Hindawi Publishing Corporation
International Journal of Analysis
Volume 2016, Article ID 3791506, 10 pages
http://dx.doi.org/10.1155/2016/3791506

Research Article

On Faster Implicit Hybrid Kirk-Multistep Schemes for Contractive-Type Operators

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Received 21 March 2016; Revised 9 July 2016; Accepted 26 July 2016

Academic Editor: Lianwen Wang

The purpose of this paper is to prove strong convergence and T-stability results of some modified hybrid Kirk-Multistep iterations for contractive-type operator in normed linear spaces. Our results show through analytical and numerical approach that the modified hybrid schemes are better in terms of convergence rate than other hybrid Kirk-Multistep iterative schemes in the literature.

1. Introduction

In the recent years, numerous papers have been published on the strong convergence and T-stability of various iterative approximations of fixed points for contractive-type operators. See [1–9]. The Picard iterative scheme defined for $x_0 \in X$

$$x_n = Tx_{n-1}, \quad n \geq 1, \tag{1}$$

was the first iteration to be proved by Banach [10] for a self-map T in a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq cd(x, y) \tag{2}$$

(called strict contraction), for all $x, y \in X$ and $c \in (0, 1)$. Picard iteration (1) which obeys (2) is said to have a fixed point in F_T, where F_T is the set of all fixed points. The Picard iteration will no longer converge to a fixed point of the operator if contractive condition (2) is weaker. Hence, there is a need to consider other iterative procedures.

Mann [11] defined a more general iteration in a Banach space E satisfying quasi-nonexpansive operators. For $x_0 \in E$, the Mann iteration is given by

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T x_{n-1}, \quad n \geq 1, \tag{3}$$

where $\{\alpha_n\}$ is a sequence of positive numbers in $[0, 1]$. Putting $\alpha_n = 1$ in (3) yields Picard iteration (1).

A double Mann iteration, called Ishikawa iteration, was introduced by Ishikawa [12]. It is defined for $x_0 \in E$ as

$$\begin{aligned} x_n &= (1 - \alpha_n) x_{n-1} + \alpha_n T y_{n-1} \\ y_{n-1} &= (1 - \beta_n) x_{n-1} + \beta_n T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$.

The three-step iteration, which is more general than Mann and Ishikawa iterations, was defined by Noor [13]. For $x_0 \in E$, the Noor iteration is given as

$$\begin{aligned} x_n &= (1 - \alpha_n) x_{n-1} + \alpha_n T y_{n-1} \\ y_{n-1} &= (1 - \beta_n) x_{n-1} + \beta_n T z_{n-1} \\ z_{n-1} &= (1 - \gamma_n) x_{n-1} + \gamma_n T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\sum \alpha_n = \infty$.

Rhoades and Soltuz [14] defined a multistep iteration in a normed linear space E as follows. For $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)\, x_{n-1} + \alpha_n T y_{n-1}^{(1)} \\ y_{n-1}^{(l)} &= \left(1-\beta_n^{(l)}\right) x_{n-1} + \beta_n^{(l)} T y_{n-1}^{l+1}, \\ &\qquad l = 1, 2, \ldots, k-2 \\ y_{n-1}^{(k-1)} &= (1-\beta_n)\, x_{n-1} + \beta_n T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n^{(l)}\}$, $l = 1, 2, \ldots, k-1$, are sequences of positive numbers in $[0,1]$ with $\sum \alpha_n = \infty$. Iteration (6) generalized (3), (4), and (5); for example, if $k = 3$ in (6), we recover the form (5); if $k = 2$, we have (4); on putting $k = 2$ and $\beta_n^{(l)} = 0$ for each l, we have (3).

Another two-step scheme, which is independent of (4), was introduced by Thianwan [15]. Let $x_0 \in E$; the sequence $\{x_n\} \subset E$ is defined as

$$\begin{aligned} x_n &= (1-\alpha_n)\, y_{n-1} + \alpha_n T y_{n-1} \\ y_{n-1} &= (1-\beta_n)\, x_{n-1} + \beta_n T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\sum \alpha_n = \infty$.

The three-step iteration of (7) called SP iteration was introduced by Phuengrattana and Suantai [16] and it was defined as follows. For $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)\, y_{n-1}^{(1)} + \alpha_n T y_{n-1}^{(1)} \\ y_{n-1}^{(1)} &= (1-\beta_n)\, y_{n-1}^{(2)} + \beta_n T y_{n-1}^{(2)} \\ y_{n-1}^{(2)} &= (1-\gamma_n)\, x_{n-1} + \gamma_n T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\sum \alpha_n = \infty$. The two-step iteration of (8) can be easily obtained when $\gamma_n = 0$.

In [17], Gürsoy et al. defined a generalized scheme of forms (7) and (8) in a Banach space. For $x_0 \in E$,

$$\begin{aligned} x_n &= (1-\alpha_n)\, y_{n-1}^{(1)} + \alpha_n T y_{n-1}^{(1)} \\ y_{n-1}^{(l)} &= \left(1-\beta_n^{(l)}\right) y_{n-1}^{(l+1)} + \beta_n^{(l)} T y_{n-1}^{(l+1)}, \\ &\qquad l = 1, 2, 3, \ldots, k-2 \\ y_{n-1}^{(k-1)} &= \left(1-\beta_n^{(k-1)}\right) x_{n-1} + \beta_n^{(k-1)} T x_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{9}$$

where $\{\alpha_n\}, \{\beta_n^{(l)}\}, \subset [0,1]$, $l = 1, 2, \ldots, k-1$, with $\sum \alpha_n = \infty$.

