Research Article

Inclusion of the Generalized Bessel Functions in the Janowski Class

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Sufficient conditions on A , B , p , b , and c are determined that will ensure the generalized Bessel function $u_{p,b,c}$ satisfies the subordination $u_{p,b,c}(z) \prec (1 + Az)/(1 + Bz)$. In particular this gives conditions for $(-4\kappa/c)(u_{p,b,c}(z) - 1)$, $c \neq 0$, to be close-to-convex. Also, conditions for $u_{p,b,c}(z)$ to be Janowski convex and $zu_{p,b,c}(z)$ to be Janowski starlike in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. If f and g are analytic in $\mathbb{D}$, then f is subordinate to g , written $f(z) \prec g(z)$, if there is an analytic self-map w of $\mathbb{D}$ satisfying $w(0) = 0$ and $f = g \circ w$. For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A, B]$ be the class consisting of normalized analytic functions $p(z) = 1 + c_1z + \dots$ in $\mathbb{D}$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}. \quad (1)$$

For instance, if $0 \leq \beta < 1$, then $\mathcal{P}[1 - 2\beta, -1]$ is the class of functions $p(z) = 1 + c_1z + \dots$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathbb{D}$.

The class $\mathcal{S}^*[A, B]$ of Janowski starlike functions [1] consists of $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B]. \quad (2)$$

For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β ; $\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}_\beta^* = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\}$; and $\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}$. These classes have been studied, for example, in [2, 3]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β [4, 5] if $\operatorname{Re}(zf'(z)/g(z)) > \beta$ for some $g \in \mathcal{S}^* := \mathcal{S}^*(0)$.

This article studies the generalized Bessel function $u_p(z) = u_{p,b,c}(z)$ given by the power series

$$u_p(z) = {}_0F_1\left(\kappa, -\frac{c}{4}z\right) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{4^k (\kappa)_k} \frac{z^k}{k!}, \quad (3)$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3, \dots$. The function $u_p(z)$ is analytic in $\mathbb{D}$ and solution of the differential equation

$$4z^2 u''(z) + 4\kappa z u'(z) + cz u(z) = 0, \quad (4)$$

if b, p, c in $\mathbb{C}$, such that $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3, \dots$, and $z \in \mathbb{D}$. This normalized and generalized Bessel function of the first kind of order p also satisfies the following recurrence relation:

$$4\kappa u_p'(z) = -cu_{p+1}(z), \quad (5)$$

which is useful tool to study several geometric properties of u_p . There have been several works [6–11] studying geometric properties of the function $u_p(z)$, for example, on its close-to-convexity, starlikeness, and convexity, and radius of starlikeness and convexity.

In Section 2 of this paper, sufficient conditions on A , B , c , and κ are determined that will ensure u_p satisfies the subordination $u_p(z) \prec (1 + Az)/(1 + Bz)$. It is to be understood

that a computationally intensive methodology with shrewd manipulations is required to obtain the results in this general framework. The benefits of such general results are that, by judicious choices of the parameters A and B , they give rise to several interesting applications, which include extending the results of previous works. Using this subordination result, sufficient conditions are obtained for $(-4\kappa/c)u'(z) \in \mathcal{P}[A, B]$, which next readily gives conditions for $(-4\kappa/c)(u_p(z) - 1)$ to be close-to-convex. Section 3 gives emphasis to the investigation of $u_p(z)$ to be Janowski convex as well as of $zu_p(z)$ to be Janowski starlike.

The following lemma is needed in sequel.

Lemma 1 (see [5, 12]). Let $\Omega \subset \mathbb{C}$ and $\Psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ satisfy

$$\Psi(i\rho, \sigma; z) \notin \Omega \quad (6)$$

whenever $z \in \mathbb{D}$, ρ is real, and $\sigma \leq -(1 + \rho^2)/2$. If p is analytic in $\mathbb{D}$, with $p(0) = 1$, and $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.

In the case $\Psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$, then the condition in Lemma 1 is generalized to

$$\Psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega. \quad (7)$$

ρ is real, $\sigma + \mu \leq 0$, and $\sigma \leq -(1 + \rho^2)/2$.

2. Close-To-Convexity of Generalized Bessel Functions

In this section, one main result on the close-to-convexity of the generalized Bessel function with several consequences is discussed in detail.

