Research Article

Hermite-Hadamard Type Inequalities Obtained via Fractional Integral for Differentiable m -Convex and (α, m) -Convex Functions

Erhan Set,¹ Süleyman Sami Karataş,¹ and Muhammad Adil Khan²

¹Department of Mathematics, Faculty of Sciences, Ordu University, 52200 Ordu, Turkey

²Department of Mathematics, University of Peshawar, Peshawar, Pakistan

Correspondence should be addressed to Erhan Set; erhanset@yahoo.com

Received 1 July 2016; Accepted 20 September 2016

Academic Editor: Ahmed Zayed

Copyright © 2016 Erhan Set et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Some Hermite-Hadamard type inequalities involving fractional integrals for m -convex and (α, m) -convex functions are obtained.

1. Introduction and Preliminaries

The following definition is well known in the literature.

Definition 1. A function $f : I \rightarrow \mathbb{R}, 0 \neq I \subseteq \mathbb{R}$, is said to be convex on the interval I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Geometrically, this means that if P, Q , and R are three points on the graph of f with Q between P and R , then Q is on or below the chord PR .

Theorem 2 (Hermite-Hadamard inequality). Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

m -convexity was defined by Toader as follows.

Definition 3 (see [1]). The function $f : [0, b] \rightarrow \mathbb{R}, b > 0$, is said to be m -convex, where $m \in [0, 1]$, if one has

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \quad (3)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. One says that f is m -concave if $-f$ is m -convex. Denote by $K_m(b)$ the class of all m -convex functions on $[0, b]$ for which $f(0) \leq 0$.

Obviously, for $m = 1$, Definition 3 recaptures the concept of standard convex functions on $[a, b]$ and for $m = 0$ the concept of starshaped functions. The notion of m -convexity has been further generalized in [2] as it is stated in the following definition.

Definition 4 (see [2]). The function $f : [0, b] \rightarrow \mathbb{R}, b > 0$, is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if one has

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y) \quad (4)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that when $(\alpha, m) \in \{(1, 1), (1, m)\}$ one obtains the following classes of functions: convex and m -convex, respectively. Note that $K_1^1(b)$ is a proper subclass of m -convex and (α, m) -functions $[0, b]$. The interested reader can find more about partial ordering of convexity in [3].

Definition 5 (see [4]). Let $f \in L_1[a, b]$. Then Riemann-Liouville integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\begin{aligned} J_{a^+}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \\ J_{b^-}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b, \end{aligned} \quad (5)$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx \quad (6)$$

is the Gamma function.

We now give the definition of the hypergeometric series which will be used in obtaining some integrals.

Definition 6 (see [4]). The integral representation of the hypergeometric functions is as follows:

$$\begin{aligned} {}_2F_1[a, b, c; z] \\ = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \end{aligned} \quad (7)$$

where $|z| < 1$, $c > b > 0$, and

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad (8)$$

is Beta function with

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (9)$$

In the present paper, we establish some new Hermite-Hadamard's type inequalities for the classes of m -convex and (α, m) -convex functions via Riemann-Liouville fractional integrals.

To prove our main results, we consider the following lemma.

Lemma 7 (see [5]). Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$. Then, for $n \in \mathbb{N}$, $k > 0$, and $x \in [a, b]$, one has

$$\begin{aligned} G(k; n; a, x, b)(f) \\ = \frac{(x-a)^{k+1}}{b-a} \left\{ \int_0^1 \frac{t^k}{2} f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) dt \right. \\ \left. - \int_0^1 \frac{t^k}{2} f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) dt \right\} \\ - \frac{(b-x)^{k+1}}{b-a} \left\{ \int_0^1 \frac{t^k}{2} f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) dt \right. \\ \left. - \int_0^1 \frac{t^k}{2} f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) dt \right\}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} G(k; n; a, x, b)(f) &= \frac{n+1}{2} \left[\frac{(x-a)^k + (b-x)^k}{b-a} f(x) \right. \\ &\quad \left. + \frac{(x-a)^k f(a) + (b-x)^k f(b)}{b-a} \right] \\ &\quad - \frac{(n+1)^{k+1} \Gamma(k+1)}{2(b-a)} \left\{ J_{x^-}^k f \left(\frac{n}{n+1}x + \frac{1}{n+1}a \right) \right. \\ &\quad \left. + J_{a^+}^k f \left(\frac{1}{n+1}x + \frac{n}{n+1}a \right) \right. \\ &\quad \left. + J_{x^+}^k f \left(\frac{n}{n+1}x + \frac{1}{n+1}b \right) \right. \\ &\quad \left. + J_{b^-}^k f \left(\frac{1}{n+1}x + \frac{n}{n+1}a \right) \right\}. \end{aligned} \quad (11)$$

