Research Article

Characterizations of $N(2, 2, 0)$ Algebras

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The so-called ideal and subalgebra and some additional concepts of $N(2, 2, 0)$ algebras are discussed. A partial order and congruence relations on $N(2, 2, 0)$ algebras are also proposed, and some properties are investigated.

1. Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [1, 2]. Recently, some authors studied fuzzy implications from different perspectives [3]. Naturally, it is meaningful to investigate the common properties of some important fuzzy implications used in fuzzy logic. Consequentially, Professor Wu [4] introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990.

In the past two decades, some authors focused on FI-algebras. Various interesting properties of FI-algebras [5, 6], regular FI-algebras [4, 7, 8], commutative FI-algebras [9], W_d -FI-algebras [10], and other kinds of FI-algebras [11] were reported, and some concepts of filter, ideal, and fuzzy filter of FI-algebras were proposed [4, 12–18]. Relationships between FI-algebra and BCK-algebra [19, 20], MV-algebras [21], and Rough set algebras [22, 23] were partly investigated, and FI-algebras were axiomatized [24]. In the recent work, the relationships between these FI-algebras and several famous fuzzy algebras were systematically discussed [25–30].

In 1900, Hilbert proposed the famous problem H_{10} , which is the tenth problem of Hilbert. H_{10} argues whether there is a useful approach to determine that the general Diophantine equation is solvable. Until 1970, this problem had been solved by Y. Matiyasevich. It is clear that there is not a method to judge whether the Diophantine equation can be solved. In 1987, Siekmann and Szabó studied the unification

problem related to H_{10} . It was concluded that unification problem of D_A -rewriting systems cannot be predicated. And it was stated that, for every axiomatic system ψ satisfying $D_A \subseteq \psi \subseteq H_{10}$, it is impossible to determine the problem of ψ -unification problem $\langle S = t \rangle_\psi$, $s, t \in T_1$, where T_1 is a set of term combined by the symbols $\oplus$ and $\otimes$ [31].

Our work constructed a new algebra system. It was called the $N(2, 2, 0)$ algebra, which is more general than the D_A -rewriting systems. We studied the basic properties of $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$. In addition, if the operation $*$ is idempotent, then $(S, *, \Delta, 0)$ is a rewriting system. In this case, it is impossible to determine the problem of $N(2, 2, 0)$ algebra unification problem. Analogously, if the operation Δ is nilpotent, then $(S, \Delta, 0)$ is associated BCI-algebra.

The present paper is organized as follows: In Section 2, we review some basic concepts and the main properties of $N(2, 2, 0)$ algebras and show relationship between the $N(2, 2, 0)$ algebras and the related fuzzy logic algebras. Then Sections 3–6 contain main results of inverse semigroup of $N(2, 2, 0)$ algebra, partial order on semigroup of $N(2, 2, 0)$ algebra, congruence on semigroup of $N(2, 2, 0)$ algebra, and ideal and subalgebra of $N(2, 2, 0)$ algebra which will be considered, respectively.

2. Basis Concepts and Properties of $N(2, 2, 0)$ Algebras

In this part, we firstly review some relevant concepts and definitions.

Definition 1 (see [4]). Let X be a universe set, and $0 \in X$, and let $\rightarrow$ be a binary operation on X . A $(2, 0)$ -type algebra $(X, \rightarrow, 0)$ is called a fuzzy implication algebra, shortly, FI-algebra, if the following five conditions hold for all $x, y, z \in X$:

- (I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
 - (I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;
 - (I3) $x \rightarrow x = 1$;
 - (I4) if $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;
 - (I5) $0 \rightarrow x = 1$,
- where $1 = 0 \rightarrow 0$.

Then $N(2, 2, 0)$ algebra is an algebra of type $(2, 2, 0)$. The notion was first formulated in 1996 by Deng and Xu and some properties were obtained (see [32]). This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu (see [4]). He proved that, in a fuzzy implication algebra $(X, \rightarrow, 0)$, the order relation $\leq$ satisfying $x \leq y$ if and only if $x \rightarrow y = 1$ is a partial order. In [32], Deng and Xu introduced a binary operation $*$ defined on fuzzy implication algebra $(X, \rightarrow, 0)$, such that, for all $a, b, u \in X$,

$$u \leq a \rightarrow b \iff a * u \leq b, \quad (1)$$

where $(*, \rightarrow)$ is an adjoint pair on X .

