Research Article

A Symmetric Algorithm for Golden Ratio in Hyper-Horadam Numbers

Mustafa Bahşı¹ and Süleyman Solak²

¹Education Faculty, Aksaray University, 68100 Aksaray, Turkey

²A. K. Education Faculty, Konya N. E. University, 42090 Konya, Turkey

Correspondence should be addressed to Süleyman Solak; ssolak42@yahoo.com

Received 3 April 2016; Revised 17 June 2016; Accepted 17 July 2016

Academic Editor: Qinghua Hu

Copyright © 2016 M. Bahşı and S. Solak. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study some ratios related to hyper-Horadam numbers such as $W_n^{(r)}/W_{n-1}^{(r)}$ while $n \rightarrow \infty$ by using a symmetric algorithm obtained by the recurrence relation $a_n^k = a_n^{k-1} + a_{n-1}^k$, where $W_n^{(r)}$ is the n th hyper-Horadam number. Also, we give some special cases of these ratios such as the golden ratio and silver ratio.

1. Introduction

The Fibonacci numbers are defined by the second-order linear recurrence relation $F_{n+1} = F_n + F_{n-1}$ ($n \geq 1$) with the initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ ($n \geq 1$) with the initial conditions $L_0 = 2$ and $L_1 = 1$. The Fibonacci sequence can be generalized as the second-order linear recurrence $W_n(a, b; p, q)$, or briefly W_n , defined by

$$W_{n+1} = pW_n + qW_{n-1}, \quad (1)$$

where $n \geq 1$, $W_0 = a$, and $W_1 = b$. This number sequence was introduced by Horadam [1]. The characteristic equation of W_n is

$$t^2 - pt - q = 0. \quad (2)$$

The roots of (2) are $\alpha = (p + \sqrt{p^2 + 4q})/2$ and $\beta = (p - \sqrt{p^2 + 4q})/2$. We think of α and β as being real, though this need not be so; that is, $p^2 + 4q \geq 0$. The Binet formula for W_n is

$$W_n = A\alpha^n + B\beta^n, \quad (3)$$

where

$$\begin{aligned} A &= \frac{b - a\beta}{\sqrt{p^2 + 4q}}, \\ B &= \frac{a\alpha - b}{\sqrt{p^2 + 4q}}. \end{aligned} \quad (4)$$

Some of the special cases of Horadam number W_n are as follows:

the Fibonacci number $F_n = W_n(0, 1; 1, 1)$,

the Lucas number $L_n = W_n(2, 1; 1, 1)$, (5)

the Pell number $P_n = W_n(0, 1; 2, 1)$.

From (3)–(5) it follows that

$$\begin{aligned} P_n &= \frac{\sqrt{2}}{4} (\alpha^n - \beta^n), \\ F_n &= \frac{\alpha_1^n - \beta_1^n}{\sqrt{5}}, \\ L_n &= \alpha_1^n + \beta_1^n, \end{aligned} \quad (6)$$

where

$$\begin{aligned}\alpha_1 &= \frac{1 + \sqrt{5}}{2}, \\ \beta_1 &= \frac{1 - \sqrt{5}}{2},\end{aligned}\quad (7)$$

that is, α_1 and β_1 are the roots of

$$t^2 - t - 1 = 0. \quad (8)$$

For the ratio W_{n+1}/W_n , (3) and (4) follow that

$$\lim_{n \rightarrow \infty} \frac{W_{n+1}}{W_n} = \tau = \begin{cases} \alpha = \frac{p + \sqrt{p^2 + 4q}}{2} & \text{if } p \geq 0 \\ \beta = \frac{p - \sqrt{p^2 + 4q}}{2} & \text{if } p < 0; \end{cases} \quad (9)$$

That is, τ is root of (2).