Theorem 2. Let $-1 \leq B \leq 3 - 2\sqrt{2} \approx 0.171573$. Suppose $B < A \leq 1$ and $c, \kappa \in \mathbb{R}$ satisfy

$$\kappa - 1 \geq \frac{(1+B)(1+A)}{4(A-B)} |c|. \quad (8)$$

$$u_p(z) = \frac{(1-A) + (1+A)p(z)}{(1-B) + (1+B)p(z)},$$

$$u'_p(z) = \frac{2(A-B)p'(z)}{((1-B) + (1+B)p(z))^2},$$

$$u''_p(z) = \frac{2(A-B)((1-B) + (1+B)p(z))p''(z) - 4(1+B)(A-B)(p'(z))^2}{((1-B) + (1+B)p(z))^3}.$$

Thus, using identities (14), the Bessel differential equation (4) can be rewritten as

$$z^2 p''(z) - \frac{2(1+B)}{(1-B) + (1+B)p(z)} (zp'(z))^2$$

Further let A, B, κ , and c satisfy either the inequality

$$\begin{aligned} & (\kappa - 1)^2 + \frac{(\kappa - 1)(1+B)}{(1-B)} \\ & - \left| \frac{(\kappa - 1)(A+B)}{2(A-B)}c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)}c \right| \\ & \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2 \end{aligned} \quad (9)$$

whenever

$$\begin{aligned} & \left| 2(\kappa - 1)(1-B)(A+B)c + (1+B)^2(1+A)c \right| \\ & \geq \frac{1}{2}(A-B)(1-B)c^2 \end{aligned} \quad (10)$$

or the inequality

$$\begin{aligned} & \left((\kappa - 1) \frac{(A+B)}{2(A-B)}c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)}c \right)^2 \\ & \leq \frac{c^2}{4} \left((\kappa - 1)^2 + (\kappa - 1) \frac{(1+B)}{1-B} - \frac{(1-AB)^2}{16(A-B)^2}c^2 \right) \end{aligned} \quad (11)$$

whenever

$$\begin{aligned} & \left| 2(\kappa - 1)(1-B)(A+B)c + (1+B)^2(1+A)c \right| \\ & < \frac{1}{2}(A-B)(1-B)c^2. \end{aligned} \quad (12)$$

If $(1+B)u_p(z) \neq (1+A)$ for all $z \in \mathbb{D}$, then $u_p(z) \in \mathcal{P}[A, B]$.

Proof. Define the analytic function $p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = -\frac{(1-A) - (1-B)u_p(z)}{(1+A) - (1+B)u_p(z)}. \quad (13)$$

Then, a computation yields

$$\begin{aligned} & + \kappa zp'(z) \\ & + \frac{((1-B) + (1+B)p(z))((1-A) + (1+A)p(z))}{8(A-B)}cz \\ & = 0. \end{aligned} \quad (15)$$

Assume $\Omega = \{0\}$, and define $\Psi(r, s, t; z)$ by

$$\begin{aligned} \Psi(r, s, t; z) \\ := t - \frac{2(1+B)}{(1-B) + (1+B)r} s^2 + \kappa s \\ + \frac{((1-B) + (1+B)r)((1-A) + (1+A)r)}{8(A-B)} cz. \end{aligned} \quad (16)$$

It follows from (15) that $\Psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$. To ensure $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, from Lemma 1, it is enough to establish $\operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z) < 0$ in $\mathbb{D}$ for any real $\rho, \sigma \leq -(1 + \rho^2)/2$, and $\sigma + \mu \leq 0$.

With $z = x + iy \in \mathbb{D}$ in (16), a computation yields

$$\begin{aligned} \operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z) \\ = \mu - \frac{2(1-B^2)}{(1-B)^2 + (1+B)^2 \rho^2} \sigma^2 + \kappa \sigma \\ - \frac{\rho(1-AB)}{4(A-B)} cy \\ + \frac{(1-B)(1-A) - (1+B)(1+A) \rho^2}{8(A-B)} cx. \end{aligned} \quad (17)$$

Since $\sigma \leq -(1 + \rho^2)/2$ and $B \in [-1, 3 - 2\sqrt{2}]$,

$$\begin{aligned} \frac{2(1-B^2)}{(1-B)^2 + (1+B)^2 \rho^2} \sigma^2 \\ \geq \frac{2(1-B^2)}{(1-B)^2 + (1+B)^2 \rho^2} \frac{(1 + \rho^2)^2}{4} \geq \frac{1+B}{2(1-B)}. \end{aligned} \quad (18)$$