In the corresponding fuzzy logic, the operation $*$ is recognized as logic connective “conjunction” and $\rightarrow$ is considered as “implication.” If the above expression holds for a product $*$, then $\rightarrow$ is the residuum of $*$. For a product $*$ the corresponding residuum $\rightarrow$ is uniquely defined by

$$a \rightarrow b = \bigvee \{x \mid a * x \leq b\}. \quad (2)$$

Let us note that $a \rightarrow b$ is the greatest element of the set $\{u \mid a * u \leq b\}$.

We proved that if $\forall a, b, u \in X$, the following hold:

$$\begin{aligned} u \rightarrow (a * b) &= b \rightarrow (u \rightarrow a), \\ (a * u) \rightarrow b &= u \rightarrow (a \rightarrow b), \end{aligned} \quad (3)$$

then $(X, *)$ is a semigroup.

In fact, the multiplication defined as above is associative.

It was shown in [4] that, for a fuzzy implication algebra $(X, \rightarrow, 0)$, considering any $a \in X$, there is $1 \rightarrow a = a$:

$$\begin{aligned} a * (b * c) &= 1 \rightarrow \\ (a * (b * c)) &= (b * c) \rightarrow \\ (1 \rightarrow a) &= (b * c) \rightarrow \\ a = c &\rightarrow (b \rightarrow a), \\ (a * b) * c &= 1 \rightarrow \\ ((a * b) * c) &= c \rightarrow \\ (1 \rightarrow (a * b)) &= c \rightarrow \\ (a * b) &= b \rightarrow (c \rightarrow a). \end{aligned} \quad (4)$$

Since $b \rightarrow (c \rightarrow a) = c \rightarrow (b \rightarrow a)$, we have $(a * b) * c = a * (b * c)$. So it can be concluded that $(X, *)$ is a semigroup.

By generalizing the expressions (3), we obtain the basic equations of $N(2, 2, 0)$ algebra.

Definition 2 (see [32]). $N(2, 2, 0)$ algebra is a nonempty set S with a constant 0 and two binary operations $*, \Delta$, for all $a, b, c \in S$, satisfying the axioms:

- (N₁) $a * (b \Delta c) = c * (a * b)$,
- (N₂) $(a \Delta b) * c = b * (a * c)$,
- (N₃) $0 * a = a$.

By substituting $*$ and Δ in expressions (N₁) and (N₂) with $\rightarrow$ and $*$, respectively, we arrive at the expressions (3).

Theorem 3 (see [32]). Let $(S, *, \Delta, 0)$ be $N(2, 2, 0)$ algebra. Then for all $a, b, c \in S$, the following holds:

- (1) $a * b = b \Delta a$;
- (2) $(a * b) * c = a * (b * c)$, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$;
- (3) $a * (b * c) = b * (a * c)$, $(a \Delta b) \Delta c = (a \Delta c) \Delta b$.

Corollary 4 (see [32, 33]). If $(S, *, \Delta, 0)$ is $N(2, 2, 0)$ algebra, then $(S, *)$ and (S, Δ) are semigroups.

Therefore the $N(2, 2, 0)$ algebra is an algebra system with a pair of dual semigroups. Several interesting properties of $N(2, 2, 0)$ algebra have been discussed earlier (see [32, 33]). Throughout this paper, we will denote the semigroup $(S, *)$ of $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$ by S .

Theorem 5. Let $(S, *, \Delta, 0)$ be $N(2, 2, 0)$ algebra and let the following hold for every x in S : $x * 0 = x$; then $(S, *)$ is a commutative semigroup.

Proof. Since $x, y \in S$, this shows we have $x * y = x * (y * 0) = y * (x * 0) = y * x$. This implies that $x * y = y * x$. This yields $(S, *)$ is a commutative semigroup.

