Research Article

Ergodicity Space for Measure-Preserving Transformations

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We introduce the concept of ergodicity space of a measure-preserving transformation and will present some of its properties as an algebraic weight for measuring the size of the ergodicity of a measure-preserving transformation. We will also prove the invariance of the ergodicity space under conjugacy of dynamical systems.

1. Introduction

In statistical mechanics [1], the ergodic hypothesis says that, over long periods of time, the time spent by a particle in some region of the phase space of microstates with the same energy is proportional to the volume of this region; that is, all accessible microstates are equiprobable over a long period of time.

The ergodic hypothesis is often assumed in statistical analysis. The analyst would assume that the average of a process parameter over time and the average over the statistical ensemble are the same. Most of the physical systems are assumed to be ergodic. But, in general, the systems are not necessarily ergodic [2]. Therefore, the more one system is near to being ergodic, the more it could be considered as a model for physical systems. So one may ask the following question: how near is a dynamical system to ergodicity? In this paper, we assign an algebraic structure to any measure-preserving map on a probability space which is invariant under conjugacy of dynamical systems. This algebraic structure is indeed a vector space which will be bigger in size as the system shows more ergodic treatments. Beside the numerical invariants such as entropy [3–7], it is an algebraic invariant in ergodic theory. It is also an algebraic weight which takes its maximum size when the systems are ergodic. The middle size represents the size of ergodicity of the system. Briefly, we apply a linear structure to demonstrate the ergodicity weight of a nonlinear system.

2. Ergodicity Space

Let $(X, \mathfrak{B}, \mu)$ be a probability space and let $T : (X, \mathfrak{B}, \mu) \rightarrow (X, \mathfrak{B}, \mu)$ be a measure-preserving transformation. Define the relation “ $\sim$ ” on $L^2(\mu)$ as follows:

$$f \sim g \quad \text{iff} \quad f - g = cte. \quad (1)$$

“ $\sim$ ” is clearly an equivalence relation on $L^2(\mu)$. Put $L_c^2(\mu) := L^2(\mu)/\sim$. We again denote the members of $L_c^2(\mu)$ by f instead of $[f]_\sim$ for simplicity. Indeed, we identify the functions in $L^2(\mu)$ whose difference is constant μ —almost everywhere.

Now we define the relation “ $\sim_T$ ” on $L_c^2(\mu)$ as follows:

$$f \sim_T g \quad \text{iff} \quad U_T(f - g) = f - g, \quad (2)$$

where $U_T(f) := f \circ T$, for $f \in L_c^2(\mu)$. “ $\sim_T$ ” is clearly a well-defined equivalence relation on $L_c^2(\mu)$. Now put

$$\Lambda_T := \frac{L_c^2(\mu)}{\sim_T} = \{[f]_T : f \in L_c^2(\mu)\}. \quad (3)$$

Define the addition operation

$$+ : \Lambda_T \times \Lambda_T \longrightarrow \Lambda_T \quad (4)$$

as

$$([f]_T, [g]_T) \longmapsto [f]_T + [g]_T := [f + g]_T \quad (5)$$

and the scalar multiplication

$$\cdot : \mathbb{C} \times \Lambda_T \longrightarrow \Lambda_T \quad (6)$$

as

$$(\lambda, [f]_T) \mapsto \lambda [f]_T = [\lambda f]_T \quad (7)$$

on Λ_T . Clearly “+” and “ $\cdot$ ” are well-defined and $(\Lambda_T, +, \cdot)$ is a vector space on $\mathbb{C}$ which is called the *ergodicity space* of T . The following theorem shows that the vector space Λ_T measures the ergodicity of T .

Theorem 1. *Let T be a measure-preserving transformation on a probability space $(X, \mathfrak{B}, \mu)$:*

- (1) *if $T = id_X$, then $\Lambda_T = \{[0]_T\}$;*
- (2) *T is ergodic if and only if $\Lambda_T = L_c^2(\mu)$.*

Proof. (i) If $T = id_X$, then $U_T(f) = f \circ T = f$ for all $f \in L_c^2(\mu)$; equivalently $f \sim_T 0$ for all $f \in L_c^2(\mu)$ and so $[f]_T = [0]_T$ for all $f \in L_c^2(\mu)$ and this is equivalent to $\Lambda_T = \{[0]_T\}$.

