Research Article

Qualitative Analysis of a Leslie-Gower Predator-Prey System with Nonlinear Harvesting in Predator

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This paper deals with the study of the stability and the bifurcation analysis of a Leslie-Gower predator-prey model with Michaelis-Menten type predator harvesting. It is shown that the proposed model exhibits the bistability for certain parametric conditions. Dulac's criterion has been adopted to obtain the sufficient conditions for the global stability of the model. Moreover, the model exhibits different kinds of bifurcations (e.g., the saddle-node bifurcation, the subcritical and supercritical Hopf bifurcations, Bogdanov-Takens bifurcation, and the homoclinic bifurcation) whenever the values of parameters of the model vary. The analytical findings and numerical simulations reveal far richer and complex dynamics in comparison to the models with no harvesting and with constant-yield predator harvesting.

1. Introduction

Marine life is a renewable natural resource that not only provides food to a large population of humans but also is involved in the regulation of the Earth's ecosystem. The growing human needs for more food and more energy have led to increased exploitation of these resources which affects the Earth's ecosystem. Thus, it is imperative to design harvesting strategies which aim at maximizing economic gains giving due consideration to the ecological health of the concerned Earth's ecological system. Mathematical modeling in harvesting of species was started by Clark [1, 2]. There are mainly three types of harvesting according to Gupta et al. [3]: (i) $h(x) = h$, constant rate harvesting (see [4–7]), (ii) $h(x) = qEx$, proportionate harvesting (see [8, 9]), and (iii) $h(x) = qEx/(m_1E + m_2x)$ (Holling type II), nonlinear harvesting (see [10–13]). Nonlinear harvesting is more realistic and exhibits saturation effects with respect to both the stock abundance and the effort level. Notice that $h(x) \rightarrow qE/m_2$ as $x \rightarrow \infty$ and $h(x) \rightarrow qE/m_1$ as $E \rightarrow \infty$.

The objective of the present work is to study dynamical behaviors of a Leslie-Gower predator-prey model in the

presence of nonlinear harvesting in predators depending on parameters of the model. There have been many papers on the Leslie-Gower predator-prey system with harvesting. For example, May et al. [14] proposed a Leslie-Gower predator-prey model in which two different kinds of constant-yield harvesting applied on both prey and predator species have been considered and this model was studied by Beddington and Cooke [15]. Beddington and May [16] proposed and studied Leslie-Gower predator-prey model when both the prey and predators were harvested with constant effort. Beddington and Cooke [15] studied a Leslie-Gower predator-prey model in which the preys are harvested at a constant-yield rate and predators are harvested with constant-effort rate. Zhu and Lan [17] studied a Leslie-Gower predator-prey model with constant-yield prey harvesting. Gong and Huang [18] studied the Bogdanov-Takens bifurcation for the model and showed that for different parameter values the model has a limit cycle or a homoclinic loop. Gupta et al. [3] discussed the bifurcation analysis of a Leslie-Gower prey-predator model in the presence of nonlinear harvesting in prey. Huang et al. [19] studied the effect of constant-yield predator

harvesting on the dynamics of a Leslie-Gower type model and showed that the model has Bogdanov-Takens (BT) singularity of codimension 3 or a weak focus of multiplicity two for some parameter values. They have shown that as the parameters change, the model exhibits saddle-node bifurcation, repelling and attracting B-T bifurcations, and supercritical, subcritical, and degenerate Hopf bifurcations.

This article is organized as follows. In Section 2, we describe the mathematical model in detail. In Section 3, we obtain the number and location of equilibrium points and the local and global stability of the equilibria are investigated. In Section 4, the existence of saddle-node, Hopf, and Bogdanov-Takens bifurcations is shown. In Section 5, we present several numerical simulations that support our theoretical results. Finally, we present some ramification of our results in Section 6.

2. Model Equations

2.1. Leslie-Gower Model. In general, the Leslie-Gower predator-prey model [20] is given as follows:

$$\begin{aligned}\frac{dX}{dT} &= r \left(1 - \frac{X}{K}\right) X - mXY, \\ \frac{dY}{dT} &= s \left(1 - \frac{Y}{nX}\right) Y, \quad (X, Y) \neq (0, 0), \\ X(0) &> 0, \\ Y(0) &> 0,\end{aligned}\quad (1)$$

where $X \equiv X(T)$ and $Y \equiv Y(T)$ are the prey density and predator density at time T , respectively; r, s, K, m , and n are positive parameters and represent the intrinsic growth rate of prey, intrinsic growth rate of predator, carrying capacity of prey in the absence of predator, maximal predator per capita consumption rate, and measure of the food quality that the prey provides for conversion into predator births. Model (1) has been studied by Hsu and Huang [21]. They showed that the unique positive interior equilibrium point of system (1) is globally asymptotically stable under all biologically admissible parameters.

2.2. Harvested Model. We assumed that only predator species is economically important and the nonlinear harvesting rate is considered. Model (1) reduces to the following:

$$\begin{aligned}\frac{dX}{dT} &= r \left(1 - \frac{X}{K}\right) X - mXY, \\ \frac{dY}{dT} &= s \left(1 - \frac{Y}{nX}\right) Y - \frac{qEY}{m_1E + m_2Y}, \\ (X, Y) &\neq (0, 0),\end{aligned}\quad (2)$$

$$X(0) > 0,$$

$$Y(0) > 0,$$

where the parameters q and E are positive and represent catchability coefficient and effort applied to harvest the

individuals, respectively, and m_1, m_2 are suitable positive constants.

Model (2) is not well defined at $(0, 0)$. In order to define system (2) at $(0, 0)$, we improve the model as given in [22]; system (2) reduces to

$$\begin{aligned}\frac{dX}{dT} &= r \left(1 - \frac{X}{K}\right) X - mXY, \\ \frac{dY}{dT} &= s \left(1 - \frac{Y}{nX}\right) Y - \frac{qEY}{m_1E + m_2Y}, \\ (X, Y) &\neq (0, 0),\end{aligned}\quad (3)$$

$$\frac{dY}{dT} = 0, \quad (X, Y) = (0, 0),$$

$$X(0) > 0,$$

$$Y(0) > 0.$$

For nondimensionalizing system (3), we introduce the following substitutions:

$$\begin{aligned}X &= Kx, \\ mY &= ry, \\ rT &= t.\end{aligned}\quad (4)$$

System (3) reduces to

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= \rho y \left(1 - \frac{\beta y}{x} - \frac{h}{c + y}\right), \quad (x, y) \neq (0, 0), \\ \frac{dy}{dt} &= 0, \quad (x, y) = (0, 0), \\ x(0) &> 0, \\ y(0) &> 0,\end{aligned}\quad (5)$$

where $\rho = s/r$, $\beta = r/mnK$, $h = qEm/srm_2$, and $c = m_1mE/m_2r$. For the existence of the biological meaning of the variables in model, system (5) is studied in the closed first quadrant, Ω , in xy -plane defined by $\Omega = \{(x, y): x \geq 0, y \geq 0\}$.

3. Equilibria and Their Local Stability

The equilibrium points of system (5) are the nonnegative real solutions of the system $dx/dt = dy/dt = 0$. It is obvious that system (5) has the trivial equilibrium point $E_0 = (0, 0)$ and the axial equilibrium point $e = (1, 0)$ and the abscissa of the positive interior equilibrium points are the roots of the quadratic equation:

$$(1 + \beta)x^2 - (1 + 2\beta + c + \beta c - h)x + \beta(1 + c) = 0, \quad (6)$$

and the ordinate of the positive interior equilibrium points is given by $y_{1,2} = 1 - x_{1,2}$, $0 \leq x_{1,2} \leq 1$.

The quadratic equation (6) has two positive roots $x_1 = (1 + 2\beta + c + \beta c - h + \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$ and $x_2 = (1 + 2\beta + c + \beta c - h - \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$ whenever $h < 1 + 2\beta + c + \beta c - 2\sqrt{\beta(1 + c + \beta + \beta c)} = \bar{h}$ (say), a double positive root $x_3 = (1 + 2\beta + c + \beta c - h)/2(1 + \beta)$ whenever $h = \bar{h}$, and no positive root whenever $h > \bar{h}$. The number and location of the equilibrium points of system (5) lying in the set Ω are given in the following lemma.

