Research Article

The Fifth Dimension Subgroup for Metabelian 2 Groups

Shalini Gupta

Department of Mathematics, Punjabi University, Patiala 147002, India

Correspondence should be addressed to Shalini Gupta; gupta_math@yahoo.com

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Given a finite metabelian 2-group G , the object of this paper is to discuss some cases under which $D_5(G) = \gamma_5(G)$. Further, some examples of 2 groups of class 3, for which $D_4(G) \neq \{1\}$ but $D_5(G) = \{1\}$, are discussed.

1. Introduction

Let $\mathbb{Z}(G)$ be the integral group ring of group G and let $I(G)$ be the augmentation ideal of $\mathbb{Z}(G)$. For $n \geq 1$, let $D_n(G) = G \cap \{1 + I^n(G)\}$ be n th dimension subgroup and let $\gamma_n(G)$ be n th term in lower central series of group G . The famous dimension subgroup conjecture states that $D_n(G) = \gamma_n(G)$ for all $n \geq 1$ and for all groups G . This conjecture has been proved for $n \leq 3$ (see, e.g., [1]). It has been proved in [2] that if G is a finite p -group, p odd prime, then $D_4(G) = \gamma_4(G)$. Keeping in view this result, Rips [3] gave an example of 2-group G for which $D_4(G) \neq \gamma_4(G)$. Further Tahara [4] gave the structure of $D_5(G)$ and proved that the exponent of $D_5(G)/\gamma_5(G)$ is divisible by 6. It has been shown in [5] that, for a metabelian group, the exponent of $D_5(G)/\gamma_5(G)$ is divisible by 2, and hence for a metabelian p -group G , p odd prime $D_5(G) = \gamma_5(G)$. Gupta [6] constructed a 2-group G , generated by four elements, such that $D_5(G) \neq \gamma_5(G)$. Now, the question arises that whether for a metabelian 2-group G , with at most three generators, $D_5(G) = \gamma_5(G)$ or not.

In this paper, we prove that if G is metabelian 2 group generated by at most two elements, then $D_5(G) = \gamma_5(G)$ (Theorem 2). Further we prove that under certain conditions, a metabelian 2-group G generated by n elements satisfies the dimension subgroup conjecture for $n = 5$ (Theorem 3). Finally, in Section 3, we give some examples of 2 groups of class 3, for which $D_5(G) = \gamma_5(G)$.

2. Main Results

We first recall the structure of $D_5(G)$ given by Tahara [4]. Let G be a finite group of class 4. Consider the lower central series $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \gamma_4(G) \supseteq \gamma_5(G) = \{1\}$ of G . Let $\{x_{1i}\gamma_2(G)\}_{1 \leq i \leq s}$ be basis of $G/\gamma_2(G)$, with $d(i)$ as the order of $x_{1i}\gamma_2(G)$: that is, $x_{1i}^{d(i)}$ belongs to $\gamma_2(G)$. Similarly, let $\{x_{2j}\gamma_3(G)\}_{1 \leq j \leq t}$ be basis of $\gamma_2(G)/\gamma_3(G)$, with $e(j)$ as the order of $x_{2j}\gamma_3(G)$, and let $\{x_{3k}\gamma_4(G)\}_{1 \leq k \leq u}$ be basis of $\gamma_3(G)/\gamma_4(G)$, with $f(k)$ as the order of $x_{3k}\gamma_4(G)$. Moreover, these basis elements are chosen in such a way that $d(i)$ divides $d(i+1)$, $e(i)$ divides $e(i+1)$, and $f(i)$ divides $f(i+1)$. Thus, we have

$$x_{1i}^{d(i)} = \prod_{1 \leq j \leq t} x_{2j}^{b_{ij}} \prod_{1 \leq k \leq u} x_{3k}^{c_{ik}} y_{4i}, \quad (1)$$

$$y_{4i} \in \gamma_4(G), \quad 1 \leq i \leq s;$$

$$x_{2k}^{e(k)} = \prod_{1 \leq l \leq u} x_{3l}^{d_{kl}} y_{4k}', \quad (2)$$

$$y_{4k}' \in \gamma_4(G), \quad 1 \leq k \leq t;$$

$$[x_{1i}^{d(i)}, x_{1j}] = \prod_{1 \leq l \leq u} x_{3l}^{\alpha_l^{(ij)}} y_{4ij}'', \quad (3)$$

$$y_{4ij}'' \in \gamma_4(G), \quad 1 \leq i \leq s.$$

The following theorem gives the structure of $D_5(G)$.