So in $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$, for any x in S , if $x * 0 = x$ holds, then $(S, *)$ and (S, Δ) are the same and are a commutative monoid as well. $\square$

Definition 6 (see [34]). A residuated poset is a structure $(A; \leq, \rightarrow, \cdot, 0, 1)$ such that

- (R₁) $(A; \leq, 0, 1)$ is a bounded poset,
- (R₂) $(A; \cdot, 1)$ is a commutative monoid,
- (R₃) it satisfies the adjointness property; that is,

$$x \cdot y \leq z \iff x \leq y \rightarrow z. \quad (5)$$

Applying Definition 6 and Theorem 5 to $N(2, 2, 0)$ algebra, we immediately obtain the following.

Remark 7. Let $(S, *, \Delta, 0)$ be $N(2, 2, 0)$ algebra. If $(S, *, 0)$ is semigroup with the induced order $\leq$, that is, $x \leq y \iff x \rightarrow y = 1$, for all $x, y \in S$, where $1 = 0 \rightarrow 0$, and $x * 0 = x$, for all $x \in S$, then semigroup $(S, *, 0)$ is a residuated poset.

Remark 8. If a fuzzy implication algebra $(X, \rightarrow, 0)$ is with a partial order “ $\leq$ ”, such that any $a, b, u \in X$, $a \leq b \Leftrightarrow a \rightarrow b = 1$, $u \leq a \rightarrow b \Leftrightarrow a * u \leq b$, and for all $a, b, u \in X$, the following conditions hold:

$$\begin{aligned} u \longrightarrow (a * b) &= b \longrightarrow (u \longrightarrow a), \\ (a * u) \longrightarrow b &= u \longrightarrow (a \longrightarrow b), \end{aligned} \quad (6)$$

then $(X, \rightarrow, *, 0)$ is $N(2, 2, 0)$ algebra.

Remark 9. Let $(S, *, \Delta, 0)$ be $N(2, 2, 0)$ algebra; then semigroups $(S, *)$ and (S, Δ) are a pair of dual semigroup. A pair of dual operations $(*, \Delta)$ form an adjoint pair $(\rightarrow, *)$; that is, $u \leq a \rightarrow b \Leftrightarrow a * u \leq b$, for every $a, b, u \in S$.

Theorem 10. Let $(S, *, \Delta, 0)$ be $N(2, 2, 0)$ algebra and for every x, y, z in S , then the following hold:

$$\begin{aligned} x\Delta(y * z) &= y * (x\Delta z), \\ x * (y\Delta z) &= y\Delta(x * z). \end{aligned} \quad (7)$$

Proof. With Theorem 3, we have $x\Delta y = y * x$ in S . Hence, $x\Delta(y * z) = (y * z) * x = y * (z * x) = y * (x\Delta z)$. Similarly, $x * (y\Delta z) = y\Delta(x * z)$. $\square$

It is well known that partial ordering and congruences play an important role in the theory of semigroups. In this paper, we have given a partial order relation and a congruence on semigroup of $N(2, 2, 0)$ algebra.

Definition 11 (see [35]). A semigroup S is called

- (i) medial if it satisfies the equation $xyz u = xzy u$;
- (ii) trimedial (dimedial, resp.) if every subsemigroup of S generated by at most three (two, resp.) elements is medial;
- (iii) left (right or middle, resp.) semimedial if it satisfies the identity $x^2 y z = x y x z (z y x^2 = z x y x, x y z x = x z y x, \text{ resp.})$;
- (iv) semimedial if it is both left and right semimedial;
- (v) strongly semimedial if it is semimedial and middle semimedial;
- (vi) exponential $(xy)^n = x^n y^n$ for every positive integer n ;
- (vii) left (right, resp.) distributive if it satisfies $xyz = xyxz (zyx = zxyx)$.

Theorem 12. (1) Every semigroup $(S, *)$ of $N(2, 2, 0)$ algebra is medial and semimedial.

(2) Every semigroup $(S, *)$ of $N(2, 2, 0)$ algebra is strongly semimedial.

(3) Every medial semigroup $(S, *)$ of $N(2, 2, 0)$ algebra is exponential.

Proof. It is easy. $\square$

3. On Inverse Semigroup of $N(2, 2, 0)$ Algebra

We will assume the reader is familiar with elementary inverse semigroup theory as described in [36]. Some properties of inverse semigroup have been discovered in literature (see [36, 37]). We present here some related definitions and examples of inverse semigroup of $N(2, 2, 0)$ algebra, as follows.