(ii) Let T be ergodic. For $f_0 \in L^2(\mu)$ we have

$$\begin{aligned} f &\in [f_0]_T \iff \\ (f - f_0) \circ T &= f - f_0 \iff \\ f - f_0 &= cte \iff \\ f &\in [f_0]_{\sim}. \end{aligned} \quad (8)$$

Therefore $[f_0]_T = [f_0]_{\sim}$ and so $\Lambda_T = L_c^2(\mu)$.

Now let $\Lambda_T = L_c^2(\mu)$. If $U_T(f) = f$, then $[f]_T = [0]_T$ and since by assumption $[f]_T = [f]_{\sim}$ for all $f \in L^2(\mu)$, then $[f]_{\sim} = [0]_{\sim}$ and hence $f = cte$; therefore T is ergodic ([8], Theorem 1.6). $\square$

In general $\{[0]_T\} \subseteq \Lambda_T \subseteq L_c^2(\mu)$, and Λ_T will attain its largest size if and only if T is ergodic, while it will be of its smallest size if $T = id_X$. The middle states of Λ_T will present the weight of ergodicity of T .

Definition 2. Let $(X, \mathfrak{B}, \mu)$ be a probability space and let T be a transformation on X . The semistable set of T is defined as follows:

$$S(T) := \{A \in \mathfrak{B} : T^{-1}(A) = A\}. \quad (9)$$

The following theorem presents the relationship between the ergodicity spaces of two measure-preserving transformations and their semistable sets.

Theorem 3. *Let T_1 and T_2 be two measure-preserving transformations on the probability space $(X, \mathfrak{B}, \mu)$. Then*

$$\begin{aligned} \Lambda_{T_1} &= \Lambda_{T_2} \iff \\ S(T_1) &= S(T_2) \iff \\ [0]_{T_1} &= [0]_{T_2}. \end{aligned} \quad (10)$$

Proof. Let $\Lambda_{T_1} = \Lambda_{T_2}$; then $[0]_{T_1} = [0]_{T_2}$. So, for $f \in L^2(\mu)$, $f \sim_{T_1} 0$ if and only if $f \sim_{T_2} 0$ or equivalently $f \circ T_1 = f$ if and only if $f \circ T_2 = f$. Now let $A \in \mathfrak{B}$. For $f = \chi_A$, $\chi_A \circ T_1 = \chi_A$ if and only if $\chi_A \circ T_2 = \chi_A$ or equivalently $\chi_{T_1^{-1}(A)} = \chi_A$ if and only if $\chi_{T_2^{-1}(A)} = \chi_A$ and hence $T_1^{-1}(A) = A$ if and only if $T_2^{-1}(A) = A$; therefore $S(T_1) = S(T_2)$.

Conversely, suppose that $S(T_1) = S(T_2)$; then $T_1^{-1}(A) = A$ if and only if $T_2^{-1}(A) = A$ for all $A \in \mathfrak{B}$ or equivalently $\chi_A \circ T_1 = \chi_A$ if and only if $\chi_A \circ T_2 = \chi_A$ for all $A \in \mathfrak{B}$. In the following, first we show that $[0]_{T_1} = [0]_{T_2}$.

As observed in the last paragraph, $f \circ T_1 = f$ if and only if $f \circ T_2 = f$ for all characteristic functions $f = \chi_A$.

Let $f = \sum_{j=1}^n c_j \chi_{A_j}$, where c_j 's are distinct real numbers and A_j 's are nonempty disjoint measurable sets. Suppose $f \circ T_1 = f$ and let $1 \leq k \leq n$ be arbitrary but fixed. Choose $x \in A_k$. By assumption we have

$$\sum_{j=1}^n c_j \chi_{T_1^{-1}(A_j)}(x) = \sum_{j=1}^n c_j \chi_{A_j}(x) = c_k. \quad (11)$$

This will give us that $x \in T_1^{-1}(A_k)$, so $A_k \subseteq T_1^{-1}(A_k)$. The opposite inclusion similarly follows and then we will have $T_1^{-1}(A_k) = A_k$. Since $S(T_1) = S(T_2)$, then $T_2^{-1}(A_k) = A_k$ and so $f \circ T_2 = f$. Similarly, if $f \circ T_2 = f$, then $f \circ T_1 = f$. Hence for simple functions we have $f \circ T_1 = f$ if and only if $f \circ T_2 = f$.