Over the past five centuries, the golden ratio has been very attractive for researchers because its occurrence is ubiquitous such as nature, art, architecture, and anatomy. From (9), we have the well-known golden ratio and silver ratio as follows:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} &= \frac{1 + \sqrt{5}}{2} = \phi \quad (\text{golden ratio}), \\ \lim_{n \rightarrow \infty} \frac{L_n}{L_{n-1}} &= \frac{1 + \sqrt{5}}{2} = \phi \quad (\text{golden ratio}), \\ \lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} &= 1 + \sqrt{2} = \Psi \quad (\text{silver ratio}).\end{aligned} \quad (10)$$

The Euler-Seidel algorithm and its analogs are useful to study some recurrence relations and identities for some numbers and polynomials [2–5]. Let (a_n) and (a^n) be two real initial sequences. Then the infinite matrix, which is called symmetric infinite matrix in [4], with entries a_n^k corresponding to these sequences is determined recursively by the following formulas:

$$\begin{aligned}a_n^0 &= a_n, \\ a_0^n &= a^n \\ (n \geq 0), \\ a_n^k &= a_n^{k-1} + a_{n-1}^k \quad (n \geq 1, k \geq 1).\end{aligned} \quad (11)$$

From (11), we can write

$$a_n^k = \sum_{s=0}^s a_s^{k-1}. \quad (12)$$

There are some applications of sequence (11) and its generalization [2–6]. For example, Bahşı et al. [6] introduced

the concepts as “hyper-Horadam” numbers and “generalized hyper-Horadam” numbers:

$$\begin{aligned}W_n^{(r)} &= \sum_{s=0}^n W_s^{(r-1)} = W_{n-1}^{(r)} + W_n^{(r-1)} \\ \text{with } W_n^{(0)} &= W_n, \quad W_0^{(n)} = W_0 = a,\end{aligned} \quad (13)$$

$$W_n^{(r)}(u, v) = uW_n^{(r-1)} + vW_{n-1}^{(r)},$$

where u and v are two nonzero real parameters, $W_n^{(0)}(u, v) = W_n(a, b; p, q) = W_n$ and $W_0^{(n)}(u, v) = W_0(a, b; p, q) = a$, and W_n is the n th Horadam number. Some of the special cases of hyper-Horadam number $W_n^{(r)}$ are as follows:

- (i) If $W_n^{(0)} = F_n = W_n(0, 1; 1, 1)$ and $W_0^{(n)} = W_0 = F_0 = 0$, then $W_n^{(r)}$ is the hyper-Fibonacci number; that is, $W_n^{(r)} = F_n^{(r)}$.
- (ii) If $W_n^{(0)} = L_n = W_n(2, 1; 1, 1)$ and $W_0^{(n)} = W_0 = L_0 = 2$, then $W_n^{(r)}$ is the hyper-Lucas number; that is, $W_n^{(r)} = L_n^{(r)}$.
- (iii) If $W_n^{(0)} = P_n = W_n(0, 1; 2, 1)$ and $W_0^{(n)} = W_0 = P_0 = 0$, then $W_n^{(r)}$ is the hyper-Pell number; that is, $W_n^{(r)} = P_n^{(r)}$.

The fundamental aim of this paper is to obtain relationships between special ratios such as the golden ratio, silver ratio, and hyper-numbers such as hyper-Fibonacci, hyper-Lucas, and hyper-Pell numbers. For this, we firstly investigate the ratio $W_n^{(r)}/W_{n-1}^{(r)}$ while $n \rightarrow \infty$ by using a symmetric algorithm obtained by the recurrence relation $a_n^k = a_n^{k-1} + a_{n-1}^k$.

2. Main Results

Theorem 1. Let the sequence a_n^k be as in (11). If $\lim_{n \rightarrow \infty} (a_n/a_{n-1}) = \ell$, then for $k \geq 0$

$$\lim_{n \rightarrow \infty} \frac{a_n^k}{a_{n-1}^k} = \ell, \quad (14)$$

where $a_0^n = a$ and a is any real number.