2. Generalized Inequalities for m -Convex Functions

Theorem 8. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}$, $k > 0$ and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|$ is an m -convex function on $[a, b]$ for some fixed $m \in (0, 1]$, then

$$\begin{aligned} |G(k; n; a, x, b)(f)| \\ \leq \frac{(x-a)^{k+1}}{2(b-a)(k+1)} \left[|f'(x)| + m \left| f' \left(\frac{a}{m} \right) \right| \right] \\ + \frac{(b-x)^{k+1}}{2(b-a)(k+1)} \left[|f'(x)| + m \left| f' \left(\frac{b}{m} \right) \right| \right], \end{aligned} \quad (12)$$

where $x \in [a, b]$.

Proof. Using Lemma 7, taking modulus and the fact that $|f'|$ is an m -convex function, we have

$$\begin{aligned} |G(k; n; a, x, b)(f)| \\ \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt \right. \\ \left. + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right] \\ + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt \right. \\ \left. + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right] \\ = \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + m \frac{1-t}{n+1} \frac{a}{m} \right) \right| dt \right. \\ \left. + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + m \frac{n+t}{n+1} \frac{a}{m} \right) \right| dt \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + m \frac{1-t}{n+1} \frac{b}{m} \right) \right| dt \right. \\
& + \left. \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + m \frac{n+t}{n+1} \frac{b}{m} \right) \right| dt \right] \\
& \leq \frac{(x-a)^{k+1}}{2(b-a)(k+1)} \left[|f'(x)| + m \left| f' \left(\frac{a}{m} \right) \right| \right] \\
& + \frac{(b-x)^{k+1}}{2(b-a)(k+1)} \left[|f'(x)| + m \left| f' \left(\frac{b}{m} \right) \right| \right].
\end{aligned} \tag{13}$$

This completes the proof. $\square$

Remark 9. Observe that if in Theorem 8 we have $m = n = 1$, the statement of Theorem 8 becomes the statement of Theorem 1 in [6].

Theorem 10. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}, k > 0$ and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|^q$ is an m -convex function on $[a, b]$ for some fixed $m \in (0, 1]$ and $1/p + 1/q = 1, q > 1$, then

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \left[\left[\frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \right. \right. \\
& \cdot \left(\frac{1}{n+1} \right)^{1/q} \\
& \times \left[\left(\frac{(2n+1)|f'(x)|^q + m|f'(a/m)|^q}{2} \right)^{1/q} \right. \\
& + \left. \left. \left(\frac{|f'(x)|^q + m(2n+1)|f'(a/m)|^q}{2} \right)^{1/q} \right] \right] \\
& + \left[\frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{1/q} \right. \\
& \times \left[\left(\frac{(2n+1)|f'(x)|^q + m|f'(b/m)|^q}{2} \right)^{1/q} \right. \\
& + \left. \left. \left(\frac{|f'(x)|^q + m(2n+1)|f'(b/m)|^q}{2} \right)^{1/q} \right] \right],
\end{aligned} \tag{14}$$

where $x \in [a, b]$.