Lemma 1. System (5) has

- four equilibrium points, trivial equilibrium point $E_0 = (0, 0)$, axial equilibrium point $e = (1, 0)$, and two interior equilibrium points $E_1 = (x_1, y_1)$ and $E_2 = (x_2, y_2)$ whenever $0 < c < h < \bar{h}$, where $x_1 = (1 + 2\beta + c + \beta c - h + \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$, $y_1 = (1 + h - c - \beta c - \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$, $x_2 = (1 + 2\beta + c + \beta c - h - \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$, $y_2 = (1 + h - c - \beta c + \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$, and $\bar{h} = 1 + 2\beta + c + \beta c - 2\sqrt{\beta(1 + c + \beta + \beta c)}$;
- three equilibrium points, trivial equilibrium point $E_0 = (0, 0)$, axial equilibrium point $e = (1, 0)$, and a double interior equilibrium point $E_3 = (x_3, y_3)$ whenever $c < h = \bar{h}$, where $x_3 = \sqrt{\beta(1 + c)/(1 + \beta)}$ and $y_3 = 1 - \sqrt{\beta(1 + c)/(1 + \beta)}$;
- three equilibrium points, trivial equilibrium point $E_0 = (0, 0)$, axial equilibrium point $e = (1, 0)$, and an interior equilibrium point $E_4 = (x_4, y_4)$ whenever $c = h$ and $\beta c < 1$, where $x_4 = \beta(1 + c)/(1 + \beta)$ and $y_4 = (1 - \beta c)/(1 + \beta)$;

- three equilibrium points, trivial equilibrium point $E_0 = (0, 0)$, axial equilibrium point $e = (1, 0)$, and an interior equilibrium point $E_5 = (x_5, y_5)$ whenever $h < c$, where $x_5 = (1 + 2\beta + c + \beta c - h - \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$ and $y_5 = (1 + h - c - \beta c + \sqrt{(1 + c + \beta c - h)^2 - 4\beta h})/2(1 + \beta)$;
- two equilibrium points, trivial equilibrium point $E_0 = (0, 0)$, and axial equilibrium point $e = (1, 0)$ whenever $\bar{h} < h$ and $c < h$.

The number and location of interior equilibrium points have been depicted in Figure 1.

Now, we shall discuss the stability of the equilibrium points. The Jacobian matrix of system (5) at the equilibrium point E_0 cannot be calculated as the term y/x is not defined at $(0, 0)$. We use the blow-up technique to analyze the stability of the equilibrium point E_0 as given in [23]. Let $x = x$ and $y = xv$; then, system (5) reduces to

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - xv), \\ \frac{dv}{dt} &= v\left(\rho - 1 + x - \rho\beta v + xv - \frac{h\rho}{c + xv}\right). \end{aligned} \quad (7)$$

System (7) has two equilibrium points $\bar{E}_0 = (0, 0)$ and $\bar{E}_1 = (0, (1/\rho\beta)(\rho - 1 - h\rho/c))$ whenever $\rho - 1 - h\rho/c > 0$. The Jacobian matrix of system (7) at the equilibrium point $\bar{E}_0$ is

$$J_{\bar{E}_0} = \begin{bmatrix} 1 & 0 \\ 0 & \rho - 1 - \frac{h\rho}{c} \end{bmatrix}. \quad (8)$$

Thus, the equilibrium point $\bar{E}_0$ of system (7) is an unstable point as $\rho - 1 - h\rho/c > 0$. The Jacobian matrix of system (7) at the equilibrium point $\bar{E}_1$ is

$$J_{\bar{E}_1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\rho\beta}\left(\rho - 1 - \frac{h\rho}{c}\right)\left(1 + \frac{1}{\rho\beta}\left(\rho - 1 - \frac{h\rho}{c}\right) + \frac{h}{\beta c^2}\left(\rho - 1 - \frac{h\rho}{c}\right)\right) - \left(\rho - 1 - \frac{h\rho}{c}\right) & \end{bmatrix}. \quad (9)$$

Thus, the equilibrium point $\bar{E}_1$ of system (7) is always saddle as $\rho - 1 - h\rho/c > 0$. Thus, we can conclude the discussion above as follows.

Theorem 2. The trivial equilibrium point E_0 of system (5) is a saddle point.

The Jacobian matrix of system (5) at the equilibrium point e is

$$J_e = \begin{bmatrix} -1 & -1 \\ 0 & \rho\left(1 - \frac{h}{c}\right) \end{bmatrix}. \quad (10)$$

The Jacobian matrix of system (5) at the positive interior equilibrium point E is

$$J_E = \begin{bmatrix} -x & -x \\ \frac{\rho\beta y^2}{x^2} & \frac{\rho y}{x}\left(\frac{x - 2\beta y - \beta c}{c + y}\right) \end{bmatrix}. \quad (11)$$

The determinant of the abovementioned Jacobian matrix $\det(J_E) = (\rho y/x(c + y))(\beta(1 + c) - (1 + \beta)x^2)$ and trace $\text{tr}(J_E) = (\rho y/x)((x - 2\beta y - \beta c)/(c + y)) - x$.

FIGURE 1: This diagram shows the number and location of the positive interior equilibrium points of system (5). The green, red, and black color curves are the predator nullclines for different values of h and the straight line is the prey nullcline: (a) $h > c$, for black color parabola $h > \bar{h}$, for red color parabola $h = \bar{h}$, and for green color parabola $h < \bar{h}$; (b) $h = c$; (c) $h < c$.

Theorem 3. (a) The equilibrium point e is asymptotically stable whenever $c < h$ and a saddle whenever $h < c$.

(b) The equilibrium point E_1 , if existent, is always a saddle point.

(c) The equilibrium point E_2 , if existent, is asymptotically stable whenever $(\rho y_2/x_2)((x_2 - 2\beta y_2 - \beta c)/(c + y_2)) - x_2 < 0$ and is unstable whenever $(\rho y_2/x_2)((x_2 - 2\beta y_2 - \beta c)/(c + y_2)) - x_2 > 0$.

(d) The equilibrium point E_3 , if existent, is a degenerate singular point.

(e) The equilibrium point E_4 , if existent, is always asymptotically stable.

(f) The equilibrium point E_5 , if existent, is asymptotically stable whenever $(\rho y_5/x_5)((x_5 - 2\beta y_5 - \beta c)/(c + y_5)) - x_5 < 0$ and is unstable whenever $(\rho y_5/x_5)((x_5 - 2\beta y_5 - \beta c)/(c + y_5)) - x_5 > 0$.

Proof. (a) The eigenvalues of the Jacobian matrix J_e are -1 and $\rho(1 - h/c)$, so the equilibrium point e is asymptotically stable whenever $c < h$ and a saddle whenever $h < c$.

(b) The determinant $\det(J_{E_1}) < 0$, so the interior equilibrium point E_1 is a saddle point.

(c) The determinant $\det(J_{E_2}) > 0$ and $\text{tr}(J_{E_2}) = (\rho y_2/x_2)((x_2 - 2\beta y_2 - \beta c)/(c + y_2)) - x_2$, so the interior equilibrium point E_2 is asymptotically stable whenever $(\rho y_2/x_2)((x_2 - 2\beta y_2 - \beta c)/(c + y_2)) - x_2 < 0$ and unstable whenever $(\rho y_2/x_2)((x_2 - 2\beta y_2 - \beta c)/(c + y_2)) - x_2 > 0$.

(d) The determinant $\det(J_{E_3}) = 0$, so the interior equilibrium point E_3 is a degenerate singular point.

(e) The determinant $\det(J_{E_4}) > 0$ as $\beta c < 1$ and $\text{tr}(J_{E_4}) < 0$ always, so the interior equilibrium point E_4 is always asymptotically stable.

