. This contradicts the fact that $G$ is $d$-self-centered graph and thus it is proven that $\text{diam}(G_k) \leq d$ for all $i$. Now, our claim is that there exists $k \in \{1, \ldots, n\}$ such that $G_k$ is $d$-self-centered graph. On the contrary, suppose that none of $G_i$ is $d$-self-centered graph. Then there exist vertices $x_l \in V(G_l)$ for all $i$ such that $\text{ecc}(x_l) = d_l < d$. Let $x = (x_1, \ldots, x_n)$. Then $\text{ecc}(x) = \max_{1 \leq i \leq n} \{d_i\} < d$, which contradicts the fact that $G$ is $d$-self-centered graph.

In the following lemma, we determine the formula for the distance between two vertices in the co-normal product of a finite number of graphs.

**Lemma 4.** Let $G = G_1 \ast G_2 \ast \cdots \ast G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$. The distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $G$ is

$$d(x, y) = \begin{cases} 1 & \text{if } x_i \sim y_i \text{ for some } i \in \{1, 2, \ldots, n\} \\ d(x_l, y_l) & \text{if } G_l = K_1, \forall j \neq i \\ 2 & \text{if } x + y, x_i \neq y_i \text{ for exactly one index } l \text{ and } G_j \neq K_1 \text{ for some } j \neq l \\ 2 & \text{if } x + y \text{ and } \exists \text{ at least two indices } k, l \text{ s.t. } x_k \neq y_k \text{ and } x_l \neq y_l. \end{cases}$$  \hfill (3)

Finally, consider the case, where, for at least two indices $k$ and $l$, $x_k \neq y_k$ and $x_l \neq y_l$; that is, for at least two indices $k$ and $l$, $G_k \neq K_1$ and $G_l \neq K_1$. Since $x + y$, $x_k + y_k$, and $x_l + y_l$, then from the connectivity of graphs $G_k$ and $G_l$ there exist vertices $z_k \in V(G_k)$ and $z_l \in V(G_l)$ such that $z_k \sim x_k$ in $G_k$ and $z_l \sim y_l$ in $G_l$. Then we have a vertex $z = (z_1, \ldots, z_k, z_l, \ldots, z_n) \in V(G)$ such that $x \sim z$ and $z \sim y$. Thus $xzy$ will be an $x$-$y$ path of length two and this proves that $d(x, y) = 2$.

The following theorem gives necessary and sufficient conditions for a co-normal product of graphs to be a self-centered graph.
Theorem 5. Let $G = G_1 \ast G_2 \ast \cdots \ast G_n$ be the co-normal product of graphs $G_1, G_2, \ldots, G_n$ with $|V(G_i)| = n_i$. Then the following hold:

(i) Let $G_i \neq K_1$ and $G_j = K_1$ for all $j \neq i$. Then $G$ is $d$-self-centered graph if and only if $G_i$ is $d$-self-centered graph.

(ii) Let there be at least two values of $i$ such that $G_i \neq K_1$. Then $G$ is 2-self-centered graph if and only if there exists an index $l$ such that $\Delta(G_l) \neq n_l - 1$, where $\Delta(G)$ is the maximum degree of a vertex in $G$.

Proof. (i) The result is true because $G$ is isomorphic to $G_i$ in this case through the isomorphism

$$f : V(G) \longrightarrow V(G_i)$$

with $f(x_1, \ldots, x_n) = x_i$.

(ii) Let $G$ be a $2$-self-centered graph. If, for all the indices $i$, $\Delta(G_i) = n_i - 1$, then there are vertices $x_j \in V(G_i), 1 \leq j \leq n$, such that $\deg(x_j) = n_j - 1$. Now, the vertex $x = (x_1, x_2, \ldots, x_n)$, ecc$(x) = 1$, which contradicts the fact that $G$ is 2-self-centered graph. Hence there exists an index $l$ such that $\Delta(G_l) \neq n_l - 1$.

Conversely, let there be an index $l$ such that $\Delta(G_l) \neq n_l - 1$. Then for any vertex $x = (x_1, x_2, \ldots, x_l, x_{l+1}, \ldots, x_n)$ in $G$ there exists another vertex $y = (y_1, y_2, \ldots, y_l, y_{l+1}, \ldots, y_n)$, where $y_j \in V(G_l)$ and $x_j \neq y_j$. Since $x \neq y$, from the third option of the distance formula given in Lemma 4, ecc$(x) = 2$. Since $x$ is an arbitrary vertex, $G$ is 2-self-centered graph. \qed

Theorem 6. Let $G = G_1 \ast G_2 \ast \cdots \ast G_n$ be the lexicographic product of graphs $G_1, G_2, \ldots, G_n$ and let $k \geq 1$ be the smallest index for which $G_k \neq K_1$. If $G_k$ is a $d$-self-centered graph, where $d \geq 2$, then $G$ is $d$-self-centered graph. The converse is true for $d \geq 3$.

Proof. For vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $G$, the following distance formula is due to Hammack et al. [6]:

$$d(x, y) = \begin{cases} d_{G_1}(x_1, y_1) & \text{if } x_i \neq y_i \\ d_{G_k}(x_r, y_r) & \text{if } d_{G_i}(x_i) = 0 \forall 1 \leq i < l \\ \min \{d_{G_i}(x_i, y_i), 2\} & \text{if } d_{G_i}(x_i) \neq 0 \text{ for some } 1 \leq i < l, \end{cases}$$

where $i$ is the smallest index for which $x_i \neq y_i$.