Theorem 1 (see [4]). With the above notations, $D_5(G)$ is equal to the subgroup generated by the elements

$$\prod_{1 \leq i < j \leq s} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}^{d(j)/d(i)}} \prod_{1 \leq i \leq s, 1 \leq k \leq t, k < l} [x_{2l}, x_{2k}]^{b_{li} v_{ik}} \cdot \prod_{1 \leq i \leq j \leq k \leq s} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}}, \quad (4)$$

with the following conditions:

$$w_{iii} = 0, \quad 1 \leq i \leq s; \quad (5)$$

$$u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} + w_{ijj} d(i) + w_{iij}'' d(j) = 0, \quad 1 \leq i < j \leq s; \quad (6)$$

$$-u_{ij} \binom{d(j)}{2} + w_{ijj} d(i) + w_{iij}' d(j) = 0, \quad 1 \leq i < j \leq s; \quad (7)$$

$$w_{ijk} d(i) + w_{ijk}' d(j) + w_{ijk}'' d(k) = 0, \quad 1 \leq i < j < k \leq s; \quad (8)$$

$$\sum_{i < h} u_{ih} b_{hk} - \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} b_{hk} + v_{ik} d(i) + v_{ik}' e(k) = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq t; \quad (9)$$

$$u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} + w_{iij} \binom{d(i)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (10)$$

$$w_{iij} \binom{d(i)}{2} + w_{iij}'' \binom{d(j)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (11)$$

$$-u_{ij} \binom{d(j)}{3} + w_{iij}' \binom{d(j)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (12)$$

$$w_{ijk} \binom{d(i)}{2}, w_{ijk}' \binom{d(j)}{2}, w_{ijk}'' \binom{d(k)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j < k \leq s; \quad (13)$$

$$v_{ik} \binom{d(i)}{2} - \sum_{h \leq i} w_{hii} b_{hk} - \sum_{i < h} w_{iij}'' b_{hk} \equiv 0 \pmod{(d(i), e(k))}, \quad 1 \leq i \leq s, \quad 1 \leq k \leq t; \quad (14)$$

$$\sum_{h \leq i} w_{hij} b_{hk} + \sum_{i < h \leq j} w_{ihj}' b_{hk} + \sum_{j < h} w_{ijk}'' b_{hk} \equiv 0 \pmod{(d(i), f(l))}, \quad 1 \leq i \leq s, \quad 1 \leq l \leq u; \quad (15)$$

$$\begin{aligned} & - \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} \alpha_l^{(hi)} + \sum_{i < h} u_{ih} c_{hi} - \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} c_{hi} \\ & - \sum_k v_{ik}' d_{kl} - \sum_{g \leq i \leq h} w_{gih} \alpha_l^{(gh)} - \sum_{g \leq h \leq i} w_{ghi} \alpha_l^{(gh)} \\ & - \sum_{i < g \leq h} w_{igh}' \alpha_l^{(gh)} \equiv 0 \pmod{(d(i), f(l))}, \quad 1 \leq i \leq s, \quad 1 \leq l \leq u; \end{aligned} \quad (16)$$

$$\sum_i v_{ik} b_{ik} \equiv 0 \pmod{e(k)}, \quad 1 \leq k \leq t; \quad (17)$$

$$\sum_i v_{ik} b_{il} + \sum_i v_{il} b_{ik} \equiv 0 \pmod{e(k)}, \quad 1 \leq k < l \leq t. \quad (18)$$

Theorem 2. Let G be metabelian 2 group with $G/\gamma_2(G)$ as sum of at most two cyclic groups. Then $D_5(G) = \gamma_5(G)$.

Proof. It is obvious that if metabelian 2 groups are such that $G/\gamma_2(G)$ is a cyclic group then $D_5(G) = \gamma_5(G)$.