Several results have been proved for the strong convergence of the explicit iteration as well as the SP-iterative scheme of fixed points for different types of contractive-like operators in various spaces. See [3, 6, 18, 19]. The stability results of explicit and SP-iterative schemes have been discussed in [2, 4, 9, 20, 21].

Kirk's iterative procedure was defined by Kirk [22]. For $x_0 \in E$, where E is a Banach space and T is a self-map of E,

$$x_{n+1} = \sum_{i=0}^{q} \alpha_i T^i x_n, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where q is a fixed integer with $q \geq 1$, $\alpha_i \geq 0$ for each i and $\sum_{i=0}^{q} \alpha_i = 1$.

In order to reduce the cost of computations, Olatinwo [8] introduced two hybrid schemes, namely, Kirk-Mann and Kirk-Ishikawa iterative schemes in a normed linear space. For $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \sum_{i=0}^{q} \alpha_{n,i} T^i x_n; \quad \sum_{i=0}^{q} \alpha_{n,i} = 1, \ n = 0, 1, 2, \ldots, \\ x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{q} \alpha_{n,i} T^i y_n, \quad \sum_{i=0}^{q} \alpha_{n,i} = 1, \\ y_n &= \sum_{i=0}^{r} \beta_{n,i} T^i x_n, \quad \sum_{i=0}^{r} \beta_{n,i} = 1; \\ &\qquad n = 0, 1, 2 \ldots, \end{aligned} \tag{11}$$

respectively, where q and r are fixed integers with $q \geq r$ and $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, and $\beta_{n,0} \neq 0$.

The Kirk-Noor iteration was introduced by Chugh and Kumar [23] as follows: for $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{q} \alpha_{n,i} T^i y_n, \quad \sum_{i=0}^{q} \alpha_{n,i} = 1, \\ y_n &= \beta_{n,0} x_n + \sum_{i=1}^{r} \beta_{n,i} T^i z_n, \quad \sum_{i=0}^{r} \beta_{n,i} = 1, \\ z_n &= \sum_{i=0}^{s} \gamma_{n,i} T^i x_n, \quad \sum_{i=0}^{s} \gamma_{n,i} = 1; \\ &\qquad n = 0, 1, 2 \ldots, \end{aligned} \tag{12}$$

where q, r, and s are fixed integers with $q \geq r \geq s$, and $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,i} \geq 0$, and $\gamma_{n,0} \neq 0$.

In an attempt to generalize (11) and (12), Gürsoy et al. [17] introduced the Kirk-Multistep iteration in an arbitrary Banach space E. For $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{q_1}\alpha_{n,i}T^i y_n, \quad \sum_{i=0}^{q_1}\alpha_{n,i} = 1, \\ y_n^{(l)} &= \beta_{n,0}^{(l)}x_n + \sum_{i=1}^{q_{l+1}}\beta_{n,i}^{(l)}T^i y_n^{(l+1)}, \\ &\qquad \sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)} = 1, \ l = 1\,(1)\,k-2 \\ y_n^{(k-1)} &= \sum_{i=0}^{q_k}\beta_{n,i}^{(k-1)}T^i x_n, \quad \sum_{i=0}^{q_k}\beta_{n,i}^{(k-1)} = 1; \ k \geq 2, \\ &\qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{13}$$

where $q_1, q_2, q_3, \ldots, q_k$ are fixed integers with $q_1 \geq q_2 \geq \cdots \geq q_k$; $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\{\beta_{n,i}^{(l)}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^{(l)} \geq 0$, and $\beta_{n,0}^{(l)} \neq 0$ for each l.

The Kirk-SP iteration was defined by Hussain et al. [24] as follows: for $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}y_n + \sum_{i=1}^{q}\alpha_{n,i}T^i y_n, \quad \sum_{i=0}^{q}\alpha_{n,i} = 1, \\ y_n &= \beta_{n,0}z_n + \sum_{i=1}^{r}\beta_{n,i}T^i z_n, \quad \sum_{i=0}^{r}\beta_{n,i} = 1, \\ z_n &= \sum_{i=0}^{s}\gamma_{n,i}T^i x_n, \quad \sum_{i=0}^{s}\gamma_{n,i} = 1; \\ &\qquad n = 0, 1, 2 \ldots, \end{aligned} \tag{14}$$

where q, r, and s are fixed integers with $q \geq r \geq s$ and $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,i} \geq 0$, and $\gamma_{n,0} \neq 0$. The Kirk-Thianwan iteration can be obtained if $s = 0$ in (14).

The Kirk-Multistep-SP, which generalized both Kirk-SP and Kirk-Thianwan schemes, was introduced by Akewe et al. [1] and it was defined as follows: for $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}y_n^{(1)} + \sum_{i=1}^{q_1}\alpha_{n,i}T^i y_n^{(1)}, \quad \sum_{i=0}^{q_1}\alpha_{n,i} = 1, \\ y_n^{(l)} &= \beta_{n,0}^{(l)}y_n^{(l+1)} + \sum_{i=1}^{q_{l+1}}\beta_{n,i}^{(l)}T^i y_n^{(l+1)}, \\ &\qquad \sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)} = 1, \ l = 1\,(1)\,k-2 \\ y_n^{(k-1)} &= \sum_{i=0}^{q_k}\beta_{n,i}^{(k-1)}T^i x_n, \quad \sum_{i=0}^{q_k}\beta_{n,i}^{(k-1)} = 1; \ k \geq 2, \\ &\qquad n = 0, 1, 2 \ldots, \end{aligned} \tag{15}$$

where $q_1, q_2, q_3, \ldots, q_k$ are fixed integers with $q_1 \geq q_2 \geq \cdots \geq q_k$; $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\{\beta_{n,i}^{(l)}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^{(l)} \geq 0$, and $\beta_{n,0}^{(l)} \neq 0$ for each l.