Thus

$$\begin{aligned} \operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z) \\ \leq (\kappa - 1) \sigma - \frac{1+B}{2(1-B)} - \frac{\rho(1-AB)}{4(A-B)} cy \\ + \frac{(1-B)(1-A) - (1+B)(1+A) \rho^2}{8(A-B)} cx \\ \leq -\frac{1}{2} (\kappa - 1) (1 + \rho^2) - \frac{1+B}{2(1-B)} - \frac{\rho(1-AB)}{4(A-B)} cy \\ + \frac{(1-B)(1-A) - (1+B)(1+A) \rho^2}{8(A-B)} cx \\ = p_1 \rho^2 + q_1 \rho + r_1 := Q(\rho), \end{aligned} \quad (19)$$

where

$$\begin{aligned} p_1 &= -\frac{1}{2} (\kappa - 1) - \frac{(1+B)(1+A) cx}{8(A-B)}, \\ q_1 &= -\frac{1-AB}{4(A-B)} cy, \\ r_1 &= -\frac{1}{2} (\kappa - 1) + \frac{(1-B)(1-A)}{8(A-B)} cx - \frac{1+B}{2(1-B)}. \end{aligned} \quad (20)$$

Condition (8) shows that

$$\begin{aligned} p_1 &= -\frac{1}{2} (\kappa - 1) - \frac{(1+B)(1+A) cx}{8(A-B)} \\ &< -\frac{1}{2} \left((\kappa - 1) - \frac{(1+B)(1+A)}{4(A-B)} |c| \right) \leq 0. \end{aligned} \quad (21)$$

Since $\max_{\rho \in \mathbb{R}} \{p_1 \rho^2 + q_1 \rho + r_1\} = (4p_1 r_1 - q_1^2)/(4p_1)$ for $p_1 < 0$, it is clear that $Q(\rho) < 0$ when

$$\begin{aligned} \frac{(1-AB)^2}{16(A-B)^2} c^2 y^2 \\ < 4 \left(-\frac{1}{2} (\kappa - 1) - \frac{(1+B)(1+A) cx}{8(A-B)} \right) \\ \cdot \left(-\frac{1}{2} (\kappa - 1) + \frac{(1-B)(1-A)}{8(A-B)} cx - \frac{1+B}{2(1-B)} \right), \end{aligned} \quad (22)$$

with $|x|, |y| < 1$. As $y^2 < 1 - x^2$, the above condition holds whenever

$$\begin{aligned} \frac{(1-AB)^2 c^2}{16(A-B)^2} (1 - x^2) \\ \leq \left((\kappa - 1) + \frac{(1+B)(1+A) cx}{4(A-B)} \right) \\ \cdot \left((\kappa - 1) - \frac{(1-B)(1-A)}{4(A-B)} cx + \frac{1+B}{1-B} \right), \end{aligned} \quad (23)$$

that is, when

$$\begin{aligned} \frac{c^2}{16} x^2 + \left((\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right) x \\ + (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - \frac{(1-AB)^2}{16(A-B)^2} c^2 \geq 0. \end{aligned} \quad (24)$$

To establish inequality (24), consider the polynomial R given by

$$R(x) := mx^2 + nx + r, \quad |x| < 1, \quad (25)$$

where

$$\begin{aligned} m &:= \frac{c^2}{16}, \\ n &:= (\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c, \\ r &:= (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} - \frac{(1-AB)^2}{16(A-B)^2} c^2. \end{aligned} \quad (26)$$

Constraint (10) yields $|n| \geq 2|m|$, and thus $R(x) \geq m + r - |n|$. Now inequality (9) readily implies that

$$\begin{aligned} R(x) &\geq m + r - |n| \\ &= \frac{c^2}{16} + (\kappa - 1)^2 + (\kappa - 1) \frac{1+B}{1-B} \\ &\quad - \frac{(1-AB)^2}{16(A-B)^2} c^2 \\ &\quad - \left| (\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right| \quad (27) \\ &= (\kappa - 1)^2 + (\kappa - 1) \frac{(1+B)}{1-B} \\ &\quad - \left| (\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{(1+B)^2(1+A)}{4(1-B)(A-B)} c \right| \\ &\quad - \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \geq 0. \end{aligned}$$

Now consider the case of constraint (12), which is equivalent to $|n| < 2m$. Then the minimum of R occurs at $x = -n/(2m)$, and (11) yields

$$R(x) \geq \frac{4mr - n^2}{4m} \geq 0. \quad (28)$$

Evidently Ψ satisfies the hypothesis of Lemma 1, and thus $\operatorname{Re} p(z) > 0$; that is,

$$-\frac{(1-A) - (1-B)u_p(z)}{(1+A) - (1+B)u_p(z)} < \frac{1+z}{1-z}. \quad (29)$$

Hence there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ such that

$$-\frac{(1-A) - (1-B)u_p(z)}{(1+A) - (1+B)u_p(z)} = \frac{1+w(z)}{1-w(z)}, \quad (30)$$

which implies that $u_p(z) < (1+Az)/(1+Bz)$. $\square$

Theorem 2 gives rise to simple conditions on c and κ to ensure $u_p(z)$ maps $\mathbb{D}$ into a half plane.