Let G be metabelian 2 group and let $G/\gamma_2(G)$ be sum of two cyclic groups. It is enough to prove the result for a group of class 4. Let $G/\gamma_2(G) = C_1 \oplus C_2$, where C_i is a cyclic group generated by x_{1i} with order of $x_{1i}\gamma_2(G) = d(i)$ and $d(1)$ divides $d(2)$. An arbitrary element g of $D_5(G)$ is of the form $[x_{11}^{d(1)}, x_{12}]^{u_{12}(d(2)/d(1))} [x_{11}^{d(1)}, x_{12}, x_{12}]^{w_{122}}$ subject to the conditions given in Theorem 1. For $1 \leq k \leq t$ and $i = 2$, (9) becomes

$$-u_{12} \frac{d(2)}{d(1)} b_{1k} + v_{2k} d(2) + v_{2k}' e(k) = 0. \quad (19)$$

Thus,

$$\begin{aligned} 1 &= \prod_{k=1}^t [x_{2k}, x_{12}]^{-u_{12}(d(2)/d(1)) b_{1k} + v_{2k} d(2) + v_{2k}' e(k)} \\ &= \prod_{k=1}^t \left\{ [x_{2k}^{b_{1k}}, x_{12}]^{-u_{12}(d(2)/d(1))} \cdot [x_{2k}, x_{12}, x_{2k}]^{u_{12}(d(2)/d(1)) \binom{b_{1k}}{2}} \right\} \\ &\quad \cdot \prod_{k=1}^t \left\{ [x_{2k}, x_{12}^{d(2)}]^{v_{2k}} [x_{2k}, x_{12}, x_{12}]^{-v_{2k} \binom{d(2)}{2}} \right\} \\ &\quad \cdot \prod_{k=1}^t \left\{ [x_{2k}^{e(k)}, x_{12}]^{v_{2k}'} [x_{2k}, x_{12}, x_{2k}]^{-v_{2k}' \binom{e(k)}{2}} \right\}. \end{aligned} \quad (20)$$

Now $[x_{2k}, x_{12}, x_{2k}]$ belongs to $\gamma_5(G)$ and $[x_{2k}, x_{12}^{d(2)}]$ belongs to $[\gamma_2(G), \gamma_2(G)]$; thus we have

$$\begin{aligned} 1 &= \prod_{k=1}^t [x_{2k}^{b_{1k}}, x_{12}]^{-u_{12}(d(2)/d(1))} \\ &\quad \cdot \prod_{k=1}^t [x_{2k}, x_{12}, x_{12}]^{-v_{2k} \binom{d(2)}{2}} \prod_{k=1}^t [x_{2k}^{e(k)}, x_{12}]^{v_{2k}'} \end{aligned}$$

$$\begin{aligned}
&= \left[\prod_{k=1}^t x_{2k}^{b_{1k}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} \\
&\cdot \prod_{k=1}^t [x_{2k}, x_{12}, x_{12}]^{-v_{2k}d\binom{d(2)}{2}} \prod_{k=1}^t \prod_{l=1}^u [x_{3l}^{d_{kl}}, x_{12}]^{v'_{2k}} \\
&= \left[\prod_{k=1}^t x_{2k}^{b_{1k}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} \\
&\cdot \prod_{k=1}^t [x_{2k}, x_{12}, x_{12}]^{-w_{122}b_{1k}} \prod_{l=1}^u [x_{3l}, x_{12}]^{\sum_k v'_{2k}d_{kl}}, \tag{21}
\end{aligned}$$

as for $i = 2$ (14) becomes $v_{2k}\binom{d(2)}{2} - w_{122}b_{1k} \equiv 0 \pmod{(d(2), e(k))}$. Now, (16), for $i = 2$, becomes

$$\begin{aligned}
&-u_{12} \frac{d(2)}{d(1)} \alpha_l^{(12)} - u_{12} \frac{d(2)}{d(1)} c_{il} - \sum_k v'_{2k} d_{kl} - 2w_{122} \alpha_l^{(12)} \\
&-w_{112} \alpha_l^{(11)} \equiv 0 \pmod{(d(2), f(l))}, \tag{22}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\prod_{l=1}^u [x_{3l}, x_{12}]^{\sum_k v'_{2k}d_{kl}} = \prod_{l=1}^u [x_{3l}, \\
&x_{12}]^{-u_{12}(d(2)/d(1))\alpha_l^{(12)} - u_{12}(d(2)/d(1))c_{il} - 2w_{122}\alpha_l^{(12)} - w_{112}\alpha_l^{(11)}} \\
&= \left[\prod_{l=1}^u x_{3l}^{c_{il}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} \left[\prod_{l=1}^u x_{3l}^{\alpha_l^{(12)}}, \right. \\
&x_{12} \left. \right]^{-u_{12}(d(2)/d(1))} \left[\prod_{l=1}^u x_{3l}^{\alpha_l^{(12)}}, x_{12} \right]^{-2w_{122}} \left[\prod_{l=1}^u x_{3l}^{\alpha_l^{(11)}}, \right. \\
&x_{12} \left. \right]^{-w_{112}} = \left[\prod_{l=1}^u x_{3l}^{c_{il}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} [x_{11}^{d(1)}, \\
&x_{12}, x_{12}]^{-u_{12}(d(2)/d(1))} [x_{11}^{d(1)}, x_{12}, x_{12}]^{-2w_{122}}.
\end{aligned} \tag{23}$$