By a standard measure theoretical argument we have $f \circ T_1 = f$ if and only if $f \circ T_2 = f$ for any complex function $f \in L^2(\mu)$.

The above arguments show that if $S(T_1) = S(T_2)$, then $[0]_{T_1} = [0]_{T_2}$. Now consider $f_0 \in L^2(\mu)$. If $f \in [f_0]_{T_1}$, then $f - f_0 \in [0]_{T_1}$ and so $f - f_0 \in [0]_{T_2}$; hence $f \in [f_0]_{T_2}$. Therefore $[f_0]_{T_1} \subseteq [f_0]_{T_2}$. Similarly $[f_0]_{T_2} \subseteq [f_0]_{T_1}$ and so $[f_0]_{T_1} = [f_0]_{T_2}$. Since f_0 was arbitrary we have $\Lambda_{T_1} = \Lambda_{T_2}$ and the proof is complete. $\square$

Corollary 4. *Let T_1 and T_2 be two injective measure-preserving transformations on a Lebesgue space $(X, \mathfrak{B}, \mu)$ such that $\Lambda_{T_1} = \Lambda_{T_2}$. Then $\text{Fix}(T_1) = \text{Fix}(T_2)$.*

Proof. Since $\Lambda_{T_1} = \Lambda_{T_2}$, then, by Theorem 3, $S(T_1) = S(T_2)$. Now we have

$$\begin{aligned} x &\in \text{Fix}(T_1) \iff \\ T_1^{-1}(\{x\}) &= \{x\} \iff \\ \{x\} &\in S(T_1) = S(T_2) \iff \\ T_2^{-1}(\{x\}) &= \{x\} \iff \\ x &\in \text{Fix}(T_2). \end{aligned} \quad (12)$$

$\square$

Theorem 5. *Let T be an invertible measure-preserving transformation on the probability space $(X, \mathfrak{B}, \mu)$. Then $\Lambda_{T^n} = \Lambda_{T^{-n}}$ for all $n \geq 1$.*

Proof. Note that $f \sim_T f_0$ if and only if $f \sim_{T^{-1}} f_0$. $\square$

3. Invariance of the Ergodicity Space

In this section we show that the ergodicity space Λ_T is an algebraic spectral isomorphism invariant and therefore is a conjugacy and isomorphism invariant. Entropy and other invariants introduced so far have mainly been numerical invariants while the ergodicity space, introduced in Section 2, is an algebraic invariant. Before stating the main theorem we recall the concept of conjugacy, isomorphism, and spectral isomorphism (see also [8]).

Definition 6. Let T_1 and T_2 be two measure-preserving transformations on probability spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively. One says that T_1 and T_2 are similar if there exists an invertible measure-preserving transformation $\varphi : X_1 \rightarrow X_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$.

Theorem 7. If T_1 and T_2 are similar measure-preserving transformations on $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively, then $\Lambda_{T_1} \cong \Lambda_{T_2}$ (i.e., Λ_{T_1} and Λ_{T_2} are isomorphic vector spaces).

Proof. Let $\varphi : X_1 \rightarrow X_2$ be an invertible measure-preserving transformation such that $\varphi \circ T_1 = T_2 \circ \varphi$. Define $\Psi : \Lambda_{T_1} \rightarrow \Lambda_{T_2}$ by $\Psi([f]_{T_1}) := [f \circ \varphi^{-1}]_{T_2}$. Since $\varphi \circ T_1 = T_2 \circ \varphi$ it is easily seen that Ψ is well-defined. Moreover, one can easily check that Ψ is a linear transformation.