Proof. We use the principle of the mathematical induction on k . It is clear that the result is true for $k = 0$; that is,

$$\lim_{n \rightarrow \infty} \frac{a_n^0}{a_{n-1}^0} = \lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = \ell. \quad (15)$$

Let us assume that it is true for $k - 1$; that is,

$$\lim_{n \rightarrow \infty} \frac{a_n^{k-1}}{a_{n-1}^{k-1}} = \ell. \quad (16)$$

Then

$$\begin{aligned}\ell &= \lim_{n \rightarrow \infty} \frac{a_n^{k-1}}{a_{n-1}^{k-1}} \\ &= \frac{\lim_{n \rightarrow \infty} ((a_0^{k-1} + a_1^{k-1} + a_2^{k-1} + \cdots + a_n^{k-1}) / (n+1))}{\lim_{n \rightarrow \infty} ((a_0^{k-1} + a_1^{k-1} + a_2^{k-1} + \cdots + a_{n-1}^{k-1}) / n)} \quad (17) \\ &= \frac{\lim_{n \rightarrow \infty} (a_n^k / (n+1))}{\lim_{n \rightarrow \infty} (a_{n-1}^k / n)} = \lim_{n \rightarrow \infty} \frac{a_n^k}{a_{n-1}^k}.\end{aligned}$$

That is, the result is true for k . Thus the proof is completed. $\square$

As an application of the Theorem 1, we have the next corollary for the hyper-Horadam numbers.

Corollary 2. Let τ be as in (9). If $r \geq 0$, then,

$$\lim_{n \rightarrow \infty} \frac{W_n^{(r)}}{W_{n-1}^{(r)}} = \tau. \quad (18)$$

Proof. Since

$$\lim_{n \rightarrow \infty} \frac{W_n^{(0)}}{W_{n-1}^{(0)}} = \lim_{n \rightarrow \infty} \frac{W_n}{W_{n-1}} = \tau \quad (19)$$

the proof is trivial from Theorem 1 if we select $a_n^0 = W_n^{(0)} = W_n$, $a_0^n = W_0 = a$, and $a_n^r = W_n^{(r)}$. $\square$

Theorem 3. Let τ be as in (9). If $r \geq 1$, then,

$$\begin{aligned}\text{(i)} \quad & \lim_{n \rightarrow \infty} \frac{W_n^{(r-1)}}{W_{n-1}^{(r)}} = \tau - 1, \\ \text{(ii)} \quad & \lim_{n \rightarrow \infty} \frac{W_{n-1}^{(r)}}{W_n^{(r-1)}} = \frac{1}{\tau - 1}, \\ \text{(iii)} \quad & \lim_{n \rightarrow \infty} \frac{W_n^{(r)}}{W_n^{(r-1)}} = \frac{\tau}{\tau - 1}.\end{aligned} \quad (20)$$

Proof. (i) From Corollary 2, we have

$$\lim_{n \rightarrow \infty} \frac{W_n^{(r)}}{W_{n-1}^{(r)}} = \tau. \quad (21)$$

Then,

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{W_n^{(r-1)}}{W_{n-1}^{(r)}} &= \lim_{n \rightarrow \infty} \frac{W_n^{(r)} - W_{n-1}^{(r)}}{W_{n-1}^{(r)}} = \lim_{n \rightarrow \infty} \left(\frac{W_n^{(r)}}{W_{n-1}^{(r)}} - 1 \right) \\ &= \lim_{n \rightarrow \infty} \frac{W_n^{(r)}}{W_{n-1}^{(r)}} - \lim_{n \rightarrow \infty} 1 = \tau - 1.\end{aligned} \quad (22)$$

(ii) From (i)

$$\lim_{n \rightarrow \infty} \frac{W_{n-1}^{(r)}}{W_n^{(r-1)}} = \frac{1}{\lim_{n \rightarrow \infty} (W_n^{(r-1)} / W_{n-1}^{(r)})} = \frac{1}{\tau - 1}. \quad (23)$$

(iii) Since $\lim_{n \rightarrow \infty} (W_n^{(r)} / W_n^{(r-1)}) = 1 / (\tau - 1)$ (from (ii)), we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{W_n^{(r)}}{W_n^{(r-1)}} &= \lim_{n \rightarrow \infty} \frac{W_n^{(r-1)} + W_{n-1}^{(r)}}{W_n^{(r-1)}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{W_{n-1}^{(r)}}{W_n^{(r-1)}} \right) = 1 + \lim_{n \rightarrow \infty} \frac{W_{n-1}^{(r)}}{W_n^{(r-1)}} \quad (24) \\ &= 1 + \frac{1}{\tau - 1} = \frac{\tau}{\tau - 1}.\end{aligned}$$