Another modified form of explicit iteration is the implicit iteration. The implicit Mann iteration and implicit Ishikawa iteration were discussed by Ćirić et al. [7] and Xue and Zhang [19], respectively. For $x_0 \in C$, C being a closed subset of normed linear space, the implicit Mann and implicit Ishikawa iterations are, respectively,

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{16}$$

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + (1-\alpha_n)Tx_n \\ y_{n-1} &= \beta_n x_{n-1} + (1-\beta_n)Ty_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{17}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

The implicit Noor iteration was defined by Chugh et al. [5] as follows: for $x_0 \in C$

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + (1-\alpha_n)Tx_n \\ y_{n-1} &= \beta_n z_{n-1} + (1-\beta_n)Ty_{n-1} \\ z_{n-1} &= \gamma_n x_{n-1} + (1-\gamma_n)Tz_{n-1}, \\ &\qquad n \geq 1, \end{aligned} \tag{18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\sum(1-\alpha_n) = \infty$. The implicit Noor iteration (18) is more general than (16) and (17).

The most generalized Banach operator used by several authors is the one proved by Zamfirescu [25].

Let X be a complete metric space and let T be a self-map of X. The operator T is Zamfirescu operator if for each pair of points $x, y \in X$, at least one of the following is true:

$$\begin{aligned} Z_1 &: d(Tx,Ty) \leq ad(x,y) \\ Z_2 &: d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)] \\ Z_3 &: d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)], \end{aligned} \tag{19}$$

where a, b, and c are nonnegative constants satisfying $a \in [0,1)$, $b, c \leq 1/2$.

The equivalence form of (19) is

$$\begin{aligned} d(Tx,Ty) \leq a \max\Big\{ & d(x,y), \\ & \frac{1}{2}[d(x,Tx) + d(y,Ty)], \\ & \frac{1}{2}[d(x,Ty) + d(y,Tx)]\Big\} \end{aligned} \tag{20}$$

for $x, y \in X$ and $a \in [0,1)$.

Berinde [2] observed that condition (20) implies

$$d(Tx,Ty) \leq 2hd(x,Tx) + hd(x,y), \tag{21}$$

where $h = \max\{a, a/(2-a)\}$.

In [26], Rhoades used a more general contractive condition than (21): for $x, y \in X$, there exists $a \in [0,1)$ such that

$$d(Tx,Ty) \le a \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], d(x,Ty), d(y,Tx)\right\}. \tag{22}$$

Osilike [20] extended and generalized the contractive condition (22): for $x, y \in X$, there exists $a \in [0,1)$ and $L \ge 0$ such that

$$d(Tx,Ty) \le Ld(x,Tx) + ad(x,y). \tag{23}$$

Imoru and Olatinwo [21] employed a more general class of operators T than (23) satisfying the following contractive conditions:

$$d(Tx,Ty) \le ad(x,y) + \varphi(d(x,Tx)) \quad \text{for } x, y \in X, \tag{24}$$

where $a \in [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function with $\varphi(0) = 0$.

The equivalence form of (24) in a normed linear space is

$$\|Tx - Ty\| \le a\|x-y\| + \varphi(\|x - Tx\|) \quad \text{for } x, y \in X \tag{25}$$

We will need the following definitions and lemmas to prove our main results.

Definition 1 (see [9]). Let (X,d) be a metric space and $T : X \to X$ a self-mapping. Suppose that $F_T = \{p \in X : Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure involving T which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, \quad n \ge 0, \tag{26}$$

where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is a function such that $\alpha_n \in [0,1]$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\epsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n})$, $n = 0,1,2,\ldots$. Then, iterative procedure (26) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty}\epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Definition 2 (see [2]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two nonnegative real sequences which converge to a and b, respectively. Let

$$l = \lim_{n\to\infty}\frac{|a_n - a|}{|b_n - b|}; \tag{27}$$

(1) if $l = 0$, then $\{a_n\}_{n=0}^{\infty}$ converges to a faster than $\{b_n\}_{n=0}^{\infty}$ to b;

(2) if $0 < l < \infty$, then both $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same convergence rate;

(3) if $l = \infty$, then $\{b_n\}_{n=0}^{\infty}$ converges to b faster than $\{a_n\}_{n=0}^{\infty}$ to a.

Lemma 3 (see [2]). *Let δ be a real number such that $\delta \in [0,1)$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of nonnegative numbers such that $\lim_{n\to\infty}\epsilon_n = 0$; then, for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta u_n + \epsilon_n, \quad \forall n \in \mathbb{N} \tag{28}$$

we have $\lim_{n\to\infty} u_n = 0$.

Lemma 4 (see [8]). *Let $(E, \|\cdot\|)$ be a normed linear space and $T : E \to E$ a map satisfying (25). Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a subadditive, monotone increasing function such that $\varphi(0) = 0$, $\varphi(Lu) = L\varphi(u)$, for $u \in \mathbb{R}^+$, $L \ge 0$. Then, for all $i \in \mathbb{N}$, $L \ge 0$ and for all $x, y \in E$*

$$\begin{aligned}\left\|T^i x - T^i y\right\| &\le \sum_{j=1}^{i}\binom{i}{j} a^{i-j}\varphi^j(\|x - Tx\|) + a^i\|x-y\|.\end{aligned} \tag{29}$$

Note that $a \in [0,1)$ in (29).