Proof. Using Lemma 7, Hölder's inequality, and the fact that $|f'|^q$ is an m -convex function,

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \Bigg] \\
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt \right. \\
& + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \Bigg] \\
& \leq \frac{(x-a)^{k+1}}{b-a} \left(\int_0^1 \left(\frac{t^k}{2} \right)^p dt \right)^{1/p} \\
& \cdot \left[\left(\int_0^1 \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{1/q} \right. \\
& + \left. \left(\int_0^1 \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{1/q} \right] \\
& + \frac{(b-x)^{k+1}}{b-a} \left(\int_0^1 \left(\frac{t^k}{2} \right)^p dt \right)^{1/p} \\
& \cdot \left[\left(\int_0^1 \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{1/q} \right. \\
& + \left. \left(\int_0^1 \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{1/q} \right] \\
& \leq \frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{1/q} \left[\left(\int_0^1 (n+t) \right. \right. \\
& \cdot \left. |f'(x)|^q + m(1-t) \left| f' \left(\frac{a}{m} \right) \right|^q dt \right)^{1/q} \\
& + \left(\int_0^1 (1-t) |f'(x)|^q + m(n+t) \right. \\
& \cdot \left. \left| f' \left(\frac{a}{m} \right) \right|^q dt \right)^{1/q} \Bigg] + \frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \\
& \cdot \left(\frac{1}{n+1} \right)^{1/q} \left[\left(\int_0^1 (n+t) |f'(x)|^q + m(1-t) \right. \right. \\
& \cdot \left. \left| f' \left(\frac{b}{m} \right) \right|^q dt \right)^{1/q} + \left(\int_0^1 (1-t) |f'(x)|^q \right. \\
& + \left. m(n+t) \left| f' \left(\frac{b}{m} \right) \right|^q dt \right)^{1/q} \Bigg] \\
& = \left[\frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{1/q} \right. \\
& \times \left[\left(\frac{(2n+1)|f'(x)|^q + m|f'(a/m)|^q}{2} \right)^{1/q} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{|f'(x)|^q + m(2n+1)|f'(a/m)|^q}{2} \right)^{1/q} \Bigg] \\
& + \left[\frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{kp+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{1/q} \right. \\
& \times \left[\left(\frac{(2n+1)|f'(x)|^q + m|f'(b/m)|^q}{2} \right)^{1/q} \right. \\
& \left. \left. + \left(\frac{|f'(x)|^q + m(2n+1)|f'(b/m)|^q}{2} \right)^{1/q} \right] \right]. \quad (15)
\end{aligned}$$

This completes the proof. $\square$

Remark 11. Observe that if in Theorem 10 we have $m = n = 1$, the statement of Theorem 10 becomes the statement of Theorem 2 in [6].

Theorem 12. Let I be on open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}$, $k > 0$ and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|^q$ is an m -convex function on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned}
& |G(k; n; a, x, b)(f)| \\
& \leq \frac{(x-a)^{k+1}}{2(b-a)(k+1)} \left(\frac{1}{(k+2)(n+1)} \right)^{1/q} \\
& \cdot \left[\left((n(k+2) + k + 1) |f'(x)|^q \right. \right. \\
& + m \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} + \left(|f'(x)|^q \right. \\
& + m \left(n(k+2) + k + 1 \right) \left| f' \left(\frac{a}{m} \right) \right|^q \Big)^{1/q} \Big] \\
& + \frac{(b-x)^{k+1}}{2(b-a)(k+1)} \left(\frac{1}{(k+2)(n+1)} \right)^{1/q} \\
& \cdot \left[\left((n(k+2) + k + 1) |f'(x)|^q \right. \right. \\
& + m \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} + \left(|f'(x)|^q \right. \\
& + m \left(n(k+2) + k + 1 \right) \left| f' \left(\frac{b}{m} \right) \right|^q \Big)^{1/q} \Big], \quad (16)
\end{aligned}$$

where $x \in [a, b]$.

Proof. Using Lemma 7, Power's mean inequality, and the fact that $|f'|^q$ is an m -convex function,