Definition 13. A semigroup $(S, *)$ of $(N, 2, 2, 0)$ algebra $(S, *, \Delta, 0)$ is called an inverse semigroup if for every $a \in S$ there exists $x \in S$ such that $(a * x) * a = a$ and $(x * a) * x = x$. Henceforth, we define x as the inverse of a .

Theorem 14. The inverses of semigroup $(S, *)$ of $(N, 2, 2, 0)$ algebra are not unique.

Proof. If a' is an inverse of a and b' is an inverse of b then $((a * b) * (a' * b')) * (a * b) = ((a * a') * (b * b')) * (a * b) = ((a * a') * a) * ((b * b') * b) = a * b$ and $((a' * b') * (a * b)) * (a' * b') = ((a' * a) * (b' * b)) * (a' * b') = ((a' * a) * a') * a * ((b' * b) * b') = a' * b'$ imply that $(a * b)' = a' * b'$.

If x and y are both inverse of a , then we can show that

$$\begin{aligned} x * a &= x * ((a * y) * a) = (a * y) * (x * a) \\ &= a * (y * (x * a)) = y * (a * (x * a)) \\ &= y * ((a * x) * a) = y * a. \end{aligned} \quad (8)$$

If $a * x = a * y$, then

$$\begin{aligned} x &= (x * a) * x = (y * a) * x = y * (a * x) \\ &= y * (a * y) = (y * a) * y = y. \end{aligned} \quad (9)$$

In general, the expression $x * a = y * a$ does not hold. In this case, the inverse is not unique. $\square$

We will give some examples in the next section.

Example 15. Let $S = \{0, a, b, c, d, e\}$ be equipped with the operation $*$ defined by the following Cayley's table:

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	a	b	c	d	e
b	b	b	b	b	b	b
c	c	c	b	c	b	e
d	b	b	b	b	b	b
e	e	e	b	e	b	e
Δ	0	a	b	c	d	e
0	0	0	b	c	b	e
a	a	a	b	c	b	e
b	b	b	b	b	b	b
c	c	c	b	c	b	e
d	d	d	b	b	b	b
e	e	e	b	e	b	e

(10)

Then $(S, *, \Delta, 0)$ is $N(2, 2, 0)$ algebra, but $(S, *)$ is not an inverse semigroup, because a, b are inverse of a or b , and c is inverse of c , and d is inverse of d , but e has no inverse.

Example 16. Let $S = \{0, a, b, c, d\}$ be equipped with the operation $*$ defined by the following Cayley's table:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	0	a	b	c	d
c	0	a	b	c	d
d	0	a	b	c	d
Δ	0	a	b	c	d
0	0	a	0	0	0
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	c	c	c
d	d	a	d	d	d

It is clear that the above system $(S, *)$ forms an inverse semigroup. Obviously, a, c, d are inverse elements of a or c or d , and 0 is inverse of 0 , and b is inverse of b in $(S, *)$.

Clearly $(S, *, \Delta, 0)$ is $N(2, 2, 0)$ algebra, and $(S, *)$ is an inverse semigroup.

4. Partial Order on Semigroup of $N(2, 2, 0)$ Algebra

We will now investigate the partial order on semigroup of $N(2, 2, 0)$ algebra.

If a, b are elements of a semigroup $(S, *)$ in $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$, then by $a \leq b$ we mean $a = e * b$, where e is an idempotent in S .

If e and f are idempotents in S , we have

$$\begin{aligned} e * f &= (e * e) * (f * f) = e * ((e * f) * f) \\ &= (e * f) * (e * f) = (e * f)^2, \end{aligned} \quad (12)$$

by repeatedly using $a * (b * c) = b * (a * c)$. That means the product of idempotents is an idempotent.

Theorem 17. *The relation $\leq$ is partial order relation on an inverse semigroup $(S, *)$ of $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$.*

Proof. Since $(S, *)$ is an inverse semigroup of $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$, for each $a \in S$, there exists $a' \in S$ such that $a = (a * a') * a$. Now $(a * a') * (a * a') = a * ((a * a') * a') = a * ((a' * a) * a') = a * a'$. This implies that $a * a'$ is an idempotent. Hence $a = e * a$ for $e = a * a'$. So $a \leq a$.