Now,

$$\begin{aligned}
&[x_{11}^{d(1)}, x_{12}, x_{12}]^{-u_{12}(d(2)/d(1))} = [x_{11}, x_{12}, x_{12}]^{-u_{12}d(2)} \\
&\cdot [x_{12}, x_{11}, x_{11}, x_{12}]^{u_{12}d(2)/d(1)} \binom{d(1)}{2} \\
&= [x_{11}, x_{12}, x_{12}^{d(2)}]^{-u_{12}} \\
&\cdot [x_{12}, x_{11}, x_{12}, x_{12}]^{-u_{12}\binom{d(2)}{2}} \\
&\cdot [x_{12}, x_{11}, x_{11}, x_{12}]^{u_{12}d(2)/d(1)} \binom{d(1)}{2}. \tag{24}
\end{aligned}$$

Using (6) and (7), we get that

$$u_{12} \frac{d(2)}{d(1)} \binom{d(1)}{2} \equiv 0 \pmod{d(1)}, \tag{25}$$

$$-u_{12} \binom{d(2)}{2} \equiv 0 \pmod{d(1)}. \tag{26}$$

Hence, (24) reduces to

$$\begin{aligned}
&[x_{11}^{d(1)}, x_{12}, x_{12}]^{-u_{12}(d(2)/d(1))} = [x_{11}, x_{12}, x_{12}^{d(2)}]^{-u_{12}} \\
&= 1, \text{ as } [x_{11}, x_{12}, x_{12}^{d(2)}] \in [\gamma_2(G), \gamma_2(G)]. \tag{27}
\end{aligned}$$

Now,

$$\begin{aligned}
&[x_{11}^{d(1)}, x_{12}, x_{12}]^{-2w_{122}} = [x_{11}, x_{12}, x_{12}]^{-2w_{122}d(1)} \\
&\cdot [x_{12}, x_{11}, x_{11}, x_{12}]^{2w_{122}\binom{d(1)}{2}} \\
&= [x_{11}, x_{12}, x_{12}]^{-2w_{122}d(1)}. \tag{28}
\end{aligned}$$

Also (7) implies that

$$-2w_{122}d(1) = -2u_{12} \binom{d(2)}{2} + 2w'_{122}d(2). \tag{29}$$

Thus, (28) becomes

$$\begin{aligned}
&[x_{11}^{d(1)}, x_{12}, x_{12}]^{-2w_{122}} = [x_{11}, x_{12}, x_{12}]^{2w'_{122}d(2)} \\
&\cdot [x_{11}, x_{12}, x_{12}]^{-u_{12}d(2)(d(2)-1)} \\
&= [x_{11}, x_{12}, x_{12}^{d(2)}]^{2w'_{122}} \\
&\cdot [x_{12}, x_{11}, x_{12}, x_{12}]^{-2w'_{122}\binom{d(2)}{2}} \\
&\cdot [x_{11}, x_{12}, x_{12}]^{-u_{12}d(2)(d(2)-1)} \\
&= [x_{11}, x_{12}, x_{12}^{d(2)}]^{-u_{12}(d(2)-1)} \\
&\cdot [x_{12}, x_{11}, x_{12}, x_{12}]^{-u_{12}((d(2)-1)\binom{d(2)}{2})} = 1,
\end{aligned} \tag{30}$$

as by (7), $-u_{12}\binom{d(2)}{2} \equiv 0 \pmod{d(1)}$.

Using (23), (27), and (30) in (21), we get that

$$\begin{aligned}
1 &= \left[\prod_{k=1}^t x_{2k}^{b_{1k}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} \\
&\cdot \left[\prod_{l=1}^u x_{3l}^{c_{il}}, x_{12} \right]^{-u_{12}(d(2)/d(1))} \\
&\cdot \left[\prod_{k=1}^t x_{2k}^{b_{1k}}, x_{12}, x_{12} \right]^{-w_{122}} \\
&= [x_{11}^{d(1)}, x_{12}]^{-u_{12}(d(2)/d(1))} [x_{11}^{d(1)}, x_{12}, x_{12}]^{-w_{122}} \\
&= g^{-1}. \tag{31}
\end{aligned}$$

Thus, $g = 1$, and hence $D_5(G) = \{1\}$. $\square$

Theorem 3. Let G be finite metabelian 2 group and $G/\gamma_2(G) = C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where C_i are cyclic groups of order $d(i)$. Let $d(1) = d(2) = \cdots = d(n-1) = 2$, $d(n) \geq 4$. Then $D_5(G) = \gamma_5(G)$.