Corollary 3. *Let $c \leq 0$ and $\kappa \geq \max\{5/4, 1 + c^2/2\}$. Then $\operatorname{Re} u_p(z) > c/(c-1)$.*

Proof. Choose $A = -(c+1)/(c-1)$ and $B = -1$ in Theorem 2. Then condition (8) is equivalent to $\kappa \geq 1$ and (10) reduces to $\kappa \geq 5/4$, and clearly both hold for $\kappa \geq \max\{5/4, 1 + c^2/2\}$. The proof will complete if hypothesis (9) holds; that is,

$$(\kappa - 1)^2 \geq \frac{1}{2} (\kappa - 1) c^2. \quad (31)$$

Since $\kappa \geq 1 + c^2/2$, it follows that

$$(\kappa - 1)^2 - \frac{1}{2} (\kappa - 1) c^2 = (\kappa - 1) \left(\kappa - 1 - \frac{c^2}{2} \right) \geq 0, \quad (32)$$

which establishes (31). $\square$

Corollary 4. *Let $c \in \mathbb{R}$. Then if $\kappa \geq |c|/2$, then $\operatorname{Re} u_p(z) > 1/2$.*

Proof. Put $A = 0$ and $B = -1$ in Theorem 2. Condition (8) reduces to $\kappa \geq 1$, which holds in all cases. It is sufficient to establish conditions (10) and (9) or, equivalently,

$$\begin{aligned} 4(\kappa - 1) - |c| &\geq 0, \\ (\kappa - 1)^2 - \frac{1}{2} (\kappa - 1) |c| &\geq 0. \end{aligned} \quad (33)$$

The hypothesis $\kappa - 1 \geq |c|/2$ implies that $4(\kappa - 1) - |c| \geq |c| \geq 0$ and $(\kappa - 1)^2 - (1/2)(\kappa - 1)|c| \geq (\kappa - 1)(\kappa - 1 - |c|/2) \geq 0$. $\square$

Next theorem gives the sufficient condition for close-to-convexity when $B \geq 3 - 2\sqrt{2}$.

Theorem 5. *Let $3 - 2\sqrt{2} \leq B < A \leq 1$ and $c, \kappa \in \mathbb{R}$ satisfy*

$$\kappa - 1 \geq \frac{(1+B)(1+A)}{4(A-B)} |c|. \quad (34)$$

Suppose A, B, κ , and c satisfy either the inequality

$$\begin{aligned} (\kappa - 1)^2 - 16(\kappa - 1) \frac{B(1-B)}{(1+B)^3} \\ - \left| (\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right| \\ \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \end{aligned} \quad (35)$$

whenever

$$\begin{aligned} \left| (\kappa - 1)(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c \right| \\ \geq \frac{c^2}{4} (A-B)(1+B)^3 \end{aligned} \quad (36)$$

or the inequality

$$\begin{aligned} \left((\kappa - 1) \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right)^2 \\ \leq \frac{c^2}{4} \left((\kappa - 1)^2 + 16(\kappa - 1) \frac{B(1-B)}{(1+B)^3} \right. \\ \left. - \frac{(1-AB)^2}{16(A-B)^2} c^2 \right) \end{aligned} \quad (37)$$

whenever

$$\begin{aligned} \left| ((\kappa - 1)(A+B)(1+B)^3 + 8B(1-B^2)(1+A))c \right| \\ < \frac{c^2}{4} (A-B)(1+B)^3. \end{aligned} \quad (38)$$

If $(1+B)u_p(z) \neq (1+A)$ for all $z \in \mathbb{D}$, then $u_p(z) \in \mathcal{S}[A, B]$.