Let $|V(G_i)| = 1$ for $i = 1, 2, \ldots, k - 1$ and let $G_k$ be $d$-self-centered graph, where $d \geq 2$. First let $k = 1$. Since $|V(G_1)| > 1$, $G_1$ is connected and degree of no vertex in $G_1$ is zero; then the second option in the distance formula will not arise. Then the above formula to calculate the distance reduces to

$$d(x, y) = \begin{cases} d_{G_i}(x_1, y_1) & \text{if } i = 1 \\ \min \{d_{G_i}(x_i, y_i), 2\} & \text{if } i \geq 2, \end{cases}$$

where $i$ is the smallest index for which $x_i \neq y_i$. For $i \geq 2$, let $r = \min\{d_{G_i}(x_i, y_i), 2\}$. Then $r \leq 2$. Since $d \geq 2$, we get $r \leq d$. Now, for $x' \in V(G)$,

$$\text{ecc}(x) = \max \{d(x, y) : y \in V(G)\} = \max \{d_{G_i}(x_1, y_1), r : y_1 \in V(G_i)\}$$

$$= d,$$

because ecc$(x_1) = d$ and there exists $y_1 \in G_1$ such that $d(x_1, y_1) = d$. This proves that ecc$(x) = d$ for all $x \in V(G)$ and hence $G$ is a $d$-self-centered graph.

Next, let $k > 1$. Since $|V(G_1)| = 1$, there is no $y_1 \in G_1$ such that $x_1 \neq y_1$. So, first option in the distance formula will not arise. Since the degree of the vertex in $G_i$ for $j = 1, 2, \ldots, k - 1$ is zero, if $i = k$ in the above distance formula then $d(x, y) = d_{G_k}(x_k, y_k)$. Since $G_k \neq K_1$ and is connected $\deg(x_k) \neq 0$. So if $i \geq k + 1$ in the above formula, $d(x, y) = \min\{d_{G_i}(x_i, y_i), 2\}$ and thus the above formula to calculate the distance reduces to

$$d(x, y) = \begin{cases} d_{G_i}(x_i, y_k) & \text{if } i = k \\ \min \{d_{G_i}(x_i, y_k), 2\} & \text{if } i \geq k + 1, \end{cases}$$

where $i$ is the smallest index for which $x_i \neq y_i$. For $i \geq k + 1$ let $r_1 = \min\{d_{G_i}(x_i, y_i), 2\}$. Then $r_1 \leq 2$. Since $d \geq 2$, we get $r_1 \leq d$. Thus, for any vertex $x \in V(G)$, we have

$$\text{ecc}(x) = \max \{d(x, y) : y \in V(G)\} = \max \{d_{G_i}(x_k, y_k), r_1 : y_k \in V(G_k)\}$$

$$= d.$$
of graphs $K_2$, $P_4$, and $K_2$ is shown in Figure 1. One can check that the eccentricity of every vertex of $G$ is two and hence $G$ is a 2-self-centered graph. However, $G_1$ is not a 2-self-centered graph.

In the theorem below, we present the general version of the 2-self-centered product graphs included in the previous example.

**Theorem 8.** Let $G = G_1 \circ G_2 \circ \cdots \circ G_n$ be the lexicographic product of graphs $G_1, G_2, \ldots, G_n$ with $|V(G_i)| = n_i$, let $G_k$ be 1-self-centered graph for some $k \in \{1, \ldots, n-1\}$, and let $G_i$ (if it exists) be $K_1$ for all $i < k$. Then $G$ is a 2-self-centered graph if and only if $\Delta(G_j) \neq n_j - 1$ for some $j \geq k + 1$.

**Proof.** First let $G$ be a 2-self-centered graph. It is given that, for some $k \in \{1, \ldots, n-1\}$, $G_k$ is 1-self-centered graph and let $G_i$ be $K_1$ for all $i < k$. Our claim is that $\Delta(G_j) \neq n_j - 1$ for some $j \geq k + 1$. On the contrary, let $\Delta(G_j) = n_j - 1$ for all $j \geq k + 1$. Then there are vertices $g_i \in G_i$ such that $ecc(g_i) = 1$ for every $i, k \leq i \leq n$. Now, by using above distance formula, for every $x = (x_1, \ldots, x_{k-1}, g_k, \ldots, g_n)$ in $G$, one gets $ecc(x) = 1$. This contradicts the fact that $G$ is a 2-self-centered graph.

Conversely, let $\Delta(G_l) \neq n_l - 1$ for some $l \geq k + 1$. Then for any vertex $x_l \in G_l$ there exists $y_l \in G_l$ such that $x_l \sim y_l$. For any vertex $x = (x_1, \ldots, x_{k-1}, x_k, \ldots, x_n)$ there exists a vertex $y = (x_1, \ldots, x_{k-1}, y_k, \ldots, x_n)$ such that $x \sim y$. So, $ecc(x) \geq 2$. Since $G_l = K_1$ for all $i < k$ (if any), the distance formula will be

$$d(x, y) = \begin{cases} d_{G_i}(x_i, y_i) & \text{if } i = k \\ \min\{d_{G_i}(x_i, y_i), 2\} & \text{if } i \geq k + 1, \end{cases}$$

(11)

where $i$ is the smallest index for which $x_i \neq y_i$. Since $G_i$ is a 1-self-centered graph, $d_{G_i}(x_i, y_i) = 1$ if $i = k$. Also, for $i \geq k + 1$, $\min\{d_{G_i}(x_i, y_i), 2\} \leq 2$. Thus eccentricity of no vertex is more than two and we get $ecc(x) = 2$ for every $x \in G$. Hence $G$ is a 2-self-centered graph.

**Competing Interests**

The authors declare that there are no competing interests regarding the publication of this paper.

**References**