(f) The determinant $\det(J_{E_5}) > 0$ and $\text{tr}(J_{E_5}) = (\rho y_5/x_5)((x_5 - 2\beta y_5 - \beta c)/(c + y_5)) - x_5$, so the interior equilibrium point E_5 is asymptotically stable whenever $(\rho y_5/x_5)((x_5 - 2\beta y_5 - \beta c)/(c + y_5)) - x_5 < 0$ and unstable whenever $(\rho y_5/x_5)((x_5 - 2\beta y_5 - \beta c)/(c + y_5)) - x_5 > 0$. $\square$

From Lemma 1, if $h = c$, system (5) has unique interior equilibrium point E_4 , and if $h < c$, system (5) has unique interior equilibrium point E_5 . In Theorem 3, it is proved that these interior equilibrium points are always asymptotically stable. Now we show that these equilibrium points are globally asymptotically stable.

Theorem 4. *The equilibrium points E_4 and E_5 , if existent, are globally asymptotically stable.*

Proof. From Lemma 1, system (5) has one trivial equilibrium point E_0 , one axial equilibrium point e , and one interior equilibrium point E_4 whenever $h = c$. Further, it has one trivial equilibrium point E_0 , one axial equilibrium point e , and one interior equilibrium point E_5 whenever $h < c$. Also, from Theorem 3, the trivial equilibrium point E_0 is always a saddle point, axial equilibrium point e is a saddle point whenever $h \leq c$, and the interior equilibrium points E_4 and E_5 are asymptotically stable. We define the following function:

$$L(x, y) = \frac{\partial}{\partial x} (M(x, y) f(x, y)) + \frac{\partial}{\partial y} (M(x, y) g(x, y)), \quad (12)$$

where $f(x, y) = x(1-x-y)$, $g(x, y) = \rho y(1-\beta y/x-h/(c+y))$, and $M(x, y) = 1/xy^2$.

After simplification, we obtain $L(x, y) = -(1/y^2)(1+(y^2+(c-h)(c+2y))/x(c+y)^2) < 0$ as $h = c$ or $h < c$. Thus, by using Dulac's criterion [24], system (5) will not have any nontrivial periodic orbit in Ω . Note that y -axis and x -axis are the stable manifolds of the trivial and the axial equilibrium points, respectively. Using this in conjunction with the Poincare-Bendixon theorem [24] gives us that the interior equilibrium points E_4 and E_5 will be globally stable. $\square$

In Theorem 3, it is shown that the equilibrium point E_3 is a degenerate singular point. Now we study the property of this point.

Theorem 5. *The equilibrium point E_3 , if existent, is*

- (a) *a saddle-node whenever $(\rho y_3/x_3)((x_3-2\beta y_3-\beta c)/(c+y_3)) - x_3 \neq 0$;*
- (b) *a cusp of codimension 2 whenever $(\rho y_3/x_3)((x_3-2\beta y_3-\beta c)/(c+y_3)) - x_3 = 0$.*

Proof. First, we shall shift the equilibrium point $E_3 = (x_3, y_3)$ to the origin by using the transformations $u_1 = x - x_3$ and $v_1 = y - y_3$; system (5) reduces into the form

$$\begin{aligned} \frac{du_1}{dt} &= -x_3 u_1 - x_3 v_1 - u_1^2 - u_1 v_1, \\ \frac{dv_1}{dt} &= \frac{\rho \beta y_3^2}{x_3^2} u_1 + \frac{\rho y_3}{x_3} \left(\frac{x_3 - 2\beta y_3 - \beta c}{c + y_3} \right) v_1 - \eta_1 u_1^2 \\ &\quad + \eta_2 u_1 v_1 + \eta_3 v_1^2 + O(|(u_1, v_1)|^3), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \eta_1 &= \frac{\rho \beta y_3^2}{x_3^3}, \\ \eta_2 &= \frac{2\rho \beta y_3}{x_3^2}, \\ \eta_3 &= \rho \left(\frac{ch}{(c+y_3)^3} - \frac{\beta}{x_3} \right). \end{aligned} \quad (14)$$

If $(\rho y_3/x_3)((x_3-2\beta y_3-\beta c)/(c+y_3)) - x_3 \neq 0$, point E_3 is a saddle point. Now we consider $(\rho y_3/x_3)((x_3-2\beta y_3-\beta c)/(c+y_3)) - x_3 = 0$; that is, $\text{tr}(J_{E_3}) = 0$. Both eigenvalues of the Jacobian matrix J_{E_3} are zero and system (13) reduces to

$$\begin{aligned} \frac{du_1}{dt} &= -x_3 u_1 - x_3 v_1 - u_1^2 - u_1 v_1, \\ \frac{dv_1}{dt} &= \frac{\rho \beta y_3^2}{x_3^2} u_1 + x_3 v_1 - \eta_1 u_1^2 + \eta_2 u_1 v_1 + \eta_3 v_1^2 \\ &\quad + O(|(u_1, v_1)|^3). \end{aligned} \quad (15)$$

Now we introduce the affine transformations $u_2 = u_1$, $v_2 = -x_3 u_1 - x_3 v_1$, to reduce system (15) as

$$\begin{aligned} \frac{du_2}{dt} &= v_2 + \frac{1}{x_3} u_2 v_2, \\ \frac{dv_2}{dt} &= \eta_4 u_2^2 + \eta_5 u_2 v_2 - \eta_6 v_2^2 + O(|(u_2, v_2)|^3), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \eta_4 &= x_3 (\eta_1 + \eta_2 - \eta_3), \\ \eta_5 &= \eta_2 - 2\eta_3 - 1, \\ \eta_6 &= \frac{\eta_3}{x_3}. \end{aligned} \quad (17)$$

Now, we consider the C^∞ change of coordinates in the small vicinity of $(0, 0)$:

$$\begin{aligned} u_3 &= u_2, \\ v_3 &= v_2 + \eta_6 u_2 v_2. \end{aligned} \quad (18)$$

Then, system (16) reduces to

$$\begin{aligned} \frac{du_3}{dt} &= v_3 + \left(\frac{1}{x_3} - \eta_6 \right) u_3 v_3 + \bar{O}(|(u_3, v_3)|^3), \\ \frac{dv_3}{dt} &= \eta_4 u_3^2 + \eta_5 u_3 v_3 + O(|(u_3, v_3)|^3). \end{aligned} \quad (19)$$

Now, we choose the C^∞ change of coordinates in the small neighbourhood of $(0, 0)$:

$$\begin{aligned} u_4 &= u_3 - \frac{1}{2} \left(\frac{1}{x_3} - \eta_6 \right) u_3^2, \\ v_4 &= v_3. \end{aligned} \quad (20)$$

System (19) reduces to

$$\begin{aligned} \frac{du_4}{dt} &= v_4 + \bar{O}(|(u_4, v_4)|^3), \\ \frac{dv_4}{dt} &= \eta_4 u_4^2 + \eta_5 u_4 v_4 + O(|(u_4, v_4)|^3). \end{aligned} \quad (21)$$

Now, we choose the final C^∞ change of coordinates in the small neighbourhood of $(0, 0)$:

$$\begin{aligned} u &= u_4, \\ v &= v_4 + \bar{O}(|(u_4, v_4)|^3). \end{aligned} \quad (22)$$

System (21) reduces to

$$\begin{aligned} \frac{du}{dt} &= v, \\ \frac{dv}{dt} &= \eta_4 u + \eta_5 uv + O(|(u, v)|^3). \end{aligned} \quad (23)$$

If $\eta_4 \eta_5 = x_3 \rho (\beta y_3^2 / x_3^3 + 2\beta y_3 / x_3^2 - hc / (c + y_3)^3 + \beta / x_3)$ ($2\rho\beta y_3 / x_3^2 - 2\rho hc / (c + y_3)^3 + 2\beta\rho / x_3 - 1 \neq 0$ (nondegeneracy condition), the origin $(0, 0)$ of (23) is a cusp of codimension 2; that is, the interior equilibrium point E_3 of system (5) is a cusp of codimension 2. $\square$

In Section 4, we shall study the bifurcations occurring in system (5).