2. Main Results

We present our main results as follows.

Let E be an arbitrary Banach space and $T : E \to E$ a self-map. Let $x_0 \in E$; we define the following iteration, namely, implicit hybrid Kirk-Multistep iterative scheme, as follows:

$$\begin{aligned}
x_n &= \alpha_{n,0}x_{n-1}^{(1)} + \sum_{i=1}^{q_1}\alpha_{n,i}T^i x_n, \quad \sum_{i=0}^{q_1}\alpha_{n,i} = 1\\
x_{n-1}^{(l)} &= \beta_{n,0}^{(l)}x_{n-1}^{(l+1)} + \sum_{i=1}^{q_{l+1}}\beta_{n,i}^{(l)}T^i x_{n-1}^{(l)},\\
&\qquad \sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)} = 1,\ l = 1(1)k-2\\
x_{n-1}^{(k-1)} &= \beta_{n,0}^{(k-1)}x_{n-1} + \sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}T^i x_{n-1}^{(k-1)},\\
&\qquad \sum_{i=0}^{q_k}\beta_{n,i}^{(k-1)} = 1,\ k \ge 2,\\
&\qquad n \ge 1,
\end{aligned} \tag{30}$$

where $q_1, q_2, q_3, \ldots, q_k$ are fixed integers with $q_1 \ge q_2 \ge q_3 \ge \cdots \ge q_k$; $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\{\beta_{n,i}^{(l)}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \ge 0$, $\alpha_{n,0} \ne 0$, $\beta_{n,i}^{(l)} \ge 0$, and $\beta_{n,0}^{(l)} \ne 0$ for each l with $\sum_{n=1}^{\infty}(1 - \alpha_{n,0}) = \infty$.

If we let $k = 3$ in (30), we obtain the implicit Kirk-Noor iteration defined by

$$\begin{aligned} x_n &= \alpha_{n,0} x_{n-1}^{(1)} + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_n, \quad \sum_{i=0}^{q_1} \alpha_{n,i} = 1 \\ x_{n-1}^{(1)} &= \beta_{n,0}^{(1)} x_{n-1}^{(2)} + \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} T^i x_{n-1}^{(1)}, \quad \sum_{i=0}^{q_2} \beta_{n,i}^{(1)} = 1 \\ x_{n-1}^{(2)} &= \beta_{n,0}^{(2)} x_{n-1} + \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} T^i x_{n-1}^{(2)}, \quad \sum_{i=0}^{q_3} \beta_{n,i}^{(2)} = 1, \\ & n \geq 1. \end{aligned} \tag{31}$$

By setting $k = 2$ in (30), we have the implicit Kirk-Ishikawa iteration and we can also obtain the implicit Kirk-Mann iteration when $k = 2$ and $q_2 = 0$ in (30).

If $q_1 = q_2 = q_3 = \cdots = q_k = 1$ in (30), then we obtain the implicit multistep iteration (19) with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(l)} = \beta_n^{(l)}$.

If $k = 3$, $q_1 = q_2 = q_3 = 1$, and $q_4 = q_5 = \cdots = q_k = 0$ in (30), we have the implicit Noor scheme (18) with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(1)} = \beta_n^{(1)}$, and $\beta_{n,1}^{(2)} = \beta_n^{(2)}$.

If $k = 2$, $q_1 = q_2 = 1$, and $q_3 = q_4 = \cdots = q_k = 0$ in (30), we have the implicit Ishikawa scheme (17) with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(1)} = \beta_n^{(1)}$.

If $k = 2$, $q_1 = 1$, and $q_2 = q_3 = q_4 = \cdots = q_k = 0$ in (30), we have the implicit Mann scheme (16) with $\alpha_{n,1} = \alpha_n$.

Throughout, the operator T will be assumed as fixed and a fixed point $p \in F_T$ with the condition (29) is unique.

Theorem 5. *Let $(E, \|\cdot\|)$ be a normed linear space. Assume T is self-map of E satisfying the contractive condition (29) with $F_T \neq \phi$. Then, for $x_0 \in E$, the sequence $\{x_n\}$ defined by (30) with $\sum_{n=1}^{\infty}(1-\alpha_{n,0}) = \infty$ converges strongly to the fixed point $p \in F_T$.*

Proof. Let $x_0 \in X$ and $p \in F_T$; then using (29) and (30) we have

$$\begin{aligned} \|x_n - p\| &\leq \alpha_{n,0} \|x_{n-1}^{(1)} - p\| + \sum_{i=1}^{q_1} \alpha_{n,i} \|T^i x_n - T^i p\| \\ &\leq \alpha_{n,0} \|x_{n-1}^{(1)} - p\| \\ &+ \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j (\|p - Tp\|) \right. \\ &\left. + a^i \|x_n - p\| \right] = \alpha_{n,0} \|x_{n-1}^{(1)} - p\| + \sum_{i=1}^{q_1} \alpha_{n,i} a^i \|x_n \\ &- p\|. \end{aligned} \tag{32}$$

This implies that

$$\|x_n - p\| \leq \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \|x_{n-1}^{(1)} - p\|. \tag{33}$$