Proof. First, proceed similarly to the proof of Theorem 2 and derive the expression of $\operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z)$ as given in (17). Now, for $\sigma \leq -(1 + \rho^2)/2$, $\rho \in \mathbb{R}$, and $B \geq 3 - 2\sqrt{2}$,

$$\begin{aligned} & \frac{2(1 - B^2)}{(1 - B)^2 + (1 + B)^2 \rho^2} \sigma^2 \\ & \geq \frac{2(1 - B^2)}{(1 - B)^2 + (1 + B)^2 \rho^2} \frac{(1 + \rho^2)^2}{4} \geq \frac{8B(1 - B)}{(1 + B)^3}, \end{aligned} \quad (39)$$

and then, with $z = x + iy \in \mathbb{D}$ and $\mu + \sigma < 0$, it follows that

$$\begin{aligned} \operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z) & \leq -\frac{1}{2}(\kappa - 1)(1 + \rho^2) \\ & \quad - \frac{(1 + B)(1 + A)\rho^2}{8(A - B)} cx \\ & \quad - \frac{\rho(1 - AB)}{4(A - B)} cy \\ & \quad + \frac{(1 - B)(1 - A)}{8(A - B)} cx \\ & \quad - \frac{8B(1 - B)}{(1 + B)^3} \\ & = p_2 \rho^2 + q_2 \rho + r_2 := Q_1(\rho), \end{aligned} \quad (40)$$

where

$$\begin{aligned} p_2 & = -\frac{1}{2}(\kappa - 1) - \frac{(1 + B)(1 + A)}{8(A - B)} cx, \\ q_2 & = -\frac{(1 - AB)cy}{4(A - B)}, \\ r_2 & = -\frac{1}{2}(\kappa - 1) + \frac{(1 - B)(1 - A)}{8(A - B)} cx - \frac{8B(1 - B)}{(1 + B)^3}. \end{aligned} \quad (41)$$

Observe that inequality (34) implies that $p_2 < 0$. Thus $Q_1(\rho) < 0$ for all $\rho \in \mathbb{R}$ provided $q_2^2 \leq 4p_2 r_2$; that is, for $|x|, |y| < 1$,

$$\begin{aligned} & \frac{(1 - AB)^2}{16(A - B)^2} c^2 y^2 \leq \left((\kappa - 1) + \frac{(1 + B)(1 + A)}{4(A - B)} cx \right) \\ & \cdot \left((\kappa - 1) - \frac{(1 - B)(1 - A)}{4(A - B)} cx + \frac{16B(1 - B)}{(1 + B)^3} \right). \end{aligned} \quad (42)$$

With $y^2 < 1 - x^2$, it is enough to show, for $|x| < 1$,

$$\begin{aligned} & \frac{(1 - AB)^2}{16(A - B)^2} c^2 (1 - x^2) \\ & \leq \left((\kappa - 1) + \frac{(1 + B)(1 + A)}{4(A - B)} cx \right) \\ & \cdot \left((\kappa - 1) - \frac{(1 - B)(1 - A)}{4(A - B)} cx + \frac{16B(1 - B)}{(1 + B)^3} \right), \end{aligned} \quad (43)$$

which is equivalent to

$$R_1(x) := m_1 x^2 + n_1 x + r_1 \geq 0, \quad (44)$$

where

$$\begin{aligned} m_1 & := \frac{c^2}{16}, \\ n_1 & := \left((\kappa - 1) \left(\frac{c(A + B)}{2(A - B)} \right) + \frac{4B(1 - B^2)(1 + A)}{(A - B)(1 + B)^3} c \right), \end{aligned} \quad (45)$$

$$r_1 := (\kappa - 1)^2 - (\kappa - 1) \frac{16B(1 - B)}{(1 + B)^3} - \frac{c^2(1 - AB)^2}{16(A - B)^2}.$$

If (36) holds, then $|n_1| \geq 2|m_1|$ and $R_1(x) \geq m_1 + r_1 - |n_1|$, which is nonnegative from (35). On the other hand, if (38) holds, then $|n_1| < 2|m_1|$, $R_1(x) \geq (4m_1 r_1 - n_1^2)/4m_1$, and (37) implies $R_1(x) \geq 0$. Either case establishes (44). $\square$

Theorem 6. Let $-1 \leq B \leq 3 - 2\sqrt{2} \approx 0.171573$. Suppose $B < A \leq 1$ and $c, \kappa \in \mathbb{R}$ with $c \neq 0$ satisfying

$$\kappa \geq \frac{(1 + B)(1 + A)}{4(A - B)} |c|. \quad (46)$$

Further let A, B, κ , and c satisfy either

$$\begin{aligned} & \kappa^2 + \kappa \frac{1 + B}{1 - B} - \left| \kappa \frac{(A + B)}{2(A - B)} c + \frac{(1 + B)^2(1 + A)}{4(1 - B)(A - B)} c \right| \\ & \geq \frac{(1 - A^2)(1 - B^2)}{16(A - B)^2} c^2 \end{aligned} \quad (47)$$