4. Bifurcation Analysis

4.1. Hopf Bifurcation. In Theorem 3, it is shown that the interior equilibrium point E_2 is stable whenever $(\rho y_2 / x_2)((x_2 - 2\beta y_2 - \beta c) / (c + y_2)) - x_2 < 0$ and unstable whenever $(\rho y_2 / x_2)((x_2 - 2\beta y_2 - \beta c) / (c + y_2)) - x_2 > 0$. Now, we consider the parametric condition $(\rho y_2 / x_2)((x_2 - 2\beta y_2 - \beta c) / (c + y_2)) - x_2 = 0$. In this parametric condition, the equilibrium point E_2 is a weak focus or a center. Hence, system (5) may enter a Hopf bifurcation at the point E_2 . In this subsection, we consider the parameter ρ as the Hopf bifurcation parameter and discuss the conditions under which the stability of E_2 will change and system (5) exhibits Hopf bifurcations.

Theorem 6. *System (5) undergoes a Hopf bifurcation with respect to parameter ρ around the equilibrium point E_2 , if existent, whenever $(\rho y_2 / x_2)((x_2 - 2\beta y_2 - \beta c) / (c + y_2)) - x_2 = 0$. Moreover,*

- the equilibrium point E_2 is a weak focus of multiplicity 1 if the parameter set (ρ, β, h, c) is in H_{sup} or H_{sub} and is stable and unstable according to whether (ρ, β, h, c) is in H_{sup} or H_{sub} ;*
- system (5) has at least one unstable limit cycle whenever (ρ, β, h, c) is in H_{sub} , $0 < \rho < \rho^*$ and $|\rho - \rho^*| \ll 1$, and at least one stable limit cycle whenever (ρ, β, h, c) is in H_{sup} , $\rho > \rho^*$ and $|\rho - \rho^*| \ll 1$.*

Proof. A critical magnitude of the bifurcation parameter $\rho = \rho^* = x_2^2(c + y_2) / y_2(1 - \beta c - (1 + 2\beta)y_2)$ exists as $1 - \beta c - (1 + 2\beta)y_2 \neq 0$ such that, at $\rho = \rho^*$, $\text{tr}(J_E) = 0$ and $\det(J_E) > 0$. In order to ensure the existence of a Hopf bifurcation, we have to check the transversality condition $(d/d\rho)(\text{tr}(J_E))|_{\rho=\rho^*} \neq 0$ (see [24]). We have $(d/d\rho)(\text{tr}(J_E))|_{\rho=\rho^*} = (y_2/x_2)((x_2 - \beta y_2)/(c + y_2) - \beta) \neq 0$ as $1 - \beta c - (1 + 2\beta)y_2 \neq 0$. Hence, the transversality condition for a Hopf bifurcation is satisfied. To

determine the direction of Hopf bifurcation and stability of E_2 , we compute the first Liapunov coefficient of system (5) at the equilibrium point E_2 .

Let $x = u - x_2$ and $y = v - y_2$; then, the equilibrium point E_2 is shifted to the origin $(0, 0)$ and system (5) can be rewritten as

$$\begin{aligned} \frac{du}{dt} &= a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 \\ &\quad + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3, \\ \frac{dv}{dt} &= b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 \\ &\quad + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + P(u, v), \end{aligned} \quad (24)$$

where $a_{10} = -x_2$, $a_{01} = -x_2$, $a_{20} = -1$, $a_{11} = -1$, $a_{02} = 0$, $a_{30} = 0$, $a_{21} = 0$, $a_{12} = 0$, $a_{03} = 0$, $b_{10} = \rho\beta y_2^2/x_2^2$, $b_{01} = (\rho y_2/x_2)((x_2 - \beta y_2)/(c + y_2) - \beta)$, $b_{20} = -\rho\beta y_2^2/x_3^3$, $b_{11} = 2\rho\beta y_2/x_2^2$, $b_{02} = \rho(ch/(c + y_2)^3 - \beta/x_2)$, $b_{30} = \rho\beta y_2^2/x_2^4$, $a_{21} = -2\rho\beta y_2/x_2^3$, $b_{12} = \rho\beta/x_2^2$, $b_{03} = \rho y_2 h/(c + y_2)^4 - \beta h/(c + y_2)^3$, and $P(u, v) = \sum_{i+j=4}^{\infty} b_{ij}u^i v^j$. Hence, using the formula of the first Lyapunov number σ at the origin of (24), we have

$$\begin{aligned} \sigma &= -\frac{3\pi}{2a_{01}\Delta^{3/2}} \{ [a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) \\ &\quad + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \\ &\quad + b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) \\ &\quad - 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) \\ &\quad + (a_{01}b_{10} - 2a_{10}^2)(b_{11}b_{02} - a_{11}a_{20})] - (a_{10}^2 \\ &\quad + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) + 2a_{10}(a_{21} + b_{12}) \\ &\quad + (b_{10}a_{12} - a_{01}b_{21})] \}, \end{aligned} \quad (25)$$

where $\Delta = (\rho y_2 / (c + y_2)) \sqrt{(1 + c + \beta c - h)^2 - 4\beta h}$. If $\sigma \neq 0$, then the origin of (24) is a weak focus of multiplicity one: also origin is stable when $\sigma < 0$ and unstable when $\sigma > 0$. Hence, in parameter space (ρ, β, h, c) , there exist surfaces $H_{\text{sub}} = \{(\rho, \beta, h, c): \sigma > 0, 0 < c < h < \bar{h}, (\rho y_2 / x_2)((x_2 - \beta y_2)/(c + y_2) - \beta) - x_2 = 0\}$ and $H_{\text{sup}} = \{(\rho, \beta, h, c): \sigma < 0, 0 < c < h < \bar{h}, (\rho y_2 / x_2)((x_2 - \beta y_2)/(c + y_2) - \beta) - x_2 = 0\}$, called subcritical and supercritical Hopf bifurcation surface, respectively, such that if the parameter set (ρ, β, h, c) is in H_{sub} , the equilibrium point E_2 of system (5) is a weak focus of multiplicity 1 and is unstable, and if the parameter set (ρ, β, h, c) is in H_{sup} , the equilibrium point E_2 of system (5) is a weak focus of multiplicity 1 and is stable.

From the discussion above, the equilibrium point E_2 of system (5) is a weak focus of multiplicity 1 and is unstable if (ρ, β, h, c) is in H_{sub} . Also from Theorem 3, the equilibrium point E_2 is stable whenever $(\rho y_2 / x_2)((x_2 - \beta y_2)/(c + y_2) - \beta) - x_2 < 0$, that is, $\rho < \rho^*$, and is unstable whenever $(\rho y_2 / x_2)((x_2 - \beta y_2)/(c + y_2) - \beta) - x_2 > 0$, that is, $\rho > \rho^*$. Thus,

the equilibrium point E_2 generates an unstable limit cycle as the bifurcation parameter ρ passes through the bifurcation value $\rho = \rho^*$. From one side of the surface H_{sub} to the other side, system (5) can undergo a subcritical Hopf bifurcation. An unstable limit cycle arises in the small neighbourhood of the equilibrium point E_2 whenever (ρ, β, h, c) is in H_{sub} , $0 < \rho < \rho^*$ and $|\rho - \rho^*| \ll 1$. Similarly, a stable limit cycle arises in the small neighbourhood of the equilibrium point E_2 whenever (ρ, β, h, c) is in H_{sup} , $\rho > \rho^*$ and $|\rho - \rho^*| \ll 1$. $\square$

4.2. Saddle-Node Bifurcation. In Section 3, it is shown that system (5) admits the double point E_3 whenever $h = \bar{h}$. In Theorem 5, it is shown that the point E_3 is a saddle node whenever $(\rho y_3/x_3)((x_3 - 2\beta y_3 - \beta c)/(c + y_3)) - x_3 \neq 0$. Now, we show that system (5) experiences a saddle-node bifurcation of codimension 1 around the equilibrium point E_3 at the threshold value of the bifurcation parameter $h = \bar{h}$ by means of Sotomayor's theorem [24].