Let $a \leq b$. This implies that $a = e * b$, where e is an idempotent. Let $b \leq a$; then $b = f * a$. If f is an idempotent, we have $e * a = e * (e * b) = (e * e) * b = e * b = a$

and $a = e * b = e * (f * a) = f * (e * a) = f * a = b$. This shows that $\leq$ is antisymmetric.

Let $a \leq b$ and $b \leq c$; then there exist idempotent e and f such that $a = e * b$ and $b = f * c$. So $a = e * b = e * (f * c) = (e * f) * c$. This implies that $a \leq c$ as $e * f$ is an idempotent. $\square$

Theorem 18. *If $(S, *)$ is an inverse semigroup of $N(2, 2, 0)$ algebra $(S, *, \Delta, 0)$, $a, b \in S$, then*

- (1) $a \leq b$ implies $c * a \leq c * b$ and $a * c \leq b * c$, for all $c \in S$,
- (2) $a \leq b$ implies $a' \leq b'$,
- (3) $a \leq b$ and $c \leq d$; then $a * c \leq b * d$.

Proof. (1) If $a \leq b$ then for an idempotent e , $a = e * b$. Now $a = e * b$ implies that $c * a = c * (e * b) = e * (c * b)$. Hence $c * a \leq c * b$. Also, since e is an idempotent, $a * c = (e * b) * c = e * (b * c)$. Hence $a * c \leq b * c$.

(2) Now $a \leq b$ implies that, for an idempotent e , $a = e * b$. So $a' = (e * b)' = e' * b'$. Since $e' = e$ is an idempotent, hence $a' \leq b'$.

(3) Let $a \leq b$ and $c \leq d$. Then we have idempotents e and f such that $a = e * b$ and $c = f * d$. Now $a * c = (e * b) * (f * d) = (e * f) * (b * d)$ implies that $a * c \leq b * d$. This completes the proof. $\square$

Theorem 19. *Let $(S, *)$ and (S, Δ) be two associative bands of $N(2, 2, 0)$ algebra, and $e, a, b \in S$. Then $e * a = e * b$ implies that $a * e = b * e$ and $a \Delta e = b \Delta e$ implies that $e \Delta a = e \Delta b$.*

Proof. Suppose that $e * a = e * b$; then $a * e = a * a * e * e = a * e * a * e = e * a * a * e = e * b * a * e = b * e * a * e = b * e * b * e = b * b * e * e = b * e$.

And in an associative band (S, Δ) , suppose that $a \Delta e = b \Delta e$; then $e \Delta a = e \Delta e \Delta a \Delta a = e \Delta a \Delta e \Delta a = e \Delta a \Delta b \Delta e = e \Delta b \Delta a \Delta e = e \Delta b \Delta b \Delta e = e \Delta e \Delta b \Delta b = e \Delta b$. $\square$

5. Congruence on Semigroup of $N(2, 2, 0)$ Algebra

Analogous characterization of the maximum idempotent-separating congruence on an eventually orthodox semigroup is given (see [38]). As important consequences, some sufficient conditions for an eventually regular subsemigroup T of S to satisfy $(S) \mid T = (T)$ are obtained, whence if S is fundamental, then so is T . Similarly to the treatises by Mitsch [39] and Luo and Li (see [38]), we consider a semigroup S of $N(2, 2, 0)$ algebra. Let $a, b, c \in S$. If apb implies that $capcb$, then the relation ρ is called left compatible. If apb implies that $acpbc$, then the relation ρ is called right compatible. If apb and cpd imply that $acpbd$ then the relation ρ is called compatible. A compatible equivalence relation is called a congruence. In this paper, we have defined a congruence relation on an inverse semigroup S of $N(2, 2, 0)$ algebra.

If ρ is a congruence relation on an inverse semigroup S of $N(2, 2, 0)$ algebra, then we can define a binary operation in S/ρ in a natural way as $(ap) * (bp) = (a * b)\rho$.

If $(ap) = (a_1\rho)$ and $(bp) = (b_1\rho)$ for $a, b, a_1, b_1 \in S$, then we have apa_1 and bpb_1 and hence $abpa_1b_1$. Thus the binary operation is well-defined. S/ρ consists of elements of the form $a\rho$, where $a \in S$.