$\square$

From these results we have some particular results for the relationships between hyper-Fibonacci, hyper-Lucas (hyper-Pell) numbers, and the golden (silver) ratio as follows:

(1) The relationships between hyper-Fibonacci (and Lucas) numbers and golden ratio ($\phi = (1 + \sqrt{5})/2$) are as follows:

$$\begin{aligned}\text{(i)} \quad & \lim_{n \rightarrow \infty} \frac{F_n^{(r)}}{F_{n-1}^{(r)}} = \lim_{n \rightarrow \infty} \frac{L_n^{(r)}}{L_{n-1}^{(r)}} = \phi, \\ \text{(ii)} \quad & \lim_{n \rightarrow \infty} \frac{F_n^{(r-1)}}{F_{n-1}^{(r)}} = \lim_{n \rightarrow \infty} \frac{L_n^{(r-1)}}{L_{n-1}^{(r)}} = \phi - 1, \\ \text{(iii)} \quad & \lim_{n \rightarrow \infty} \frac{F_{n-1}^{(r)}}{F_n^{(r-1)}} = \lim_{n \rightarrow \infty} \frac{L_{n-1}^{(r)}}{L_n^{(r-1)}} = \phi, \\ \text{(iv)} \quad & \lim_{n \rightarrow \infty} \frac{F_n^{(r)}}{F_n^{(r-1)}} = \lim_{n \rightarrow \infty} \frac{L_n^{(r)}}{L_n^{(r-1)}} = 1 + \phi.\end{aligned} \quad (25)$$

(2) The relationships between hyper-Pell numbers and silver ratio ($\Psi = 1 + \sqrt{2}$) are as follows:

$$\text{(i)} \quad \lim_{n \rightarrow \infty} \frac{P_n^{(r)}}{P_{n-1}^{(r)}} = \Psi, \quad (26)$$

$$\text{(ii)} \quad \lim_{n \rightarrow \infty} \frac{P_n^{(r-1)}}{P_{n-1}^{(r)}} = \Psi - 1, \quad (27)$$

$$\text{(iii)} \quad \lim_{n \rightarrow \infty} \frac{P_{n-1}^{(r)}}{P_n^{(r-1)}} = \frac{\Psi - 1}{2}, \quad (28)$$

$$\text{(iv)} \quad \lim_{n \rightarrow \infty} \frac{P_n^{(r)}}{P_n^{(r-1)}} = \frac{\Psi + 1}{2}. \quad (29)$$

Competing Interests

The authors declare that they have no competing interests.

References

- [1] A. F. Horadam, "Basic properties of a certain generalized sequence of numbers," *The Fibonacci Quarterly*, vol. 3, pp. 161–176, 1965.

- [2] N.-N. Cao and F.-Z. Zhao, "Some properties of hyperfibonacci and hyperlucas numbers," *Journal of Integer Sequences*, vol. 13, article 10.8.8, 2010.
- [3] A. Dil, V. Kurt, and M. Cenkci, "Algorithms for Bernoulli and allied polynomials," *Journal of Integer Sequences*, vol. 10, no. 5, Article 07.5.4, 14 pages, 2007.
- [4] A. Dil and I. Mezö, "A symmetric algorithm for hyperharmonic and Fibonacci numbers," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 942–951, 2008.
- [5] D. Dumont, Matrices d. Euler. Seidel, Seminaire Lotharingien de Combinatoire, B05c, 1981, <http://www.emis.de/journals/SLC/opapers/s05dumont.pdf>.
- [6] M. Bahşı, I. Mezö, and S. Solak, "A symmetric algorithm for hyper-Fibonacci and hyper-Lucas numbers," *Annales Mathematicae et Informaticae*, vol. 43, pp. 19–27, 2014.