Also, from (30), we have

$$\begin{aligned} \|x_{n-1}^{(1)} - p\| &\leq \beta_{n,0}^{(1)} \|x_{n-1}^{(2)} - p\| + \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} \|T^i x_{n-1}^{(1)} - T^i p\| \\ &\leq \beta_{n,0}^{(1)} \|x_{n-1}^{(2)} - p\| \\ &+ \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j (\|p - Tp\|) \right. \\ &\left. + a^i \|x_{n-1}^{(1)} - p\| \right] = \beta_{n,0}^{(1)} \|x_{n-1}^{(2)} - p\| \\ &+ \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i \|x_{n-1}^{(1)} - p\|. \end{aligned} \tag{34}$$

This becomes

$$\|x_{n-1}^{(1)} - p\| \leq \frac{\beta_n^{(1)}}{1 - \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i} \|x_{n-1}^{(2)} - p\|. \tag{35}$$

From (30) again, we have

$$\begin{aligned} \|x_{n-1}^{(2)} - p\| &\leq \beta_{n,0}^{(2)} \|x_{n-1}^{(3)} - p\| + \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} \|T^i x_{n-1}^{(2)} - T^i p\| \\ &\leq \beta_{n,0}^{(2)} \|x_{n-1}^{(3)} - p\| \\ &+ \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j (\|p - Tp\|) \right. \\ &\left. + a^i \|x_{n-1}^{(2)} - p\| \right] = \beta_{n,0}^{(2)} \|x_{n-1}^{(3)} - p\| \\ &+ \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i \|x_{n-1}^{(2)} - p\| \end{aligned} \tag{36}$$

which implies

$$\|x_{n-1}^{(2)} - p\| \leq \frac{\beta_n^{(2)}}{1 - \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i} \|x_{n-1}^{(3)} - p\|. \tag{37}$$

Continuing this way up to $k-1$ in (30), we have

$$\begin{aligned}\left\|x_{n-1}^{(k-1)}-p\right\| &\le \beta_{n,0}^{(k-1)}\left\|x_{n-1}-p\right\|+\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}\left\|T^i x_{n-1}^{(k-1)}\right.\\ &\left.-T^i p\right\| \le \beta_{n,0}^{(k-1)}\left\|x_{n-1}-p\right\|\\ &+\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi^j\left(\|p-Tp\|\right)\right.\\ &\left.+a^i\left\|x_{n-1}^{(k-1)}-p\right\|\right]=\beta_{n,0}^{(k-1)}\left\|x_{n-1}-p\right\|\\ &+\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i\left\|x_{n-1}^{(k-1)}-p\right\|\end{aligned} \tag{38}$$

implying that

$$\left\|x_{n-1}^{(k-1)}-p\right\| \le \frac{\beta_n^{(k-1)}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i}\left\|x_{n-1}-p\right\|. \tag{39}$$

Substituting (35)–(39) into (33) it becomes

$$\begin{aligned}\|x_n-p\| &\le \left(\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\right)\left(\frac{\beta_n^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\right)\\ &\cdot\left(\frac{\beta_n^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}\right)\\ &\times\cdots\left(\frac{\beta_n^{(k-1)}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i}\right)\left\|x_{n-1}-p\right\|.\end{aligned} \tag{40}$$

Let $\lambda_n=\alpha_{n,0}/(1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i)$; then

$$\begin{aligned}1-\lambda_n &= 1-\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}=\frac{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i-\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\\ &\ge 1-\left(\sum_{i=1}^{q_1}\alpha_{n,i}a^i+\alpha_{n,0}\right).\end{aligned} \tag{41}$$

Therefore,

$$\lambda_n \le \sum_{i=1}^{q_1}\alpha_{n,i}a^i+\alpha_{n,0}=\sum_{i=0}^{q_1}\alpha_{n,i}a^i<\sum_{i=0}^{q_1}\alpha_{n,i}=1. \tag{42}$$

Similarly, we can easily obtain the following from (40):

$$\begin{aligned}\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i} &\le \sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i<\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}=1\\ \frac{\beta_{n,0}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i} &\le \sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i<\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}=1\\ \frac{\beta_{n,0}^{(3)}}{1-\sum_{i=1}^{q_4}\beta_{n,i}^{(3)}a^i} &\le \sum_{i=1}^{q_4}\beta_{n,i}^{(3)}a^i<\sum_{i=1}^{q_4}\beta_{n,i}^{(3)}=1\\ &\vdots\\ \frac{\beta_{n,0}^{(k-1)}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i} &\le \sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i<\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}=1.\end{aligned} \tag{43}$$

Applying (42) and (43) in (40) and letting $a^i \le a < 1$ for each i, we have

$$\begin{aligned}\|x_n-p\| &\le \left(\sum_{i=1}^{q_1}\alpha_{n,i}a^i+\alpha_{n,0}\right)\|x_{n-1}-p\|\\ &\le\left[\left(1-\alpha_{n,0}\right)a+\alpha_{n,0}\right]\|x_{n-1}-p\|\\ &=\left[1-\left(1-\alpha_{n,0}\right)(1-a)\right]\|x_{n-1}-p\|\\ &\le\left[1-\left(1-\alpha_{n,0}\right)(1-a)\right]\left[1-\left(1-\alpha_{n-1,0}\right)(1-a)\right]\\ &\cdot\|x_{n-2}-p\|\\ &\vdots\\ &\le\prod_{r=1}^{n}\left[1-\left(1-\alpha_{r,0}\right)(1-a)\right]\|x_0-p\|\\ &\le e^{-(1-a)\sum_{r=1}^{n}(1-\alpha_{r,0})}\|x_0-p\|.\end{aligned} \tag{44}$$

As $n\to\infty$, $\sum_{r=1}^{\infty}(1-\alpha_{r,0})=\infty$. Hence, $\lim_{n\to\infty}\|x_n-p\|=0$.