Theorem 7. *System (5) undergoes a saddle-node bifurcation with respect to the parameter h around the equilibrium point E_3 whenever $h = \bar{h} = 1 + 2\beta + c + \beta c - 2\sqrt{\beta(1 + c + \beta + \beta c)}$ and $(\rho y_3/x_3)((x_3 - 2\beta y_3 - \beta c)/(c + y_3)) - x_3 \neq 0$.*

Proof. It has been shown that if $h = \bar{h} = 1 + 2\beta + c + \beta c - 2\sqrt{\beta(1 + c + \beta + \beta c)}$ and $(\rho y_3/x_3)((x_3 - 2\beta y_3 - \beta c)/(c + y_3)) - x_3 \neq 0$, one eigenvalue of the Jacobian matrix J_{E_3} is zero and the other has nonzero real part. Suppose V and W are the eigenvectors corresponding to the zero eigenvalues of the Jacobian matrix J_{E_3} and transverse matrix J'_{E_3} , respectively; then, $V = [1 \ -1]'$ and $W = [\rho\beta y_3^2/x_3^3 \ 1]'$. We have $f_h(E_3, \bar{h}) = [\rho y_3/(c + y_3)]$, $D^2(f(E_3, \bar{h}))(V, V) = [-2\rho\beta y_3^2/x_3^3 - 4\rho\beta y_3/x_3^2 + 2\rho(-\beta/x + \bar{h}c)/(c + y_3)^3]$. Therefore, $W'f_h(E_3, \bar{h}) = \rho y_3/(c + y_3) > 0$ and $W'[D^2(f(E_3, \bar{h}))(V, V)] = -2\rho(\beta/x_3^3 - \bar{h}c/(c + y_3)^3)$. Since $\bar{h}/(c + y_3)^2 = \beta/x^2$, $W'[D^2(f(E_3, \bar{h}))(V, V)] = -\rho\beta(1 + c)y_3/x_3^3(c + y_3) < 0$. Thus, Sotomayor's theorem confirms that system (5) experiences a saddle-node bifurcation of codimension 1 around interior equilibrium point E_3 . This means that there are no equilibrium points for $h > \bar{h}$ while there are two coexistence equilibrium points E_1 and E_2 for $h > \bar{h}$, one of which is saddle point and the other is a stable node. $\square$

4.3. Bogdanov-Takens Bifurcation. In Theorem 5, we have proved that the interior equilibrium point E_3 is a cusp of codimension 2 whenever $(\rho y_3/x_3)((x_3 - 2\beta y_3 - \beta c)/(c + y_3)) - x_3 = 0$ and $\eta_4\eta_5 \neq 0$, which implies that there may exist the Bogdanov-Takens bifurcation in system (5). The parameters h and ρ are taken as the bifurcation parameters as they are important from biological point of view and by means of a series of transformations as given in Xiao and Ruan [5], we derive a normal form.

Theorem 8. *System (5) undergoes a Bogdanov-Takens bifurcation with respect to the bifurcation parameters h and ρ around*

the double equilibrium point E_3 if $c < h = \bar{h}$, $(\rho y_3/x_3)((x_3 - 2\beta y_3 - \beta c)/(c + y_3)) - x_3 = 0$, and $\eta_4\eta_5 \neq 0$.

Proof. We consider that the parameters h and ρ vary in a small neighbourhood of the BT point (h_0, ρ_0) . Let $(h_0 + \lambda_1, \rho_0 + \lambda_2)$ be a point of this neighbourhood, where λ_1, λ_2 are small. System (5) becomes

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= (\rho_0 + \lambda_1) \left(1 - \frac{\beta y}{x} - \frac{h_0 + \lambda_2}{c + y} \right) y. \end{aligned} \quad (26)$$

Translating the equilibrium point E_3 into the origin by using the transformations $u_1 = x - x_3$ and $u_2 = y - y_3$ and then using Taylor's series expansion, system (26) reduces to

$$\begin{aligned} \frac{du_1}{dt} &= -x_3 u_1 - x_3 u_2 - u_1^2 - u_1 u_2, \\ \frac{du_2}{dt} &= \alpha_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_1^2 + \alpha_4 u_1 u_2 + \alpha_5 u_2^2 \\ &\quad + R_1(u_1, u_2), \end{aligned} \quad (27)$$

where $\alpha_0 = -(\rho_0 + \lambda_1)\lambda_2 y_3/(c + y_3)$, $\alpha_1 = \beta(\rho_0 + \lambda_1)y_3^2/x_3^2$, $\alpha_2 = -(\rho_0 + \lambda_1)(\beta y_3/x_3 + (c\lambda_2 - h_0 y_3)/(c + y_3)^2)$, $\alpha_3 = -\beta(\rho_0 + \lambda_1)y_3^2/x_3^3$, $\alpha_4 = 2\beta(\rho_0 + \lambda_1)y_3/x_3^2$, $\alpha_5 = (\rho_0 + \lambda_1)(-\beta/x_3 + c(h_0 + \lambda_2)/(c + y_3)^3)$, and $R_1(u_1, u_2)$ is a power series in (u_1, u_2) with powers $u_1^i u_2^j$ satisfying $i + j \geq 3$.

Making the affine transformations $v_1 = u_1$ and $v_2 = -x_3 u_1 - x_3 u_2$, then system (27) reduces to

$$\begin{aligned} \frac{dv_1}{dt} &= v_2 + \frac{1}{x_3} v_1 v_2, \\ \frac{dv_2}{dt} &= \beta_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_1^2 + \beta_4 v_1 v_2 + \beta_5 v_2^2 \\ &\quad + R_2(v_1, v_2), \end{aligned} \quad (28)$$

where $\beta_0 = -\alpha_0 x_3$, $\beta_1 = x_3(\alpha_2 - \alpha_1)$, $\beta_2 = \alpha_2 - x_3$, $\beta_3 = x_3(\alpha_4 - \alpha_5 - \alpha_3)$, $\beta_4 = \alpha_4 - 2\alpha_5 - 1$, $\beta_5 = -\alpha_5/x_3$, and $R_2(v_1, v_2)$ is a power series in (v_1, v_2) with powers $v_1^i v_2^j$ satisfying $i + j \geq 3$.

Consider the C^∞ change of coordinates in the small neighbourhood of $(0, 0)$:

$$\begin{aligned} w_1 &= v_1 - \frac{1}{2x_3} v_1^2, \\ w_2 &= v_2 - \beta_5 v_1 v_2; \end{aligned} \quad (29)$$

then, system (28) reduces to

$$\begin{aligned} \frac{dw_1}{dt} &= w_2 + R_3(w_1, w_2), \\ \frac{dw_2}{dt} &= \gamma_0 + \gamma_1 w_1 + \gamma_2 w_2 + \gamma_3 w_1^2 + \gamma_4 w_1 w_2 \\ &\quad + R_4(w_1, w_2), \end{aligned} \quad (30)$$

where $\gamma_0 = \beta_0$, $\gamma_1 = \beta_1 - \beta_0\beta_5$, $\gamma_2 = \beta_2$, $\gamma_3 = (\beta_1 - \beta_0\beta_5)/2x_3 + \beta_3 - \beta_1\beta_5$, $\gamma_4 = \beta_4$, and $R_3(w_1, w_2)$ and $R_4(w_1, w_2)$ are the power series in (w_1, w_2) with powers $w_1^i w_2^j$ satisfying $i + j \geq 3$.

Consider the C^∞ change of coordinates in the small neighbourhood of $(0, 0)$:

$$\begin{aligned} z_1 &= w_1, \\ z_2 &= w_2 + R_3(w_1, w_2); \end{aligned} \quad (31)$$

then, system (30) reduces to

$$\begin{aligned} \frac{dz_1}{dt} &= z_2, \\ \frac{dz_2}{dt} &= \gamma_0 + \gamma_1 z_1 + \gamma_3 z_1^2 + F_1(z_1) + \gamma_2 z_2 + \gamma_4 z_1 z_2 \\ &\quad + z_2 F_2(z_1) + z_2^2 F_3(z_1, z_2), \end{aligned} \quad (32)$$

where F_1, F_2 , and F_3 are the power series in z_1 and (z_1, z_2) with powers $z_1^{k_1}, z_2^{k_2}$, and $z_1^i z_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$, and $i + j \geq 1$, respectively.