Let (ap) and (bp) be in S/ρ ; then $(ap) * (bp) = (a * b)\rho$ is in S/ρ . Now if $a = (a * a') * a$ then $(ap) = ((a * a') * a)\rho = ((ap) * (a'\rho)) * (ap)$. If $a' = (a' * a) * a'$, then $(a'\rho) = ((a' * a) * a')\rho = ((a'\rho) * ap) * (a'\rho)$. This shows that ap and $a'\rho$ are inverse of each other. So S/ρ is an inverse semigroup on condition that S is an inverse semigroup.

Let e be an idempotent in an inverse semigroup S ; then ep belongs to S/ρ and $(ep) * (ep) = e^2\rho = ep$ implies that ep is idempotent in S/ρ .

Theorem 20. Let ρ be a congruence relation on an inverse semigroup S of $N(2, 2, 0)$ algebra. If (ap) is an idempotent in S/ρ , then there exists an idempotent $e \in S$ such that $(ap) = (ep)$.

Proof. If (ap) is an idempotent in S/ρ then $(ap) * (ap) = (a^2\rho) = (ap)$. This implies that apa^2 and a^2pa . Let x be an inverse of a^2 in S . Then $(a^2 * x) * a^2 = a^2$ and $(x * a^2) * x = x$.

If $e = (a * x) * a$, then $e^2 = ((a * x) * a) * ((a * x) * a) = x * (a^2 * x * a^2) = x * a^2 = a * x * a = e$. So $e = a * x * a$ is an idempotent. Now apa^2 implies $(a * x)\rho(a^2 * x)$. Again apa^2 and $(a * x)pa^2$ imply that $((a * x) * a)\rho(a^2 * x) * a^2$ or $ep(a^2 * x) * a^2$ or epa^2 . So epa^2 and a^2pa imply that $(ap) = (ep)$. $\square$

6. Ideal and Subalgebra on Semigroup of $N(2, 2, 0)$ Algebra

Recently, Jun and Song (see [40]) considered int-soft (generalized) bi-ideals of semigroups, further properties and characterizations of int-soft left (right) ideals are studied, and the notion of int-soft (generalized) bi-ideals is introduced. Relations between int-soft generalized bi-ideals and int-soft semigroups are discussed, and characterizations of (int-soft) generalized bi-ideals and int-soft bi-ideals are considered. Refer to the literature (see [37, 40–42]); in this paper, we focus our attention on the ideal and subalgebra on semigroup of $N(2, 2, 0)$ algebra.

The semigroup $(S, *)$ of $N(2, 2, 0)$ algebra determines an order relation on S : $a \leq b \Leftrightarrow a * b = 0$, where $0 = a * a$ is an element that does not depend on the choice of $a \in S$.

Definition 21. A nonempty subset A of $N(2, 2, 0)$ algebra S is called an ideal of S if it satisfies (1) $0 \in A$, (2) $(\forall x \in S) (\forall y \in A) (x * y \in A \Rightarrow x \in A)$.

Lemma 22. An ideal A of $N(2, 2, 0)$ algebra S has the following property: $(\forall x \in S) (\forall y \in A) (x \leq y \Rightarrow x \in A)$.

Proof. If $\forall x \in S, \forall y \in A, x \leq y$, then $y = 0 * y = (x * y) * y = x * (y * y) = x * 0$; by Definition 21, we get $x \in A$. $\square$

Definition 23. Let S be $N(2, 2, 0)$ algebra. For any $a, b \in S$, let $G(a, b) = \{x \in S \mid x * a \leq b\}$. Then S is said to be complicated if for any $a, b \in S$, the set $G(a, b)$ has a greatest element.

Theorem 24. Let S be $N(2, 2, 0)$ algebra and let I be a subset of S . Then I is an ideal of S if and only if $G(x, y) \subseteq I$ for all $x, y \in I$.

Proof. $(\Rightarrow)$ I is an ideal and x, y are elements of I . For any $z \in G(x, y)$, we have $z * x \leq y$. Hence we have $z * x \in I$ with Lemma 22. Then we obtain $z \in I$ since I is an ideal.