Therefore, implicit Kirk-Multistep scheme (30) converges strongly to $p\in F_T$. □

Corollary 6. *Let $(E,\|\cdot\|)$ be a normed linear space. Assume T is self-map of E satisfying the contractive condition (29) with $F_T\neq\phi$. Then, for $x_0\in E$, the implicit Kirk-Noor, the implicit Kirk-Ishikawa, and the implicit Kirk-Mann schemes with $\sum_{n=1}^{\infty}(1-\alpha_{n,0})=\infty$ converge strongly to the fixed point $p\in F_T$.*

Remark 7. The strong convergence results for implicit Noor, implicit Ishikawa, and implicit Mann schemes are obvious from Theorem 5.

Theorem 8. *Let $(E,\|\cdot\|)$ be a normed linear space and T is a self-map of E satisfying contractive condition (29) with $F_T\neq\phi$. Then, for $x_0\in E$ and $p\in F_T$, the sequence $\{x_n\}$ defined by (30) is T-stable.*

Proof. Let $\{y_n\} \in E$ be an arbitrary sequence and let $\epsilon_n = \|y_n - \alpha_{n,0} z_{n-1}^{(1)} - \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_n\|$, where

$$\begin{aligned} z_{n-1}^{(1)} &= \beta_{n,0}^{(1)} z_{n-1}^{(2)} + \sum_{i=1}^{q_2} \beta_{n,i}^{(1)} T^i z_{n-1}^{(1)}, \quad \sum_{i=0}^{q_2} \beta_{n,i}^{(1)} = 1, \\ z_{n-1}^{(l)} &= \beta_{n,0}^{(l)} z_{n-1}^{(l+1)} + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} T^i z_{n-1}^{(l)}, \\ &\qquad \sum_{i=0}^{q_{l+1}} \beta_{n,i}^{(l)} = 1, \ l = 1\,(1)\,k-2 \\ z_{n-1}^{(k-1)} &= \beta_{n,0}^{(k-1)} y_{n-1} + \sum_{i=1}^{q_k} \beta_{n,i}^{(l)} T^i z_{n-1}^{(k-1)}, \quad \sum_{i=0}^{q_k} \beta_{n,i}^{(k-1)} = 1. \end{aligned} \tag{45}$$

Suppose $\lim_{n\to\infty} \epsilon_n = 0$ and $p \in F_T$; by (29) we have

$$\begin{aligned} \|y_n - p\| &\le \left\| y_n - \alpha_{n,0} z_{n-1}^{(1)} - \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_n \right\| \\ &\quad + \left\| \alpha_{n,0} z_{n-1}^{(1)} + \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_n - p \right\| \\ &\le \epsilon_n + \alpha_{n,0} \left\| z_{n-1}^{(1)} - p \right\| + \sum_{i=1}^{q_1} \alpha_{n,i} \left\| T^i y_n - p \right\| \\ &\le \epsilon_n + \alpha_{n,0} \left\| z_{n-1}^{(1)} - p \right\| + \sum_{i=1}^{q_1} \alpha_{n,i} a^i \|y_n - p\|. \end{aligned} \tag{46}$$

This implies that

$$\begin{aligned} \|y_n - p\| &\le \frac{\epsilon_n}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \\ &\quad + \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left\| z_{n-1}^{(1)} - p \right\|. \end{aligned} \tag{47}$$

From inequalities (42) and (43), one can easily obtain the following:

$$\begin{aligned} \left\| z_{n-1}^{(1)} - p \right\| &\le \left\| z_{n-1}^{(2)} - p \right\| \le \left\| z_{n-1}^{(3)} - p \right\| \le \cdots \\ &\le \left\| z_{n-1}^{(k-1)} - p \right\| \le \|y_{n-1} - p\|. \end{aligned} \tag{48}$$

Then, inequality (47) becomes

$$\begin{aligned} \|y_n - p\| &\le \frac{\epsilon_n}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \\ &\quad + \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \|y_{n-1} - p\|. \end{aligned} \tag{49}$$

Letting $\delta = \alpha_{n,0}/(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i) < 1$ and by Lemma 3, we have

$$\|y_n - p\| = 0. \tag{50}$$

Conversely, suppose $\|y_n - p\| = 0$ for $p \in F_T$; then

$$\begin{aligned} \epsilon_n &= \left\| y_n - \alpha_{n,0} z_{n-1}^{(1)} - \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_n \right\| \\ &\le \|y_n - p\| + \left\| p - \left(\alpha_{n,0} z_{n-1}^{(1)} + \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_n \right) \right\| \\ &\le \|y_n - p\| + \alpha_{n,0} \left\| z_{n-1}^{(1)} - p \right\| + \sum_{i=1}^{q_1} \alpha_{n,i} \left\| T^i y_n - T^i p \right\| \\ &\le \|y_n - p\| + \alpha_{n,0} \|y_{n-1} - p\| + \sum_{i=1}^{q_1} \alpha_{n,i} a^i \|y_n - p\| \\ &\le \left(1 + \sum_{i=1}^{q_1} \alpha_{n,i} a^i \right) \|y_n - p\| + \alpha_{n,0} \|y_{n-1} - p\|. \end{aligned} \tag{51}$$

Since $\|y_n - p\| \to 0$, then $\lim_{n\to\infty} \epsilon_n = 0$.