Proof. It is enough to prove the result for a group G of class 4. Any element g of $D_5(G)$ is of the form

$$g = \prod_{1 \leq i < j \leq n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}(d(j)/d(i))} \cdot \prod_{1 \leq i < j < k \leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}}. \quad (32)$$

$$\prod_{1 \leq i < j \leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}} = A \cdot B \cdot C \text{ say.}$$

Since for $1 \leq i < j < k \leq n$, $d(i) = d(j) = 2$, (13) reduces to

$$w_{ijk} \equiv 0 \pmod{d(j)} \quad (33)$$

and hence $B = 1$. Now,

$$A = \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}(d(j)/d(i))} \cdot \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}(d(n)/d(i))} = \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \quad (34)$$

$$\cdot \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}(d(n)/d(i))}$$

as for $1 \leq i < j \leq n-1$, $d(i) = d(j)$.

Also for $1 \leq i < j \leq n$, (7) implies that $-u_{ij} + 2w_{ijj} + 2w'_{ijj} = 0$, which gives that

$$\begin{aligned} & \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{2w_{ijj} + 2w'_{ijj}}, \text{ as } d(j) = 2 \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{d(j)(w_{ijj} + w'_{ijj})} \\ &= \prod_{1 \leq i < j \leq n-1} \left\{ [x_{1i}^{d(i)}, x_{1j}^{d(j)}]^{(w_{ijj} + w'_{ijj})} \right\} \end{aligned}$$

$$\begin{aligned} & \cdot [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-(w_{ijj} + w'_{ijj}) \binom{d(j)}{2}} \Big\} \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}} \\ & \cdot \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w'_{ijj}} \\ & \text{ as } [x_{1i}^{d(i)}, x_{1j}^{d(j)}] \in [\gamma_2(G), \gamma_2(G)], \quad d(j) = 2. \end{aligned} \quad (35)$$

Now for $1 \leq i < j \leq n$, (12) implies that $w'_{ijj} \equiv 0 \pmod{d(j)}$, which gives that

$$\begin{aligned} & \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}}. \end{aligned} \quad (36)$$

Thus, (34) reduces to

$$A = \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}(d(n)/d(i))} \cdot \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}}. \quad (37)$$

Hence,

$$g = \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}(d(n)/d(i))} \cdot \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{w_{inn}}. \quad (38)$$

Now, for $i = n$, (9) becomes

$$S = -\sum_{j=1}^{n-1} u_{jn} \frac{d(n)}{d(j)} b_{jk} + v_{nk} d(n) + v'_{nk} e(k) = 0, \quad (39)$$

which gives

$$\begin{aligned} 1 &= \prod_{k=1}^t [x_{2k}, x_{1n}]^S = \prod_{k=1}^t [x_{2k}, x_{1n}]^{-\sum_{j=1}^{n-1} u_{jn}(d(n)/d(j)) b_{jk}} \\ & \cdot \prod_{k=1}^t [x_{2k}, x_{1n}]^{v_{nk} d(n)} \prod_{k=1}^t [x_{2k}, x_{1n}]^{v'_{nk} e(k)} \\ &= \prod_{k=1}^t \left[\prod_{j=1}^{n-1} x_{2k}^{b_{jk}}, x_{1n} \right]^{-u_{jn}(d(n)/d(j))} \\ & \cdot \prod_{k=1}^t \left\{ [x_{2k}, x_{1n}^{d(n)}]^{v_{nk}} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk} \binom{d(n)}{2}} \right\} \\ & \cdot \prod_{k=1}^t [x_{2k}^{e(k)}, x_{1n}]^{v'_{nk}} \end{aligned}$$

$$\begin{aligned}
&= \prod_{k=1}^t \left[\prod_{j=1}^{n-1} x_{2k}^{b_{jk}}, x_{1n} \right]^{-u_{jn}(d(n)/d(j))} \\
&\cdot \prod_{k=1}^t [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk} \binom{d(n)}{2}} \prod_{k=1}^t \prod_{l=1}^u [x_{3l}^{d_{kl}}, x_{1n}]^{v'_{nk}} \\
&= \prod_{k=1}^t \left[\prod_{j=1}^{n-1} x_{2k}^{b_{jk}}, x_{1n} \right]^{-u_{jn}(d(n)/d(j))} \\
&\cdot \prod_{k=1}^t [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk} \binom{d(n)}{2}} \prod_{l=1}^u [x_{3l}, x_{1n}]^{\sum_k v'_{nk} d_{kl}}.
\end{aligned} \quad (40)$$