whenever

$$\begin{aligned} & |2\kappa(1 - B)(A + B)c + (1 + B)^2(1 + A)c| \\ & \geq \frac{1}{2}(A - B)(1 - B)c^2 \end{aligned} \quad (48)$$

or the inequality

$$\begin{aligned} & \left(\kappa \frac{(A + B)}{2(A - B)} + \frac{(1 + B)^2(1 + A)}{4(1 - B)(A - B)} c \right)^2 \\ & \leq \frac{c^2}{4} \left(\kappa^2 + \frac{\kappa(1 + B)}{1 - B} - \frac{(1 - AB)^2}{16(A - B)^2} c^2 \right) \end{aligned} \quad (49)$$

when

$$\begin{aligned} & |2\kappa(1 - B)(A + B)c + (1 + B)^2(1 + A)c| \\ & < \frac{1}{2}(A - B)(1 - B)c^2. \end{aligned} \quad (50)$$

If $(1 + B)u_p(z) \neq (1 + A)$ for all $z \in \mathbb{D}$, then $(-4\kappa/c)u'_p(z) \in \mathcal{P}[A, B]$.

Theorem 7. Let $3 - 2\sqrt{2} < B < A \leq 1$. Suppose $c, \kappa \in \mathbb{R}$ and $a \neq 0$, such that

$$\kappa \geq \frac{(1+B)(1+A)}{4(A-B)} |c|. \quad (51)$$

Suppose A, B, κ , and c satisfy either

$$\begin{aligned} & \kappa^2 + 16\kappa \frac{B(1-B)}{(1+B)^3} \\ & - \left| \kappa \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right| \\ & \geq \frac{(1-A^2)(1-B^2)}{16(A-B)^2} c^2 \end{aligned} \quad (52)$$

whenever

$$\begin{aligned} & \left| \kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c \right| \\ & \geq \frac{c^2}{4} (A-B)(1+B)^3 \end{aligned} \quad (53)$$

or the inequality

$$\begin{aligned} & \left(\kappa \frac{(A+B)}{2(A-B)} c + \frac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)} c \right)^2 \\ & \leq \frac{c^2}{4} \left(\kappa^2 + \frac{16\kappa B(1-B)}{(1+B)^3} - \frac{(1-AB)^2}{16(A-B)^2} c^2 \right) \end{aligned} \quad (54)$$

when

$$\begin{aligned} & \left| 2\kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c \right| \\ & < \frac{c^2}{4} (A-B)(1+B)^3. \end{aligned} \quad (55)$$

If $(1+B)u_p(z) \neq (1+A)$ for all $z \in \mathbb{D}$, then $(-4\kappa/c)u_p'(z) \in \mathcal{P}[A, B]$.

Corollary 8. Let $c \leq -1$ and

$$\kappa \geq \max \left\{ \frac{c(c+1)}{2}, \frac{c}{2(c+1)} \right\}. \quad (56)$$

Then $(-4\kappa/c)(u_p(z) - 1)$ is close-to-convex of order $(c+1)/c$ with respect to the identity function.

Corollary 9. Let c be a nonzero real number and $\kappa \geq |c|/2$. Then

$$\operatorname{Re} \left(-\frac{4\kappa}{c} \right) u_p'(z) > \frac{1}{2}. \quad (57)$$

3. Janowski Starlikeness of Generalized Bessel Functions

This section contributes to finding conditions to ensure a normalized and generalized Bessel function $zu_p(z)$ in the

class of Janowski starlike functions. For this purpose, first sufficient conditions for $u_p(z)$ to be Janowski convex are determined, and then an application of relation (5) yields conditions for $zu_p(z) \in \mathcal{S}^*[A, B]$.

Theorem 10. Let $c, \kappa \in \mathbb{R}$ be such that $(A-B)u_p'(z) \neq (1+B)zu_p''(z)$ for all $z \in \mathbb{D}$ and $-1 \leq B < A \leq 1$. Suppose

$$\kappa(1+B) \geq \frac{(1+B)^2}{4(A-B)} |c| - (1+A-B). \quad (58)$$

Further let A, B, κ , and c satisfy

$$\begin{aligned} & (1+A-B+\kappa(1+B))(1-A+B+\kappa(1-B)) \\ & \geq \frac{(1-B^2)^2}{16(A-B)^2} c^2 \\ & + \left| \frac{-A(B^2+1)+B(3+B^2)+(1-B^2)B\kappa c}{2(A-B)} \right|. \end{aligned} \quad (59)$$