$$\begin{aligned}
& |G(k; n; a, x, b)(f)| \\
& \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt \right. \\
& + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \Big] \\
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt \right. \\
& + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \Big] \\
& \leq \frac{(x-a)^{k+1}}{b-a} \left(\int_0^1 \frac{t^k}{2} dt \right)^{1-1/q} \\
& \cdot \left[\left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{1/q} \right. \\
& + \left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{1/q} \Big] \\
& + \frac{(b-x)^{k+1}}{b-a} \left(\int_0^1 \frac{t^k}{2} dt \right)^{1-1/q} \\
& \cdot \left[\left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{1/q} \right. \\
& + \left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{1/q} \Big] \\
& \leq \frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \left(\frac{1}{n+1} \right)^{1/q} \\
& \cdot \left\{ \left(\int_0^1 t^k \left[(n+t) |f'(x)|^q + m(1-t) \left| f' \left(\frac{a}{m} \right) \right|^q \right] dt \right)^{1/q} \right. \\
& + \left(\int_0^1 t^k \left[(1-t) |f'(x)|^q + m(n+t) \left| f' \left(\frac{a}{m} \right) \right|^q \right] dt \right)^{1/q} \Big\} \\
& + \frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \left(\frac{1}{n+1} \right)^{1/q} \\
& \cdot \left\{ \left(\int_0^1 t^k \left[(n+t) |f'(x)|^q + m(1-t) \left| f' \left(\frac{b}{m} \right) \right|^q \right] dt \right)^{1/q} \right. \\
& + \left(\int_0^1 t^k \left[(1-t) |f'(x)|^q + m(n+t) \left| f' \left(\frac{b}{m} \right) \right|^q \right] dt \right)^{1/q} \Big\} \\
& = \frac{(x-a)^{k+1}}{2(b-a)(k+1)} \left(\frac{1}{(k+2)(n+1)} \right)^{1/q} \\
& \cdot \left[\left((n(k+2) + k + 1) |f'(x)|^q + m \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \right. \\
& + \left(|f'(x)|^q + m(n(k+2) + k + 1) \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \Big] \\
& + \left[\left((n(k+2) + k + 1) |f'(x)|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right. \\
& + \left(|f'(x)|^q + m(n(k+2) + k + 1) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(b-x)^{k+1}}{2(b-a)(k+1)} \left(\frac{1}{(k+2)(n+1)} \right)^{1/q} \\
& \cdot \left[\left((n(k+2) + k + 1) |f'(x)|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right. \\
& \left. + \left(|f'(x)|^q + m(n(k+2) + k + 1) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right].
\end{aligned} \tag{17}$$

This completes the proof. $\square$

Remark 13. Observe that if in Theorem 12 we have $m = n = 1$, the statement of Theorem 12 becomes the statement of Theorem 3 in [6].

3. Generalized Inequalities for (α, m) -Convex Functions

Theorem 14. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}, k > 0$, and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|$ is (α, m) -convex function on $[a, b]$ for some fixed $(\alpha, m) \in (0, 1]^2$, then

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \frac{(x-a)^{k+1}}{2(b-a)(n+1)^\alpha} \left[A |f'(x)| \right. \\
& + m(B-A) \left| f' \left(\frac{a}{m} \right) \right| \\
& + \frac{(b-x)^{k+1}}{2(b-a)(n+1)^\alpha} \left[A |f'(x)| \right. \\
& \left. + m(B-A) \left| f' \left(\frac{b}{m} \right) \right| \right],
\end{aligned} \tag{18}$$

where $x \in [a, b]$ and

$$\begin{aligned}
A & = \frac{n^\alpha}{k+1} {}_2F_1 \left(-\alpha, k+1, k+2; -\frac{1}{n} \right) \\
& + \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \\
B & = \frac{2(n+1)^\alpha}{k+1}.
\end{aligned} \tag{19}$$

Proof. Using Lemma 7 and taking modulus and the fact that $|f'|$ is (α, m) -convex function, we have

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt \right. \\
& \left. + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt \right. \\
& \left. + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right] \\
& \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)| \right. \right. \\
& + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{a}{m} \right) \right| \Big) dt \\
& + \int_0^1 \frac{t^k}{2} \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)| + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \right. \\
& \left. \cdot \left| f' \left(\frac{a}{m} \right) \right| \right) dt \Big] \\
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)| \right. \right. \\
& + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{b}{m} \right) \right| \Big) dt \\
& + \int_0^1 \frac{t^k}{2} \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)| + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \right. \\
& \left. \cdot \left| f' \left(\frac{b}{m} \right) \right| \right) dt \Big] \\
& = \frac{(x-a)^{k+1}}{2(b-a)(n+1)^\alpha} \left[A |f'(x)| + m(B-A) \right. \\
& \cdot \left| f' \left(\frac{a}{m} \right) \right| \Big] + \frac{(b-x)^{k+1}}{2(b-a)(n+1)^\alpha} \left[A |f'(x)| \right. \\
& \left. + m(B-A) \left| f' \left(\frac{b}{m} \right) \right| \right].
\end{aligned} \tag{20}$$

This completes the proof. $\square$

Remark 15. Observe that if in Theorem 14 we have $\alpha = 1$, the statement of Theorem 14 becomes the statement of Theorem 8.