Equation (14) for $i = n$ becomes

$$v_{nk} \binom{d(n)}{2} \equiv \sum_{j=1}^{n-1} w_{jnn} b_{jk} \pmod{(d(n), e(k))}. \quad (41)$$

Thus,

$$\begin{aligned}
&\prod_{k=1}^t [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk} \binom{d(n)}{2}} \\
&= \prod_{i=1}^{n-1} \left[\prod_k x_{2k}^{b_{ik}}, x_{1n}, x_{1n} \right]^{-w_{inn}} \\
&= \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-w_{inn}}.
\end{aligned} \quad (42)$$

Using (16), we get that

$$\begin{aligned}
&\prod_{l=1}^u [x_{3l}, x_{1n}]^{\sum_k v'_{nk} d_{kl}} \\
&= \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-u_{in}(d(n)/d(i))} \\
&\cdot \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-2w_{inn}} \\
&\cdot \prod_{i=1}^{n-1} \left[\prod_l x_{3l}^{c_{il}}, x_{1n} \right]^{-u_{in}(d(n)/d(i))}.
\end{aligned} \quad (43)$$

Now

$$\begin{aligned}
&\prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-u_{in}(d(n)/d(i))} \\
&= \prod_{i=1}^{n-1} \left\{ [x_{1i}, x_{1n}, x_{1n}^{d(n)}]^{-u_{in}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\cdot [x_{1n}, x_{1i}, x_{1i}, x_{1n}]^{u_{in}(d(n)/d(i)) \binom{d(i)}{2}} \\
&\cdot [x_{1n}, x_{1i}, x_{1n}, x_{1n}]^{u_{in} \binom{d(n)}{2}} \Big\} = 1, \\
&\text{as } 2 = d(i) \text{ divides } \frac{d(n)}{d(i)}.
\end{aligned} \quad (44)$$

Also it can be seen easily from (7) that

$$\prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-2w_{inn}} = 1. \quad (45)$$

Now using (38), (40), (43), (44), and (45), we get that

$$\begin{aligned}
1 &= \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-u_{in}(d(n)/d(i))} \\
&\cdot \prod_{i=1}^{n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-w_{inn}} = g^{-1}.
\end{aligned} \quad (46)$$

Hence, $g = 1$. $\square$

3. Groups of Class 3

It has been proved that if G is a finite p -group of class 3, p odd prime, then $D_4(G) = \{1\}$. Clearly for such a group G , $D_5(G) = \{1\}$. Now the question arises that if G is 2 group of class 3 with $D_4(G) \neq \{1\}$, then is it possible that $D_5(G) = \{1\}$? We will discuss some examples of 2 groups of class 3 in which $D_4(G) \neq \{1\}$, but $D_5(G) = \{1\}$.

3.1. Example 1. Let G be a group generated by four elements r, a, b, c . Set $x_0 = y_0 = z_0 = r$ and $x_i = [x_{i-1}, a]$, $y_i = [y_{i-1}, b]$, and $z_i = [z_{i-1}, c]$, $i = 1, 2, 3$. Suppose that G satisfies the relations

- (i) $r^{2^7} = 1$, $a^{2^6} = y_1^4 z_1^2$, $b^{2^4} = x_1^{-4} z_1$, and $c^{2^2} = x_1^{-2} y_1^{-1}$;
- (ii) $z_2 = y_2^4$, $y_2 = x_2^4$;
- (iii) $x_3 = y_3 = z_3 = 1$;
- (iv) $[a, b, g] = 1$, $[b, c, g] = [a, c, g] = 1 \ \forall g \in G$;
- (v) $[x_i, b] = [x_i, c] = [y_i, a] = [y_i, c] = [z_i, a] = [z_i, b] = 1 \ \forall i \geq 1$;
- (vi) $[x_i, x_j] = [x_i, y_j] = [x_i, z_j] = [y_i, y_j] = [y_i, z_j] = [z_i, z_j] = 1 \ \forall i, j \geq 0$. It has been proved in [6] that G is 2 group of class 3, and $D_4(G) \neq \{1\}$. we will show that $D_5(G) = \{1\}$.