Applying the Malgrange Preparation theorem [25], we have

$$\begin{aligned} \gamma_0 + \gamma_1 z_1 + \gamma_3 z_1^2 + F_1(z_1) \\ = \left(z_1^2 + \frac{\gamma_1}{\gamma_3} z_1 + \frac{\gamma_0}{\gamma_3} \right) B(z_1, \lambda), \end{aligned} \quad (33)$$

where $B(0, \lambda) = \gamma_3$ and B is a power series in z_1 whose coefficients depend on parameters (λ_1, λ_2) .

The sign of γ_3 is tedious to determine. So, we use numerical computation. Here, we consider $\gamma_3 > 0$ as $\lambda_1, \lambda_2 \rightarrow 0$. We take the transformation

$$\begin{aligned} X_1 &= z_1, \\ X_2 &= \frac{z_2}{\sqrt{\gamma_3}}, \\ dT &= \sqrt{\gamma_3} dt; \end{aligned} \quad (34)$$

then, system (32) reduces to

$$\begin{aligned} \frac{dX_1}{dT} &= X_2, \\ \frac{dX_2}{dT} &= \frac{\gamma_0}{\gamma_3} + \frac{\gamma_1}{\gamma_3} X_1 + X_1^2 + \frac{\gamma_2}{\sqrt{\gamma_3}} X_2 + \frac{\gamma_4}{\sqrt{\gamma_3}} X_1 X_2 \\ &\quad + S_1(X_1, X_2, \lambda), \end{aligned} \quad (35)$$

where $S_1(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^i X_2^j$ satisfying $i + j \geq 3$ and $j \geq 2$.

Applying the parameter dependent affine transformations $Y_1 = X_1 + \gamma_1/2\gamma_3$ and $Y_2 = X_2$ in system (35), we obtain

$$\begin{aligned} \frac{dY_1}{dT} &= Y_2, \\ \frac{dY_2}{dT} &= \frac{\gamma_0}{\gamma_3} - \frac{\gamma_1^2}{4\gamma_3^2} + \frac{1}{\sqrt{\gamma_3}} \left(\gamma_2 - \frac{\gamma_1\gamma_4}{2\gamma_3} \right) Y_2 + Y_1^2 \\ &\quad + \frac{\gamma_4}{\sqrt{\gamma_3}} Y_1 Y_2 + S_2(Y_1, Y_2, \lambda), \end{aligned} \quad (36)$$

where $S_2(Y_1, Y_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ and $j \geq 2$.

By means of C^∞ transformation,

$$\begin{aligned} X &= \frac{\gamma_4^2}{\gamma_3} Y_1, \\ Y &= \frac{\gamma_4^3}{(\gamma_3)^{3/2}} Y_2, \\ dT &= \frac{\sqrt{\gamma_3}}{\gamma_4} d\tau. \end{aligned} \quad (37)$$

System (36) reduces to

$$\begin{aligned} \frac{dX}{d\tau} &= Y, \\ \frac{dY}{d\tau} &= \frac{\gamma_0\gamma_4^4}{\gamma_3^3} - \frac{\gamma_1^2\gamma_4^4}{4\gamma_3^4} + \left(\frac{\gamma_2\gamma_4}{\gamma_3} - \frac{\gamma_4^2\gamma_1}{2(\gamma_3)^2} \right) Y + X^2 \\ &\quad + XY + S_3(X, Y, \lambda), \end{aligned} \quad (38)$$

where $S_3(X, Y, 0)$ is a power series in (X, Y) with powers $X^i Y^j$ satisfying $i + j \geq 3$ and $j \geq 2$.

The system above is topologically equivalent to the normal form of the Bogdanov-Takens bifurcation which is given by

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2) y + x^2 + xy. \end{aligned} \quad (39)$$

The system undergoes a Bogdanov-Takens bifurcation if $J(\mu_1, \mu_2/\lambda_1, \lambda_2)_{(\lambda_1=0, \lambda_2=0)} \neq 0$. When system (5) undergoes Bogdanov-Takens bifurcation at $\lambda_1 = \lambda_2 = 0$, three bifurcation curves in $\lambda_1\lambda_2$ plane through the BT point are given by

- (1) saddle-node bifurcation curve: $SN = \{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0, \mu_2(\lambda_1, \lambda_2) \neq 0\}$;
- (2) Hopf bifurcation curve: $H = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = \sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_1(\lambda_1, \lambda_2) < 0\}$;
- (3) Homoclinic bifurcation curve: $HL = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = (5/7)\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_1(\lambda_1, \lambda_2) < 0\}$.

□

5. Numerical Simulation Results

In this section, we will present the numerical simulation results which will support our analytical findings.

(a) $\beta = 0.1$, $c = 0.01$, and $h = 0.5$. System (5) has two interior equilibrium points $E_1 = (0.435558, 0.564442)$ and $E_2 = (0.210806, 0.789194)$, trivial equilibrium point $E_0 = (0, 0)$, and an axial equilibrium point $e = (1, 0)$. If $\rho = 0.75$, the interior equilibrium point E_2 is asymptotically stable (see Figure 2(a)) and if $\rho = 0.9$, the interior equilibrium point E_2 is unstable (see Figure 2(b)). The axial equilibrium point e is an asymptotically stable point and the interior equilibrium point E_1 is always a saddle point. If $h = 0.01$ and $\rho = 0.9$, system (5) has only one positive interior equilibrium point $E_4 = (0.091818, 0.908182)$ which is globally asymptotically stable (see Figure 2(c)). If $h = 0.05$ and $\rho = 0.9$, system (5) has only one positive interior equilibrium point $E_5 = (0.0913611, 0.908639)$ which is also globally asymptotically stable (see Figure 2(d)). $\beta = 0.1$, $c = 0.01$, and $\bar{h} = 0.544367 < h = 0.6$. System (5) has no positive interior equilibrium points and the phase portrait diagram is shown in Figure 2(e). The trivial equilibrium point is always a saddle.

(b) $\beta = 0.1$, $c = 0.01$, $h = 0.5$, and then $\rho^* = 0.865973$. System (5) undergoes Hopf bifurcation and the first Lyapunov coefficient $\sigma = 214.922 > 0$, so an unstable limit cycle arises around the equilibrium point $E_2 = (0.210806, 0.789194)$. Here, we take $\rho = 0.854$ (see Figure 3(a)). $\beta = 0.02$, $c = 0.1$, $h = 0.8$, and then $\rho^* = 0.173169$. System (5) undergoes Hopf bifurcation and the first Lyapunov coefficient $\sigma = -762.219 < 0$, so a stable limit cycle arises around the equilibrium point $E_2 = (0.0867959, 0.913204)$. Here, we take $\rho = 0.175$ (see Figure 3(b)).

(c) $\beta = 0.5$, $c = 0.1$, $\rho = 0.4$, and $\bar{h} = 0.33341$. System (5) has a double interior equilibrium point $\bar{E} = (0.60553, 0.39447)$ which is a saddle node (see Figure 4(a)). The phase portrait diagram is shown in Figure 4(a) and the saddle-node bifurcation is shown in Figures 4(b) and 4(c).

(d) $\beta = 0.12$ and $c = 0.1$; then, we obtain $h_0 = 0.583001$ and $\rho_0 = 0.781849$. Thus, system (5) is

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= (0.781849 + \lambda_1) \\ &\quad \cdot \left(1 - \frac{0.12y}{x} - \frac{(0.583001 + \lambda_2)}{0.1 + y} \right) y. \end{aligned} \quad (40)$$

System (40) has a unique positive interior equilibrium point $E_3 = (0.343303, 0.656697)$.