Theorem 16. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}, k > 0$, and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|^q$ is (α, m) -convex function on $[a, b]$ for some fixed $(\alpha, m) \in (0, 1]^2$ and $1/p + 1/q = 1, q > 1$, then

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \frac{(x-a)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \\
& \cdot \left(\frac{1}{n+1} \right)^{\alpha/q} \\
& \cdot \left[\left(C |f'(x)|^q + m(D-C) \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left((D-C) |f'(x)|^q + mC \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \quad \text{where } x \in [a, b] \text{ and} \\
& + \frac{(b-x)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{\alpha/q} \\
& \cdot \left[\left(C |f'(x)|^q + m(D-C) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right. \\
& \left. + \left((D-C) |f'(x)|^q + mC \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right], \quad (21)
\end{aligned}$$

$$\begin{aligned}
C &= \frac{(n+1)^\alpha - n^\alpha}{\alpha + 1} \\
D &= (n+1)^\alpha.
\end{aligned} \quad (22)$$

Proof. Using Lemma 7, Hölder's inequality, and the fact that $|f'|^q$ is (α, m) -convex function,

$$\begin{aligned}
|G(k; n; a, x, b)(f)| &\leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right] \\
&+ \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right] \\
&\leq \frac{(x-a)^{k+1}}{b-a} \left(\int_0^1 \left(\frac{t^k}{2} \right)^p dt \right)^{1/p} \left[\int_0^1 \left(\left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{1/q} + \int_0^1 \left(\left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{1/q} \right] \\
&+ \frac{(b-x)^{k+1}}{b-a} \left(\int_0^1 \left(\frac{t^k}{2} \right)^p dt \right)^{1/p} \left[\int_0^1 \left(\left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{1/q} + \int_0^1 \left(\left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{1/q} \right] \\
&\leq \frac{(x-a)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \left\{ \left[\int_0^1 \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right) dt \right]^{1/q} \right. \\
&+ \left. \left[\int_0^1 \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} + \frac{(b-x)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \\
&\cdot \left\{ \left[\int_0^1 \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right. \\
&+ \left. \left[\int_0^1 \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} = \frac{(x-a)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \left(\frac{1}{n+1} \right)^{\alpha/q} \\
&\cdot \left[\left(C |f'(x)|^q + m(D-C) \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} + \left((D-C) |f'(x)|^q + mC \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{1/q} \right] + \frac{(b-x)^{k+1}}{b-a} \left(\frac{1}{pk+1} \right)^{1/p} \\
&\cdot \left(\frac{1}{n+1} \right)^{\alpha/q} \left[\left(C |f'(x)|^q + m(D-C) \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} + \left((D-C) |f'(x)|^q + mC \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right]. \quad (23)
\end{aligned}$$

This completes the proof. $\square$

Remark 17. Observe that if in Theorem 16 we have $\alpha = 1$, the statement of Theorem 16 becomes the statement of Theorem 10.

Theorem 18. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $n \in \mathbb{N}, k > 0$, and $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|^q$ is

(α, m) -convex function on $[a, b]$ for some fixed $(\alpha, m) \in (0, 1]^2$ and $q \geq 1$, then

$$\begin{aligned}
|G(k; n; a, x, b)(f)| &= \frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \\
&\cdot \left(\frac{1}{n+1} \right)^{\alpha/q} \left\{ \left[\left(A - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) |f'(x)|^q \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + m \left(\frac{B}{2} - A + \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{a}{m} \right) \right|^q \Big]^{1/q} \\
& + \left[\frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} |f'(x)|^q \right. \\
& + m \left(\frac{B}{2} - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{a}{m} \right) \right|^q \Big]^{1/q} \Big\} \\
& + \frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \left(\frac{1}{n+1} \right)^{\alpha/q} \\
& \cdot \left\{ \left[\left(A - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) |f'(x)|^q \right. \right. \\
& \left. \left. + m \left(\frac{B}{2} - A + \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right]^{1/q} \right. \\
& \left. + \left[\frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} |f'(x)|^q \right. \right. \\
& \left. \left. + m \left(\frac{B}{2} - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}, \tag{24}
\end{aligned}$$

where A and B are given by (19) and $x \in [a, b]$.