From the above relations we conclude that

- (i) $[a, b]^{2^4} = x_2^4$, $[b, c]^{2^2} = y_2 = x_2^4$, and $[a, c]^{2^2} = x_2^2$;
- (ii) $o(x_1) = 2^7$, $o(x_2) = 2^6$, $o(y_1) = 2^5$, $o(y_2) = 2^4$, $o(z_1) = 2^3$, and $o(z_2) = 2^2$, where $o(g)$ denotes the order of an element g .

Consider

$$\begin{aligned}
 x_{11} &= c, \\
 x_{12} &= b, \\
 x_{13} &= a, \\
 x_{14} &= r; \\
 x_{21} &= [a, c], \\
 x_{22} &= [b, c], \\
 x_{23} &= [r, c], \\
 x_{24} &= [a, b], \\
 x_{25} &= [r, b], \\
 x_{26} &= [r, a]; \\
 x_{31} &= [r, a, a] = x_2.
 \end{aligned} \tag{47}$$

It is easy to see that $d(1) = 2^2, d(2) = 2^4, d(3) = 2^6, d(4) = 2^7, e(1) = 2^2, e(2) = 2^2, e(3) = 2^2, e(4) = 2^4, e(5) = 2^4, e(6) = 2^6$, and $f(1) = 2^6$. Also with the help of (1) and (2), we get that $b_{15} = -1, b_{16} = -2, b_{23} = 1, b_{26} = -4, b_{33} = 2$, and $b_{35} = 4$.

Since G is a group of class 3, so an arbitrary element of $D_5(G)$ can be written as

$$\begin{aligned}
 g &= \prod_{1 \leq i < j \leq 4} [x_{1i}, x_{1j}]^{u_{ij}d(j)} = [c, b]^{16u_{12}} [c, a]^{64u_{13}} \\
 &\cdot [c, r]^{128u_{14}} [b, a]^{64u_{23}} [b, r]^{128u_{24}} [a, r]^{128u_{34}} \\
 &= x_2^{-16u_{12}-32u_{13}-16u_{23}}.
 \end{aligned} \tag{48}$$

Equation (9) for $i = 1$ and $k = 6$ becomes

$$\begin{aligned}
 0 &= u_{12}b_{26} + u_{13}b_{36} + u_{14}b_{46} + v_{16}d(1) + v'_{16}e(6) \\
 &= -4u_{12} + 4v_{16} + 64v'_{16},
 \end{aligned} \tag{49}$$

which gives that

$$-u_{12} + v_{16} \equiv 0 \pmod{4}. \tag{50}$$

Also (17), for $k = 6$, gives that

$$v_{16} + 2v_{26} \equiv 0 \pmod{16}. \tag{51}$$

For $k = 3$ and $l = 6$, (18) becomes

$$v_{26} \equiv 0 \pmod{2}. \tag{52}$$

Now, (50), (51), and (52) imply that

$$u_{12} \equiv 0 \pmod{4}. \tag{53}$$

Again, for $i = 3$ and $k = 6$, (9) becomes

$$32u_{13} + 16u_{23} \equiv 0 \pmod{64}. \tag{54}$$

Since order of x_2 is 64, combining (53) and (54), we get that $g = 1$.

3.2. Example 2. Let G be a group generated by four elements x_1, x_2, x_3, x_4 . Set $y_1 = [x_2, x_3], y_2 = [x_1, x_2], y_3 = [x_1, x_3]$, and $y_4 = [x_1, x_4]$. Suppose that G satisfies the following relations:

$$\begin{aligned}
 [x_2, y_2] &= y_1^4, \\
 [x_3, y_3] &= y_1^{-16}, \\
 [x_4, y_4] &= y_1^{64}; \\
 [x_1, y_q] &= 1, \quad 1 \leq q \leq 4; \\
 [x_2, y_q] &= 1, \quad 1 \leq q \leq 4, q \neq 2; \\
 [x_3, y_q] &= 1, \quad 1 \leq q \leq 4, q \neq 3; \\
 [x_4, y_q] &= 1, \quad 1 \leq q \leq 4, q \neq 4; \\
 x_1^{2^{k+l+4}} &= y_1^{-2^{k+l+4}} y_2^{2^{k+l+3}}; \\
 x_2^{2^{k+4}} &= y_3^{2^k} y_4^{-2^{k-1}} y_1^{3 \cdot 2^{k+3}}; \\
 x_3^{2^{k+2}} &= y_1^{-5 \cdot 2^{k+1}} y_2^{2^k} y_4^{2^{k-2}}; \\
 x_4^{2^k} &= y_1^{2^k} y_2^{2^{k-1}} y_3^{2^{k-2}}.
 \end{aligned} \tag{55}$$