If $0 \notin u_p'(\mathbb{D})$ and $0 \notin u_p''(\mathbb{D})$, then

$$1 + \frac{zu_p''(z)}{u_p'(z)} < \frac{1+Az}{1+Bz}. \quad (60)$$

Proof. Define an analytic function $p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) := \frac{(A-B)u_p'(z) + (1-B)zu_p''(z)}{(A-B)u_p'(z) - (1+B)zu_p''(z)}. \quad (61)$$

Then

$$\frac{zu_p''(z)}{u_p'(z)} = \frac{(A-B)(p(z)-1)}{(p(z)+1)+B(p(z)-1)}, \quad (62)$$

$$\begin{aligned} & \frac{z^2 u_p'''(z) + zu_p''(z)}{zu_p''(z)} - \frac{zu_p''(z)}{u_p'(z)} \\ & = \frac{zp'(z)}{(p(z)-1)} - \frac{(1+B)zp'(z)}{(p(z)+1)+B(p(z)-1)} \\ & = \frac{zp'(z)((p(z)+1)+B(p(z)-1)-(1+B)(p(z)-1))}{(p(z)-1)((p(z)+1)+B(p(z)-1))}. \end{aligned} \quad (63)$$

A rearrangement of (63) yields

$$\begin{aligned} & \frac{zu_p'''(z)}{u_p''(z)} = \frac{2zp'(z)}{(p(z)-1)((p(z)+1)+B(p(z)-1))} \\ & - 1 + \frac{zu_p''(z)}{u_p'(z)}. \end{aligned} \quad (64)$$

Thus,

$$\begin{aligned} & \left(\frac{zu_p''''(z)}{u_p''(z)} \right) \left(\frac{zu_p''(z)}{u_p'(z)} \right) \\ &= \frac{2(A-B)(p(z)-1)zp'(z)}{(p(z)-1)((p(z)+1)+B(p(z)-1))^2} \\ & \quad - \frac{(A-B)(p(z)-1)}{(p(z)+1)+B(p(z)-1)} \\ & \quad + \frac{(A-B)^2(p(z)-1)^2}{((p(z)+1)+B(p(z)-1))^2}. \end{aligned} \quad (65)$$

Now a differentiation of (4) leads to

$$4z^2u_p''''(z) + 4(\kappa+1)zu_p''(z) + czu_p'(z) = 0, \quad (66)$$

which gives

$$\left(\frac{zu_p''''(z)}{u_p''(z)} \right) \left(\frac{zu_p''(z)}{u_p'(z)} \right) + (\kappa+1) \frac{zu_p''(z)}{u_p'(z)} + \frac{c}{4}z = 0. \quad (67)$$

Using (62) and (65), (67) yields

$$\begin{aligned} & \frac{2(A-B)zp'(z)}{((p(z)+1)+B(p(z)-1))^2} \\ & + \frac{(A-B)^2(p(z)-1)^2}{((p(z)+1)+B(p(z)-1))^2} \\ & + \frac{(A-B)(p(z)-1)\kappa}{(p(z)+1)+B(p(z)-1)} + \frac{c}{4}z = 0, \end{aligned} \quad (68)$$

and equivalently

$$\begin{aligned} & zp'(z) \\ & + \left(\frac{A-B}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)} \right) (p(z))^2 \\ & - \left(A-B+\kappa B - \frac{c(1-B^2)}{4(A-B)}z \right) p(z) \\ & + \left(\frac{A-B}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)} \right) = 0. \end{aligned} \quad (69)$$

Define

$$\begin{aligned} \Psi(p(z), zp'(z), z) &:= zp'(z) + F_1(p(z))^2 \\ & \quad + F_2p(z) + F_3, \end{aligned} \quad (70)$$

where

$$\begin{aligned} F_1 &= \frac{(A-B)}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)}, \\ F_2 &= -(A-B) - \kappa B + \frac{c(1-B^2)}{4(A-B)}z, \\ F_3 &= \frac{(A-B)}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)}. \end{aligned} \quad (71)$$

Thus, (69) yields $\Psi(p(z), zp'(z), z) \in \Omega = \{0\}$. Now, with $z = x + iy \in \mathbb{D}$, let

$$\begin{aligned} G_1 &:= \operatorname{Re}(F_1) = \frac{A-B}{2} + \frac{\kappa(1+B)}{2} + \frac{cx(1+B)^2}{8(A-B)} \\ &= \frac{1}{2} \left(A-B+\kappa(1+B) + \frac{cx(1+B)^2}{4(A-B)} \right), \\ G_2 &:= \operatorname{Re}(iF_2) = -\frac{c(1-B^2)}{4(A-B)}y, \\ G_3 &:= \operatorname{Re}(F_3) = \frac{A-B}{2} - \frac{\kappa(1-B)}{2} + \frac{cx(1-B)^2}{8(A-B)} \\ &= \frac{1}{2} \left(A-B-\kappa(1-B) + \frac{cx(1-B)^2}{4(A-B)}x \right). \end{aligned} \quad (72)$$