Now if $\Psi([f]_{T_1}) = [0]_{T_2}$, then $[f \circ \varphi^{-1}]_{T_2} = [0]_{T_2}$ or $f \circ \varphi^{-1} \sim_{T_2} 0$; hence $f \circ \varphi^{-1} \circ T_2 = f \circ \varphi^{-1}$ and so $f \circ T_1 \circ \varphi^{-1} = f \circ \varphi^{-1}$. Since φ is bijective we have $f \circ T_1 = f$ and thus $[f]_{T_1} = [0]_{T_1}$; therefore Ψ is one-to-one. Finally, for $[g]_{T_2} \in \Lambda_{T_2}$, put $f := g \circ \varphi$; then $\Psi([f]_{T_1}) = [g]_{T_2}$ and so Ψ is onto. Therefore Λ_{T_1} is isomorphic to Λ_{T_2} . $\square$

Remark 8. If the measure-preserving transformation $\varphi : X_1 \rightarrow X_2$ in Definition 6 is onto but not necessarily invertible, then Λ_{T_2} is isomorphic to a subspace of Λ_{T_1} . The map $\Psi : \Lambda_{T_2} \rightarrow \Lambda_{T_1}$, defined by $\Psi([f]_{T_2}) := [f \circ \varphi]_{T_1}$, is the injective isomorphism embedding Λ_{T_2} into Λ_{T_1} .

Definition 9. Let T_1 and T_2 be two measure-preserving transformations on probability spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively. We say that T_1 is isomorphic to T_2 if there exist $M_1 \in \mathfrak{B}_1$ and $M_2 \in \mathfrak{B}_2$ with $\mu_1(M_1) = \mu_2(M_2) = 1$ such that

- (i) $T_i(M_i) \subseteq M_i$ ($i = 1, 2$);
- (ii) there is an invertible measure-preserving transformation $\phi : M_1 \rightarrow M_2$ with $\phi \circ T_1(x) = T_2 \circ \phi(x)$ for all $x \in M_1$.

The following is an example of isomorphism of transformations [8].

Example 10. Let $T_1 : [0, 1) \rightarrow [0, 1)$ be the transformation $T_1(x) = 2x \bmod 1$, where $[0, 1)$ is equipped by the Borel σ -algebra and Lebesgue measure. Let, also, $T_2 = \sigma : \Sigma_2 \rightarrow \Sigma_2$ be the 1-sided $(1/2, 1/2)$ -shift map. Define $\phi : \Sigma_2 \rightarrow [0, 1)$ by

$$\phi(a_1, a_2, a_3, \dots) := \sum_{i=1}^{+\infty} \frac{a_i}{2^i}. \quad (13)$$

Let $M_1 = \Sigma_2 \setminus D_1$, where D_1 is the set of points of Σ_2 which have eventually constant coordinates, and $M_2 = [0, 1) \setminus D_2$, where D_2 is the set of dyadic rational numbers in $[0, 1)$. Clearly both M_1 and M_2 have full measure and ϕ maps M_1 to M_2 bijectively. Also $\phi \circ T_2(x) = T_1 \circ \phi(x)$ for $x \in M_1$, so T_1 is isomorphic to T_2 .

Definition 11. Let T_1 and T_2 be two measure-preserving transformations on probability spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively. We say that T_1 is conjugate to T_2 if there is a measure algebra isomorphism $\Phi : (\mathfrak{B}_2, \widetilde{\mu_2}) \rightarrow (\mathfrak{B}_1, \widetilde{\mu_1})$ such that $\Phi \circ \widetilde{T_2}^{-1} = \widetilde{T_1}^{-1} \circ \Phi$.

Definition 12. Measure-preserving transformations T_i on $(X_i, \mathfrak{B}_i, \mu_i)$ ($i = 1, 2$) are spectrally isomorphic, if there is a linear operator $W : L^2(\mu_2) \rightarrow L^2(\mu_1)$ such that

- (i) W is invertible;
- (ii) $\langle Wf, Wg \rangle = \langle f, g \rangle$ for all $f, g \in L^2(\mu_2)$;
- (iii) $U_{T_1} W = W U_{T_2}$.