Translating the equilibrium point E_3 into the origin by using the transformations $u_1 = x - 0.343303$ and $u_2 = y - 0.656697$, by Taylor's series expansion, system (40) reduces to

$$\begin{aligned} \frac{du_1}{dt} &= -0.343303u_1 - 0.343303u_2 - u_1^2 - u_1u_2, \\ \frac{du_2}{dt} &= \alpha_0(\lambda) + \alpha_1(\lambda)u_1 + \alpha_2(\lambda)u_2 + \alpha_3(\lambda)u_1^2 \\ &\quad + \alpha_4(\lambda)u_1u_2 + \alpha_5(\lambda)u_2^2 + R_1(u_1, u_2), \end{aligned} \quad (41)$$

where $\alpha_0(\lambda) = -0.678525\lambda_2 - 0.867847\lambda_1\lambda_2$, $\alpha_1(\lambda) = 0.343303 + 0.439092\lambda_1$, $\alpha_2(\lambda) = 0.343303 + 0.439092\lambda_1 - 0.136546\lambda_2 - 0.174645\lambda_1\lambda_2$, $\alpha_3(\lambda) = -1 - 1.27902\lambda_1$, $\alpha_4(\lambda) = 1.04555 + 1.33727\lambda_1$, $\alpha_5(\lambda) = -0.168089 - 0.214989\lambda_1 + 0.18045\lambda_2 + 0.230799\lambda_1\lambda_2$, and $R_1(u_1, u_2)$ is a power series in (u_1, u_2) with powers $u_1^i u_2^j$ satisfying $i + j \geq 3$.

Making the affine transformations $v_1 = u_1$ and $v_2 = -0.343303u_1 - 0.343303u_2$, then system (41) reduces to

$$\begin{aligned} \frac{dv_1}{dt} &= v_2 + \frac{1}{0.343303}v_1v_2, \\ \frac{dv_2}{dt} &= \beta_0(\lambda) + \beta_1(\lambda)v_1 + \beta_2(\lambda)v_2 + \beta_3(\lambda)v_1^2 \\ &\quad + \beta_4(\lambda)v_1v_2 + \beta_5(\lambda)v_2^2 + R_2(v_1, v_2), \end{aligned} \quad (42)$$

where $\beta_0(\lambda) = 0.23294\lambda_2 + 0.297935\lambda_1\lambda_2$, $\beta_1(\lambda) = -0.0468767\lambda_2 - 0.0599562\lambda_1\lambda_2$, $\beta_2(\lambda) = 0.439092\lambda_1 - 0.136546\lambda_2 - 0.174645\lambda_1\lambda_2$, $\beta_3(\lambda) = 0.759948 + 0.971988\lambda_1 - 0.0619491\lambda_2 - 0.0792341\lambda_1\lambda_2$, $\beta_4(\lambda) = 0.381724 + 1.76725\lambda_1 - 0.3609\lambda_2 - 0.461599\lambda_1\lambda_2$, and $\beta_5(\lambda) = 0.489622 + 0.626236\lambda_1 - 0.525629\lambda_2 - 0.67229\lambda_1\lambda_2$.

Performing the C^∞ change of coordinates $w_1 = v_1 - (1/2x_3)v_1^2$, $w_2 = v_2 - \beta_5v_1v_2$ and $z_1 = w_1$, $z_2 = w_2 + R_3(w_1, w_2)$ in the small neighbourhood of $(0, 0)$, system (42) reduces to

$$\begin{aligned} \frac{dz_1}{dt} &= z_2, \\ \frac{dz_2}{dt} &= \gamma_0(\lambda) + \gamma_1(\lambda)z_1 + \gamma_3(\lambda)z_1^2 + F_1(z_1) \\ &\quad + \gamma_2(\lambda)z_2 + \gamma_4(\lambda)z_1z_2 + z_2F_2(z_1) \\ &\quad + z_2^2F_3(z_1, z_2), \end{aligned} \quad (43)$$

where $\gamma_0(\lambda) = 0.23294\lambda_2 + 0.297935\lambda_1\lambda_2$, $\gamma_1(\lambda) = -0.160929\lambda_2 - 0.351707\lambda_1\lambda_2 - 0.186578\lambda_1^2\lambda_2 + 0.12244\lambda_2^2 + 0.313206\lambda_2\lambda_2^2 + 0.200298\lambda_1^2\lambda_2^2$, $\gamma_2(\lambda) = 0.439092\lambda_1 - 0.136546\lambda_2 - 0.174645\lambda_1\lambda_2$, $\gamma_3(\lambda) = 0.759948 + 0.971988\lambda_1 - 0.273381\lambda_2 - 0.532762\lambda_1\lambda_2 - 0.234192\lambda_1^2\lambda_2 + 0.153687\lambda_2^2 + 0.393136\lambda_1\lambda_2^2 + 0.251414\lambda_1^2\lambda_2^2$, $\gamma_4(\lambda) = 0.381724 + 1.76725\lambda_1 - 0.3609\lambda_2 - 0.461599\lambda_1\lambda_2$, and F_1, F_2 , and F_3 are the power series in z_1 and (z_1, z_2) with powers $z_1^{k_1} z_2^{k_2}$, and $z_1^i z_2^j$ satisfying $k_1 \geq 3$, $k_2 \geq 2$, and $i + j \geq 1$, respectively.

Applying the Malgrange Preparation theorem, we have

$$\begin{aligned} &\gamma_0 + \gamma_1z_1 + \gamma_3z_1^2 + F_1(z_1) \\ &= \left(z_1^2 + \frac{\gamma_1}{\gamma_3}z_1 + \frac{\gamma_0}{\gamma_3} \right) B(z_1, \lambda), \end{aligned} \quad (44)$$

where $B(0, \lambda) = \gamma_3$ and B is a power series in z_1 whose coefficients depend on parameters (λ_1, λ_2) .

FIGURE 2: For ((a) and (b)) $\beta = 0.1, c = 0.01, h = 0.5$. (a) $\rho = 0.75$: the interior equilibrium point $E_2 = (0.210806, 0.789194)$ is asymptotically stable, axial equilibrium point $e = (1, 0)$ is asymptotically stable, and interior equilibrium point $E_1 = (0.435558, 0.564442)$ is a saddle point. (b) $\rho = 0.9$: the point E_2 is unstable, e is asymptotically stable, and E_1 is a saddle. (c) $\beta = 0.1, c = 0.01, h = 0.01$, and $\rho = 0.9$: the interior equilibrium point $E_4 = (0.091818, 0.908182)$ is asymptotically stable; axial equilibrium point $e = (1, 0)$ is saddle. (d) $\beta = 0.1, c = 0.01, h = 0.005$, and $\rho = 0.9$: the interior equilibrium point $E_5 = (0.0913611, 0.908639)$ is asymptotically stable; axial equilibrium point $e = (1, 0)$ is saddle. (e) The phase portrait diagram when no interior equilibrium exists. The origin is always a saddle.

FIGURE 3: (a) $\beta = 0.1$, $c = 0.01$, $h = 0.5$, $\rho^* = 0.865973$, and $\rho = 0.854$: an unstable limit cycle arises around the interior equilibrium point E_2 . (b) $\beta = 0.02$, $c = 0.1$, $h = 0.8$, $\rho^* = 0.173169$, and $\rho = 0.175$: a stable limit cycle arises around the interior equilibrium point E_2 .

FIGURE 4: $\beta = 0.5$, $c = 0.1$, $\bar{h} = 0.33341$, and $\rho = 0.4$. System (5) has a double interior equilibrium point E_3 . (a) The point E_3 is stable from above of the saperatrix curve and a saddle from below. ((b) and (c)) Saddle-node bifurcation diagram.

We have $\gamma_3 = 0.759948 > 0$ as $\lambda_1, \lambda_2 \rightarrow 0$. We take the transformation

$$\begin{aligned} X_1 &= z_1, \\ X_2 &= \frac{z_2}{\sqrt{\gamma_3}}, \\ dT &= \sqrt{\gamma_3} dt; \end{aligned} \quad (45)$$

then, system (43) reduces to

$$\begin{aligned} \frac{dX_1}{dT} &= X_2, \\ \frac{dX_2}{dT} &= \frac{\gamma_0}{\gamma_3} + \frac{\gamma_1}{\gamma_3} X_1 + X_1^2 + \frac{\gamma_2}{\sqrt{\gamma_3}} X_2 + \frac{\gamma_4}{\sqrt{\gamma_3}} X_1 X_2 \\ &\quad + S_1(X_1, X_2, \lambda), \end{aligned} \quad (46)$$

where $S_1(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^i X_2^j$ satisfying $i + j \geq 3$ and $j \geq 2$.