Therefore, iterative scheme (30) is T-stable. □

Corollary 9. *Let $(E, \|\cdot\|)$ be a normed linear space and T is a self-map of E satisfying the contractive condition (29) with $F_T \neq \phi$. Then, for $x_0 \in E$ and $p \in F_T$, the sequence $\{x_n\}$ defined by implicit Kirk-Mann, implicit Kirk-Ishikawa, and implicit Kirk-Noor schemes are T-stable.*

Remark 10. The stability results for implicit Mann, implicit Ishikawa, and implicit Noor schemes with contractive condition (29) are special cases of Corollary 9.

2.1. Comparison of Several Iterative Schemes. We compare our iterative schemes with others by using the following example.

Example 11. Let $T : [0,1] \to [0,1]$ and $Tx = x/2$ with $x_0 \neq 0$ and fixed point $p = 0$ using $\alpha_{n,0} = \beta_{n,0}^{(l)} = 1 - 2/\sqrt{n}$, $\alpha_{n,i} = \beta_{n,i}^{(l)} = 4/\sqrt{n}$, for each l, $n \ge 25$, and $q_1 = q_2 = q_3 = q_4 = 2$.

For the implicit Kirk-Mann iteration (IKM), we have

$$\begin{aligned} x_n &= \alpha_{n,0} x_{n-1} + \sum_{i=1}^{2} \alpha_{n,i} T^i x_n \\ &= \left(1 - \frac{2}{\sqrt{n}} \right) x_{n-1} + \frac{2}{\sqrt{n}} x_n + \frac{1}{\sqrt{n}} x_n. \end{aligned} \tag{52}$$

This implies that

$$x_n\,(\mathrm{IKM}) = \frac{2\sqrt{n} - 4}{2\sqrt{n} - 3} x_{n-1} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r} - 4}{2\sqrt{r} - 3} \right) x_0. \tag{53}$$

Also, for implicit Kirk-Ishikawa iteration (IKI), we have

$$x_n = \frac{2\sqrt{n} - 4}{2\sqrt{n} - 3} x_{n-1}^{(1)} \tag{54}$$

with

$$x_{n-1}^{(1)} = \frac{2\sqrt{n} - 4}{2\sqrt{n} - 3} x_{n-1}. \tag{55}$$

Hence,

$$x_n(\mathrm{IKI}) = \left(\frac{2\sqrt{n}-4}{2\sqrt{n}-3}\right)^2 x_{n-1} = \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)^2 x_0. \quad (56)$$

Similarly, implicit Kirk-Noor iteration (IKN) implies

$$x_n(\mathrm{IKN}) = \left(\frac{2\sqrt{n}-4}{2\sqrt{n}-3}\right)^3 x_{n-1} = \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)^3 x_0 \quad (57)$$

while the implicit multistep Kirk iteration (IMK) gives

$$x_n(\mathrm{IMK}) = \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)^k x_0. \quad (58)$$

Now, using Definition 2, we compare the implicit Kirk type iterations as follows: for $k \geq 4$, we have

$$\begin{aligned}\frac{|x_n(\mathrm{IMK})-0|}{|x_n(\mathrm{IKN})-0|} &= \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)^{k-3} \\ &= \prod_{r=25}^{n}\left(1-\frac{1}{2\sqrt{r}-3}\right)^{k-3}\end{aligned} \quad (59)$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{2\sqrt{r}-3}\right)^{k-3} \\ &\leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)^{k-3} \\ &= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^{k-3} \\ &= \left(\lim_{n\to\infty}\frac{24}{n}\right)^{k-3} = 0, \quad \forall k \geq 4.\end{aligned} \quad (60)$$

Remark 12. The implicit multistep Kirk iteration (IMK) converges faster than the implicit Kirk-Noor iteration (IKN) for $k = 4, 5, \ldots$.

Also,

$$\begin{aligned}\frac{|x_n(\mathrm{IKN})-0|}{|x_n(\mathrm{IKI})-0|} &= \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right) \\ &= \prod_{r=25}^{n}\left(1-\frac{1}{2\sqrt{r}-3}\right)\end{aligned} \quad (61)$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{2\sqrt{r}-3}\right) \leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right) \\ &= \lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n} = \lim_{n\to\infty}\frac{24}{n} = 0.\end{aligned} \quad (62)$$

Remark 13. The implicit Kirk-Noor iteration $x_n(\mathrm{IKN})$ converges to $p = 0$ faster than the implicit Kirk-Ishikawa iteration $x_n(\mathrm{IKI})$ to $p = 0$.

Similarly, using Definition 2, we have that

$$\lim_{n\to\infty}\frac{|x_n(\mathrm{IKI})-0|}{|x_n(\mathrm{IKM})-0|} = 0 \quad (63)$$

which implies that the implicit Kirk-Ishikawa iteration $x_n(\mathrm{IKI})$ converges faster than the implicit Kirk-Mann iteration $x_n(\mathrm{IKM})$.