For $\sigma \leq -(1+\rho^2)/2$, $\rho \in \mathbb{R}$,

$$\begin{aligned} \operatorname{Re} \Psi(i\rho, \sigma, z) &= \sigma - G_1\rho^2 + G_2\rho + G_3 \\ &\leq -\frac{1+2G_1}{2}\rho^2 + G_2\rho + \frac{2G_3-1}{2} \\ &:= Q(\rho). \end{aligned} \quad (73)$$

Note that condition (58) implies $(1+2G_1)/2 > 0$. In this case, Q has a maximum at $\rho = G_2/(1+2G_1)$. Thus $Q(\rho) < 0$ for all real ρ provided

$$G_2^2 \leq (1+2G_1)(1-2G_3), \quad |x|, |y| < 1. \quad (74)$$

Since $y^2 < 1-x^2$, it is left to show that

$$\begin{aligned} & \frac{(1-B^2)^2}{16(A-B)^2}c^2(1-x^2) \\ & \leq \left(1+A-B+\kappa(1+B) + \frac{c(1+B)^2}{4(A-B)}x \right) \\ & \quad \cdot \left(1-A+B-\kappa(-1+B) - \frac{c(1-B)^2}{4(A-B)}x \right), \end{aligned} \quad (75)$$

$|x| < 1$. The above inequality is equivalent to

$$H(x) := h_2(A, B)x + h_3(A, B) \geq 0, \quad (76)$$

where

$$\begin{aligned} h_2(A, B) &= \frac{(-A(B^2 + 1) + B(3 + B^2) + (1 - B^2)B\kappa)c}{2(A - B)}, \\ h_3(A, B) &= (1 + A - B + \kappa(1 + B))(1 - A + B - \kappa(B - 1)) \\ &\quad - \frac{(1 - B^2)^2}{16(A - B)^2}c^2. \end{aligned} \quad (77)$$

Since $|x| < 1$, the left-hand side of inequality (76) satisfies

$$h_2(A, B)x + h_3(A, B) \geq -|h_2(A, B)| + h_3(A, B). \quad (78)$$

Now it is evident from (59) that $H(x) \geq 0$ which establishes inequality (76).

Thus Ψ satisfies the hypothesis of Lemma 1, and hence $\operatorname{Re} p(z) > 0$, or equivalently

$$\frac{(A - B)u'_p + (1 - B)zu''_p}{(A - B)u'_p - (1 + B)zu''_p} < \frac{1 + z}{1 - z}. \quad (79)$$

By definition of subordination, there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ and

$$\frac{(A - B)u'_p(z) + (1 - B)zu''_p(z)}{(A - B)u'_p(z) - (1 + B)zu''_p(z)} = \frac{1 + w(z)}{1 - w(z)}. \quad (80)$$

A simple computation shows that

$$1 + \frac{zu''_p(z)}{u'_p(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (81)$$

and hence

$$1 + \frac{zu''_p(z)}{u'_p(z)} < \frac{1 + Az}{1 + Bz}. \quad (82)$$

□

Relation (5) also shows that

$$\frac{z(zu_p(z))'}{zu_p(z)} = 1 + \frac{zu''_{p-1}(z)}{u'_{p-1}(z)}. \quad (83)$$

Together with Theorem 10, relation (83) immediately yields the following result for $zu_p(z) \in \mathcal{S}^*[A, B]$.

Theorem 11. Let c and κ be real numbers such that $(A - B)u'_{p-1}(z) \neq (1 + B)zu''_{p-1}(z)$ for all $z \in \mathbb{D}$ and $-1 \leq B < A \leq 1$. Suppose

$$\kappa(1 + B) \geq \frac{(1 + B)^2}{4(A - B)}|c| - (A - 2B). \quad (84)$$

Further let A, B, κ , and c satisfy

$$\begin{aligned} &(A - 2B + \kappa(1 + B))(2B - A + \kappa(1 - B)) \\ &\geq \frac{(1 - B^2)^2}{16(A - B)}c^2 \\ &\quad + \left| \frac{B^3 - (A - B)(1 + B^2) + (1 - B^2)B\kappa}{2(A - B)}c \right|. \end{aligned} \quad (85)$$

Then $zu_p(z) \in \mathcal{S}^*[A, B]$.

Competing Interests

The authors declare that they have no competing interests.

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