Applying the parameter dependent affine transformations $Y_1 = X_1 + \gamma_1/2\gamma_3$ and $Y_2 = X_2$ in system (46), we obtain

$$\begin{aligned} \frac{dY_1}{dT} &= Y_2, \\ \frac{dY_2}{dT} &= \frac{\gamma_0}{\gamma_3} - \frac{\gamma_1^2}{4\gamma_3^2} + \frac{1}{\sqrt{\gamma_3}} \left(\gamma_2 - \frac{\gamma_1\gamma_4}{2\gamma_3} \right) Y_2 + Y_2^2 \\ &\quad + \frac{\gamma_4}{\sqrt{\gamma_3}} Y_1 Y_2 + S_2(Y_1, Y_2, \lambda), \end{aligned} \quad (47)$$

where $S_2(Y_1, Y_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ and $j \geq 2$.

Consider the C^∞ transformation

$$\begin{aligned} X &= \frac{\gamma_2^2}{\gamma_3} Y_1, \\ Y &= \frac{\gamma_4^3}{(\gamma_3)^{3/2}} Y_2, \\ dT &= \frac{\sqrt{\gamma_3}}{\gamma_4} d\tau. \end{aligned} \quad (48)$$

Then, system (47) reduces to

$$\begin{aligned} \frac{dX}{d\tau} &= Y, \\ \frac{dY}{d\tau} &= \mu_1 + \mu_2 Y + X^2 + XY + S_3(X, Y, \lambda), \end{aligned} \quad (49)$$

where $\mu_1 = \gamma_0\gamma_4^4/\gamma_3^3 - \gamma_1^2\gamma_4^4/4\gamma_3^4$, $\mu_2 = (\gamma_2\gamma_4/\gamma_3 - \gamma_4^2\gamma_1/2(\gamma_3)^2)$, and $S_3(X, Y, 0)$ is a power series in (X, Y) with powers $X^i Y^j$ satisfying $i + j \geq 3$ and $j \geq 2$. Also $J(\mu_1, \mu_2/\lambda_1, \lambda_2)_{(\lambda_1=0, \lambda_2=0)} = -0.154391$. Thus, system (40) undergoes Bogdanov-Takens bifurcation.

The local representations of the bifurcation curves are as follows:

- (a) SN = $\{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0, \mu_2(\lambda_1, \lambda_2) \neq 0\}$.
- (b) H = $\{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = \sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_1(\lambda_1, \lambda_2) < 0\}$.
- (c) HL = $\{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = (5/7)\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_1(\lambda_1, \lambda_2) < 0\}$.

These bifurcation curves are sketched in a small neighbourhood of the origin in the $\lambda_1\lambda_2$ plane (see Figure 5(a)). The bifurcation curves divide the parameter plane into four parts: I, II, III, and IV. The saddle-node bifurcation curve (SN) is the horizontal axis, that is, $\lambda_2 = 0$ axis, and the homoclinic bifurcation curve (HL) and Hopf bifurcation curve (H) lie in the fourth quadrant. When $(\lambda_1, \lambda_2) = (0, 0)$, then system (5) has a unique interior equilibrium point which is a cusp of codimension 2 (see Figure 5(b)). When the parameters λ_1 and λ_2 vary and lie in region I, then system (5) has no interior equilibrium point and all the solution trajectories go to the axial equilibrium point; that is, the axial equilibrium point is globally stable (see Figure 5(c)). Hence, in this case, the predator will go extinct and prey will approach the carrying capacity. When the parameters λ_1 and λ_2 pass the SN bifurcation curve and lie in region II, the cusp of codimension 2 breaks into a hyperbolic saddle and an unstable focus (see Figure 5(d)). When the parameters λ_1 and λ_2 lie in region III, the cusp of codimension 2 breaks into a stable focus and a hyperbolic saddle. The change of stability of the focus yields an unstable limit cycle (see Figure 5(e)). Further, when the parameters λ_1 and λ_2 lie in region IV, the cusp of codimension 2 breaks into a saddle point and a stable focus (see Figure 5(f)). Hence, an open set of initial population densities exists for which both predator and prey approach a stable steady state.

6. Discussion and Conclusion

In this paper, a Leslie-Gower predator-prey model has been analyzed in the presence of nonlinear predator harvesting. It is shown that the system has at most four equilibrium points in Ω , in which the trivial equilibrium point and the axial equilibrium point always exist while the positive interior equilibrium point changes from two to zero. The qualitative properties of solutions in the vicinity of $(0, 0)$ have been studied by using a blow-up technique, and it is found that the origin will never be stable; ecologically speaking, the two species cannot go to extinction together. The axial equilibrium point is either asymptotically stable or globally stable or a saddle depends on the parametric conditions. Thus, ecologically speaking, the prey species never goes to extinction whatever the choice of initial population density. If two interior equilibrium points exist, then one is always a saddle point while the other is either asymptotically stable or unstable or the system undergoes a Hopf bifurcation around this point; that is, for certain parametric domain, the system exhibits a bistability phenomenon as well as oscillatory coexistence of both the populations. The stability of limit cycles has been studied and validated through numerical simulations by calculating the first Lyapunov number. It is also found that for certain parametric domain the proposed

FIGURE 5: $\beta = 0.12$, $c = 0.1$, $h_0 = 0.583001$, and $\rho_0 = 0.781849$. (a) BT bifurcation diagram in the $\lambda_1\lambda_2$ plane. (b) The equilibrium point E_3 is a cusp when $(\lambda_1, \lambda_2) = (0, 0)$. (c) No interior equilibrium point exists when $(\lambda_1, \lambda_2) = (0.04, 0.001)$ lies in region I. (d) An unstable focus when $(\lambda_1, \lambda_2) = (0.03, -0.001)$ lies in region II. (e) An unstable limit cycle when $(\lambda_1, \lambda_2) = (0.03, -0.002)$ lies in region III. (f) A stable focus when $(\lambda_1, \lambda_2) = (0.03, -0.004)$ lies in region IV.

system has a unique interior equilibrium which is globally asymptotically stable.

It is shown that the system can have zero, one, or two interior equilibrium points through saddle-node bifurcation as the bifurcation parameter h crosses a certain threshold value. By means of Sotomayor's theorem, the existence of saddle-node bifurcation has been shown. Ecologically speaking, a maximum threshold value of h exists, below which both species coexist and above which predator species suddenly collapse to extinction. The proposed system undergoes Bogdanov-Takens bifurcation near degenerate equilibria. We have considered the parameters h and ρ as bifurcation parameters and reduced the system into normal form. Ecologically speaking, a small perturbation in the bifurcation parameters is a cause of extinction, coexistence, and oscillation.

The dynamical analysis of model (5) shows complex and rich dynamics as compared to the model with no harvesting [21] and with constant-yield predator harvesting [19]. The model without harvesting has a unique interior equilibrium point which is globally asymptotically stable [21] while model (5) has zero to two interior equilibrium points and model (5) undergoes a series of bifurcations (Hopf bifurcation, saddle-node bifurcation, and Bogdanov-Takens bifurcation). The model with constant-yield predator harvesting has no axial equilibrium point, and the solutions once touching the x -axis will leave the first quadrant [21]. But the model with nonlinear predator harvesting has one trivial equilibrium point $E_0 = (0, 0)$ and one axial equilibrium point $e = (1, 0)$ and the solutions touching the x -axis will go to the axial equilibrium point e . Further, if no interior equilibrium point exists, then the axial equilibrium point e is globally asymptotically stable (see Figure 2(e)). Ecologically speaking, in the absence of predator species, the prey density approaches the carrying capacity. Although the model with constant-yield predator harvesting undergoes a series of bifurcations, like Hopf bifurcation, saddle-node bifurcation, and Bogdanov-Takens bifurcation [19], the model with nonlinear harvesting gives more general parametric conditions for the occurrence of these bifurcations. Thus, the results of nonlinear harvesting model can explain the real life situation in a more effective and realistic manner.

Competing Interests

The authors declare that they have no competing interests.

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