For the Kirk-Mann iteration (KM), we have the following estimate:

$$\begin{aligned}x_n &= \alpha_{n,0}x_{n-1} + \sum_{i=1}^{2}\alpha_{n,i}T^i x_{n-1} \\ &= \left(1-\frac{2}{\sqrt{n}}\right)x_{n-1} + \frac{2}{\sqrt{n}}x_{n-1} + \frac{1}{\sqrt{n}}x_{n-1}.\end{aligned} \quad (64)$$

This implies that

$$x_n(\mathrm{KM}) = \left(1+\frac{\sqrt{n}}{n}\right)x_{n-1} = \prod_{r=25}^{n}\left(1+\frac{\sqrt{r}}{r}\right)x_0. \quad (65)$$

The estimates for Kirk-Thianwan (KT), Kirk-SP (KSP), and Kirk-Multistep-SP (KMSP) iterations are, respectively,

$$\begin{aligned}x_n(\mathrm{KT}) &= \prod_{r=25}^{n}\left(1+\frac{\sqrt{r}}{r}\right)^2 x_0, \\ x_n(\mathrm{KSP}) &= \prod_{r=25}^{n}\left(1+\frac{\sqrt{r}}{r}\right)^3 x_0, \\ x_n(\mathrm{KMSP}) &= \prod_{r=25}^{n}\left(1+\frac{\sqrt{r}}{r}\right)^k x_0.\end{aligned} \quad (66)$$

We compare Kirk-Mann (KM), Kirk-Thianwan (KT), Kirk-SP (KSP), and Kirk-Multistep-SP (KMSP) iterations with our iterative schemes as follows.

Again, using Definition 2, we have

$$\begin{aligned}\frac{|x_n(\mathrm{IKM})-0|}{|x_n(\mathrm{KM})-0|} &= \prod_{r=25}^{n}\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)\left(\frac{r}{\sqrt{r}+r}\right) \\ &= \prod_{r=25}^{n}\left(\frac{2r^{3/2}-4r}{2r^{3/2}-3r^{1/2}-r}\right) \\ &= \prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\end{aligned} \quad (67)$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right) \\ &\leq \lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right) = \lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n} \\ &= \lim_{n\to\infty}\frac{24}{n} = 0.\end{aligned} \quad (68)$$

Remark 14. The implicit Kirk-Mann iteration $x_n(\mathrm{IKM})$ converges to $p = 0$ faster than the Kirk-Mann iteration $x_n(\mathrm{KM})$ to $p = 0$.

For the comparison of implicit Kirk-Ishikawa iteration $x_n(\mathrm{IKI})$ and Kirk-Thianwan iteration $x_n(\mathrm{KT})$, we have

$$\begin{aligned}\frac{|x_n(\mathrm{IKI})-0|}{|x_n(\mathrm{KT})-0|} &= \prod_{r=25}^{n}\left[\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)\left(\frac{r}{\sqrt{r}+r}\right)\right]^2 \\ &= \left[\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^2\end{aligned} \tag{69}$$

with

$$\begin{aligned}0 &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^2 \\ &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^2 \\ &= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^2 = \left(\lim_{n\to\infty}\frac{24}{n}\right)^2 \\ &= 0.\end{aligned} \tag{70}$$

Remark 15. The implicit Kirk-Ishikawa iteration $x_n(\mathrm{IKI})$ converges faster than the Kirk-Thianwan iteration $x_n(\mathrm{KT})$.

For the comparison of implicit Kirk-Noor iteration $x_n(\mathrm{IKN})$ and Kirk-SP iteration $x_n(\mathrm{KSP})$, we have

$$\begin{aligned}\frac{|x_n(\mathrm{IKN})-0|}{|x_n(\mathrm{KSP})-0|} &= \prod_{r=25}^{n}\left[\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)\left(\frac{r}{\sqrt{r}+r}\right)\right]^3 \\ &= \left[\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^3\end{aligned} \tag{71}$$

with

$$\begin{aligned}0 &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^3 \\ &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^3 \\ &= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^3 = \left(\lim_{n\to\infty}\frac{24}{n}\right)^3 \\ &= 0.\end{aligned} \tag{72}$$

Remark 16. The implicit Kirk-Noor iteration $x_n(\mathrm{IKN})$ has better convergence rate than the Kirk-SP iteration $x_n(\mathrm{KSP})$.

For the comparison of implicit Kirk-Multistep iteration $x_n(\mathrm{IKM})$ and Kirk-Multistep-SP iteration $x_n(\mathrm{KMSP})$, we have

$$\begin{aligned}\frac{|x_n(\mathrm{IKM})-0|}{|x_n(\mathrm{KMSP})-0|} &= \prod_{r=25}^{n}\left[\left(\frac{2\sqrt{r}-4}{2\sqrt{r}-3}\right)\left(\frac{r}{\sqrt{r}+r}\right)\right]^k \\ &= \left[\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^k\end{aligned} \tag{73}$$

with

$$\begin{aligned}0 &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{3\left(r-r^{1/2}\right)}{2r^{3/2}-3r^{1/2}-r}\right)\right]^k \\ &\le \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^k \\ &= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^k = \left(\lim_{n\to\infty}\frac{24}{n}\right)^k \\ &= 0.\end{aligned} \tag{74}$$

Remark 17. The implicit Kirk-Multistep iteration $x_n(\mathrm{IKM})$ has better convergence rate than the Kirk-Multistep-SP iteration $x_n(\mathrm{KMSP})$ for $k \ge 4$.

3. Conclusion

We have established and proved strong convergence and T-stability results for implicit Kirk-Multistep, implicit Kirk-Noor, implicit Kirk-Ishikawa, and implicit Kirk-Mann iterative schemes of fixed points with contractive-type operators in normed linear spaces. These iterative schemes have better convergence rate when compared with other iterative schemes, namely, multistep Kirk-SP iteration, Kirk-Multistep scheme, Kirk-SP iteration, Kirk-Thianwan scheme, Kirk-Noor scheme, Kirk-Ishikawa scheme, Kirk-Mann scheme, implicit Noor iteration, implicit Ishikawa iteration, implicit Mann iteration, and many more iterative schemes of fixed point in the literature.

Disclosure

Authors agreed to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

Competing Interests

The authors hereby declare that there are no competing interests.

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