Proof. Using Lemma 7, Power's mean inequality, and the fact that $|f'|^q$ is (α, m) -convex function,

$$\begin{aligned}
|G(k; n; a, x, b)(f)| & \leq \frac{(x-a)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right| dt + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right| dt \right] \\
& + \frac{(b-x)^{k+1}}{b-a} \left[\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right| dt + \int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right| dt \right] \leq \frac{(x-a)^{k+1}}{b-a} \left(\int_0^1 \frac{t^k}{2} dt \right)^{1-1/q} \\
& \cdot \left[\left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}a \right) \right|^q dt \right)^{1/q} + \left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}a \right) \right|^q dt \right)^{1/q} \right] \\
& + \frac{(b-x)^{k+1}}{b-a} \left(\int_0^1 \frac{t^k}{2} dt \right)^{1-1/q} \left[\left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{n+t}{n+1}x + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{1/q} + \left(\int_0^1 \frac{t^k}{2} \left| f' \left(\frac{1-t}{n+1}x + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{1/q} \right] \\
& \leq \frac{(x-a)^{k+1}}{b-a} \left(\frac{1}{2(k+1)} \right)^{1-1/q} \left\{ \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right) dt \right]^{1/q} \right. \\
& + \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right) dt \right]^{1/q} \Big\} + \frac{(b-x)^{k+1}}{b-a} \left(\frac{1}{2(k+1)} \right)^{1-1/q} \\
& \cdot \left\{ \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{n+t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{n+t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right. \\
& + \left[\int_0^1 \frac{t^k}{2} \left(\left(\frac{1-t}{n+1} \right)^\alpha |f'(x)|^q + m \left(1 - \left(\frac{1-t}{n+1} \right)^\alpha \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \Big\} = \frac{(x-a)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \left(\frac{1}{n+1} \right)^{\alpha/q} \\
& \cdot \left\{ \left[\left(A - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) |f'(x)|^q + m \left(\frac{B}{2} - A + \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right]^{1/q} \right. \\
& + \left[\frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} |f'(x)|^q + m \left(\frac{B}{2} - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{a}{m} \right) \right|^q \right]^{1/q} \Big\} + \frac{(b-x)^{k+1}}{2(b-a)} \left(\frac{1}{k+1} \right)^{1-1/q} \\
& \cdot \left(\frac{1}{n+1} \right)^{\alpha/q} \left\{ \left[\left(A - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) |f'(x)|^q + m \left(\frac{B}{2} - A + \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right. \\
& + \left[\frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} |f'(x)|^q + m \left(\frac{B}{2} - \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} \right) \left| f' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \Big\}.
\end{aligned} \tag{25}$$

This completes the proof. $\square$

Remark 19. Observe that if in Theorem 18 we have $\alpha = 1$, the statement of Theorem 18 becomes the statement of Theorem 12.

Competing Interests

The authors declare that they have no competing interests.

References

- [1] G. Toader, "On a generalization of the convexity," *Mathematica*, vol. 30, no. 53, pp. 83–87, 1988.
- [2] V. G. Miheșan, *A Generalization of the Convexity, Seminar of Functional Equations, Approx. and Convex*, Cluj-Napoca, Romania, 1993.
- [3] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.
- [4] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.
- [5] M. A. Noor, K. I. Noor, M. V. Mihai, and M. U. Awan, "Fractional Hermite-Hadamard inequalities for differentiable s-Godunova-Levin functions," *Filomat*, In press.
- [6] M. V. Mihai and F.-C. Mitroi, "Hermite-Hadamard type inequalities obtained via Riemann-Liouville fractional calculus," *Acta Mathematica Universitatis Comenianae*, vol. 83, no. 2, pp. 209–215, 2014.

Hindawi

Submit your manuscripts at
<http://www.hindawi.com>