Definition 13. Let $(X, \mathfrak{B}, \mu)$ be a probability space such that $L^2(\mu)$ is separable. An invertible measure-preserving transformation T on $(X, \mathfrak{B}, \mu)$ is said to have countable Lebesgue spectrum if there is a sequence $\{f_j\}_{j=0}^{\infty}$, with $f_0 \equiv 1$, of members of $L^2(\mu)$ such that $\{f_0\} \cup \{U_T^n f_j \mid j \geq 1, n \in \mathbb{Z}\}$ is an orthogonal basis of $L^2(\mu)$.

Theorem 14 (see [8]). Let T_1 and T_2 be two measure-preserving transformations on probability spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively. Then

- (i) if T_1 is isomorphic to T_2 , then T_1 is conjugate to T_2 ;
- (ii) if T_1 and T_2 are conjugate, then T_1 and T_2 are spectrally isomorphic;
- (iii) any two invertible measure-preserving transformations with countable Lebesgue spectrum are spectrally isomorphic.

Now we are ready to prove the invariance of Λ_T .

Theorem 15. Let T_1 and T_2 be two measure-preserving transformations on probability spaces $(X_1, \mathfrak{B}_1, \mu_1)$ and $(X_2, \mathfrak{B}_2, \mu_2)$, respectively. If T_1 and T_2 are spectrally isomorphic, then $\Lambda_{T_1} \cong \Lambda_{T_2}$.

Proof. Let $W : L^2(\mu_2) \rightarrow L^2(\mu_1)$ be an isomorphism of Hilbert spaces such that $U_{T_1} W = W U_{T_2}$. Define $\Psi : \Lambda_{T_1} \rightarrow \Lambda_{T_2}$ by $\Psi([f]_{T_1}) := [W^{-1}f]_{T_2}$. Since $U_{T_1} W = W U_{T_2}$, then Ψ is well-defined. Obviously Ψ is a linear transformation. Moreover, if $\Psi([f]_{T_1}) = [0]_{T_2}$, then $[W^{-1}f]_{T_2} = [0]_{T_2}$ so $W^{-1}f \sim_{T_2} 0$ and hence $U_{T_2} W^{-1}f = W^{-1}f$. Thus $W^{-1}U_{T_1}f = W^{-1}f$. Since W is bijective, then $U_{T_1}f = f$ and so $[f]_{T_1} = [0]_{T_1}$; therefore Ψ is injective. Finally, for $[g]_{T_2} \in \Lambda_{T_2}$, let $f := Wg$. Then $\Psi([f]_{T_1}) = [g]_{T_2}$; hence Ψ is onto and therefore an isomorphism. $\square$

Corollary 16. If $T_1 : (X_1, \mathfrak{B}_1, \mu_1) \rightarrow (X_1, \mathfrak{B}_1, \mu_1)$ and $T_2 : (X_2, \mathfrak{B}_2, \mu_2) \rightarrow (X_2, \mathfrak{B}_2, \mu_2)$ are conjugate, then $\Lambda_{T_1} \cong \Lambda_{T_2}$.

Corollary 17. Any two invertible measure-preserving transformations with countable Lebesgue spectrum have isomorphic ergodicity spaces.

So far, we have assigned a linear space to any given measure-preserving transformation T on a probability space $(X, \mathfrak{B}, \mu)$ which is spectrally isomorphism invariant. Any basis of the ergodicity space of T is denoted by $\mathcal{B}_T$.

In the following theorem we observe that the ergodicity space decreases in size as we compose T with itself.

Theorem 18. Let T be a measure-preserving transformation on a probability space $(X, \mathfrak{B}, \mu)$. Then Λ_{T^2} is isomorphic to a subspace of Λ_T .