$(\Leftarrow)$ If $G(x, y) \subseteq I$, for all $x, y \in I$, we have $0 \in I$ since $0 \in G(x, y)$. For any $b \in I$ and $a \in S$, let $a * b \in I$. Then $G(a * b, b) \subseteq I$. Hence, since $a * (a * b) \leq b$, we obtain $a \in G(a * b, b) \subseteq I$. Hence, $a \in I$. $\square$

Theorem 25. Let S be $N(2, 2, 0)$ algebra, for any $a, b, c, d \in S$, defining $G(a, b) = \{x \in S \mid x * a \leq b\}$, the following statements are true:

- (1) If $0 \in G(a, b)$, then $a \leq b$.
- (2) If $a \in G(a, b)$, then $b = 0$.
- (3) If $b \in G(a, b)$, then $a * 0 = 0$.
- (4) If $x \in G(a, b)$, and $y \in G(c, d)$, then $x * y \in G(a * c, b * d)$.

Proof. (1) If $0 \in G(a, b)$, then $a = 0 * a \leq b$, $a * b = (0 * a) * b = 0$, $a \leq b$.

(2) If $a \in G(a, b)$, then $a * a \leq b$, so $0 = (a * a) * b = 0 * b = b$; hence $b = 0$.

(3) Assume that $b \in G(a, b)$ implies $0 = (b * a) * b = b * (a * b) = a * (b * b) = a * 0$. Thus $a * 0 = 0$.

(4) If $x \in G(a, b)$, then $x * a \leq b$; we have $(x * a) * b = 0$, $y \in G(c, d)$; then $y * c \leq d$, and we have $(y * c) * d = 0$; we get $x * y * a * c * b * d = (x * a * b) * (y * c * d) = 0 * 0 = 0$. Hence $x * y \in G(a * c, b * d)$. $\square$

Definition 26. Let S be $N(2, 2, 0)$ algebra and U a nonempty subset of S . Then U is called a subalgebra of S , if $x * y \in U$ and $y * x \in U$ whenever $x, y \in U$.

Theorem 27. Let S be $N(2, 2, 0)$ algebra, for any $x \in S$, if $x * 0 = 0$; then $G(a, b)$ is a subalgebra of S .

Proof. Let $\forall x, y \in G(a, b)$, then $x * a \leq b, y * a \leq b$; that is, $(x * a) * b = 0$ and $(y * a) * b = 0$ imply $((x * y) * a) * b = x * ((y * a) * b) = x * 0 = 0$, so $(x * y) * a \leq b$; that is, $x * y \in G(a, b)$, $y * x \in G(a, b)$ can be proved in a similar way. Therefore, if $x * 0 = 0$, then $G(a, b)$ is a subalgebra of S . $\square$

Theorem 28. Let S be $N(2, 2, 0)$ algebra, T is a subalgebra of S , if the condition $x * 0 = 0, \forall x \in T$, is satisfied, and then any ideal I of T is $N(2, 2, 0)$ algebra.

Proof. For any $x, y \in I$, we have $(x * y) * x = y * (x * x) = y * 0 = 0$, if I is ideal of T ; we obtain $x * y \in I$ with $(x * y) * x = y * (x * x) = (y * x) * x = 0$; we get $y * x \in I$. Hence $x * y, y * x \in I$. So I is $N(2, 2, 0)$ algebra, where I is an ideal of T . $\square$

Theorem 29. Let S be $N(2, 2, 0)$ algebra and $a \in S$. Then the set $H(a) = \{x \in S \mid x * a = a\}$ is ideal of S and is a subalgebra of S as well.

Proof. (1) For any $a \in S$, we have $0 * a = a$, so $0 \in H(a)$. $\forall x \in S, \forall y \in H(a)$, there is $x * y \in H(a)$, and $y * a = a$; we get $a = x * y * a = x * a$, so we have $x \in H(a)$. Hence, $H(a)$ is ideal of S .

(2) If $x \in H(a), y \in H(a)$, then $x * a = a, y * a = a$, then $(x * y) * a = x * (y * a) = x * a = a$, and $x * y \in H(a)$. Similarly it can be proved that $y * x \in H(a)$. Hence, $H(a)$ is a subalgebra of S . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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