Consider

$$\begin{aligned}
 x_{11} &= x_4, \\
 x_{12} &= x_3, \\
 x_{13} &= x_2, \\
 x_{14} &= x_1; \\
 x_{21} &= y_1, \\
 x_{22} &= y_4, \\
 x_{23} &= y_3, \\
 x_{24} &= y_{12}; \\
 x_{31} &= [x_2, y_2] = y_1^4.
 \end{aligned} \tag{56}$$

It is easy to see that $d(1) = 2^k, d(2) = 2^{k+2}, d(3) = 2^{k+4}, d(4) = 2^{k+l+4}, e(1) = 4, e(2) = 2^k, e(3) = 2^{k+2}, e(4) = 2^{k+4}$, and $f(1) = 2^{k+2}$. Also we have $b_{11} = 2^k, b_{13} = 2^{k-2}, b_{14} = 2^{k-1}, b_{21} = -5 \cdot 2^{k+1}, b_{22} = 2^{k-2}, b_{24} = 2^k, b_{31} = 3 \cdot 2^{k+3}, b_{32} = -2^{k-1}, b_{33} = 2^k, b_{41} = -2^{k+l+4}$, and $b_{44} = 2^{k+l+3}$. It has been proved in [7] that G is a group of class 3 and $D_4(G) \neq 1$. We will prove that $D_5(G) = \{1\}$.

An arbitrary element g of $D_5(G)$ can be written as

$$\begin{aligned}
 g &= \prod_{1 \leq i < j \leq 4} [x_{1i}, x_{1j}]^{u_{ij}d(j)} = [x_4, x_3]^{u_{12}2^{k+2}} \\
 &\cdot [x_4, x_2]^{u_{13}2^{k+4}} [x_4, x_1]^{u_{14}2^{k+l+4}} [x_3, x_2]^{u_{23}2^{k+4}} \\
 &\cdot [x_3, x_1]^{u_{24}2^{k+l+4}} [x_2, x_1]^{u_{34}2^{k+l+4}} \\
 &= y_1^{2^{k+4}(u_{12}-2u_{13}-u_{23}+2^{l+1}u_{34})}.
 \end{aligned} \tag{57}$$

To show that $g = 1$, it is enough to show that $u_{12} - 2u_{13} - u_{23} + 2^{l+1}u_{34} \equiv 0 \pmod{4}$, since order of y_1 is 2^{k+6} .

Now, for $i = 3$ and $k = 4$, (9) becomes

$$\begin{aligned} 0 &= u_{34}b_{44} - u_{13}\frac{d(3)}{d(1)}b_{14} - u_{23}\frac{d(3)}{d(2)}b_{24} + v_{34}d(3) \\ &\quad + v'_{34}e(4) \\ &= 2^{k+2}\left(2^{l+1}u_{34} - 2u_{13} - u_{23} + 4v_{34} + 4v'_{34}\right), \end{aligned} \quad (58)$$

which implies that

$$2^{l+1}u_{34} - 2u_{13} - u_{23} \equiv 0 \pmod{4}. \quad (59)$$

Also (9) for $i = 1$ and $k = 2$ gives that

$$\begin{aligned} 0 &= u_{12}b_{22} + u_{13}b_{32} + u_{14}b_{42} + v_{12}d(1) + v'_{12}e(2) \\ &= 2^{k-2}u_{12} - 2^{k-1}u_{13} + 2^kv_{12} + 2^kv'_{12}. \end{aligned} \quad (60)$$

That is, we have

$$u_{12} = 2u_{13} - 4v_{12} - 4v'_{12}. \quad (61)$$

Again (9), for $i = 1$, $k = 3$, gives that

$$u_{13} \equiv -v_{13} \pmod{4}. \quad (62)$$

Now (17) for $k = 3$ gives that

$$v_{13} \equiv 0 \pmod{4}. \quad (63)$$

Combining (61), (62), and (63), we get that

$$u_{13} \equiv 0 \pmod{4}, \quad (64)$$

which together with (59) gives that $g = 1$.

Competing Interests

The author declares no competing interests.

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