Proof. Let $\mathcal{B}_T = \{[f_\alpha]_T\}_{\alpha \in J}$ be a basis for Λ_T . Then for $f \in L^2(\mu)$ we have $[f]_T = \sum_{i=1}^k \lambda_i [f_{\alpha_i}]_T$ for some $k \in \mathbb{N}$ and $\lambda_i \in \mathbb{C}$. Hence $[f]_{T^2} = \sum_{i=1}^k \lambda_i [f_{\alpha_i}]_{T^2}$. So $\{[f_\alpha]_{T^2}\}_{\alpha \in J}$ generates the linear space Λ_{T^2} . Therefore $\text{card}(\mathcal{B}_{T^2}) \leq \text{card}(\mathcal{B}_T)$ and so Λ_{T^2} is isomorphic to a subspace of Λ_T . $\square$

Analogously we have the following corollary.

Corollary 19. Let T be a measure-preserving transformation on a probability space $(X, \mathfrak{B}, \mu)$ and $m, n \in \mathbb{N}$. If $m \mid n$ then Λ_{T^n} is isomorphic to a subspace of Λ_{T^m} .

Roughly speaking, Corollary 19 states that if $m \mid n$, then T^m is more ergodic than T^n .

Theorem 20. If T_1 and T_2 are two invertible measure-preserving transformations on the probability space $(X, \mathfrak{B}, \mu)$, then $\Lambda_{T_1 \circ T_2} \cong \Lambda_{T_2 \circ T_1}$.

Proof. Let $\{[f_\alpha]_{T_2 \circ T_1}\}_{\alpha \in J}$ be a basis for $\Lambda_{T_2 \circ T_1}$. For $f \in L^2(\mu)$ we have

$$[f \circ T_2^{-1}]_{T_2 \circ T_1} = \sum_{i=1}^k \lambda_i [f_{\alpha_i}]_{T_2 \circ T_1} \quad (14)$$

for some $\lambda_i \in \mathbb{C}$ and $\alpha_i \in J$. In other words $f \circ T_2^{-1} \sim_{T_2 \circ T_1} \sum_{i=1}^k \lambda_i f_{\alpha_i}$; hence

$$\left(f \circ T_2^{-1} - \sum_{i=1}^k \lambda_i f_{\alpha_i}\right) \circ T_2 \circ T_1 = f \circ T_2^{-1} - \sum_{i=1}^k \lambda_i f_{\alpha_i}. \quad (15)$$

Thus

$$\left(f - \sum_{i=1}^k \lambda_i f_{\alpha_i} \circ T_2\right) \circ T_1 = \left(f - \sum_{i=1}^k \lambda_i f_{\alpha_i} \circ T_2\right) \circ T_2^{-1} \quad (16)$$

or

$$\left(f - \sum_{i=1}^k \lambda_i f_{\alpha_i} \circ T_2\right) \circ T_1 \circ T_2 = f - \sum_{i=1}^k \lambda_i f_{\alpha_i} \circ T_2. \quad (17)$$

Therefore $f \sim_{T_1 \circ T_2} \sum_{i=1}^k \lambda_i f_{\alpha_i} \circ T_2$ and this means that

$$[f]_{T_1 \circ T_2} = \sum_{i=1}^k \lambda_i [f_{\alpha_i} \circ T_2]_{T_1 \circ T_2}. \quad (18)$$

Hence $\{[f_\alpha \circ T_2]_{T_1 \circ T_2}\}_{\alpha \in J}$ generates $\Lambda_{T_1 \circ T_2}$ and so $\Lambda_{T_1 \circ T_2}$ is isomorphic to a subspace of $\Lambda_{T_2 \circ T_1}$. Similarly $\Lambda_{T_2 \circ T_1}$ is isomorphic to a subspace of $\Lambda_{T_1 \circ T_2}$. $\square$

Corollary 21. If T_1 and T_2 are two invertible measure-preserving transformations on the probability space $(X, \mathfrak{B}, \mu)$, then $\Lambda_{(T_1 \circ T_2)^n} \cong \Lambda_{(T_2 \circ T_1)^n}$ for all $n \in \mathbb{Z}$.

Proof. Since T_1 and T_2 are invertible then any composition of them is also invertible. For $n \in \mathbb{N}$, applying Theorem 20 to T_2 and $T_1 \circ (T_2 \circ T_1)^{n-1}$, we will have

$$\begin{aligned} \Lambda_{(T_2 \circ T_1)^n} &= \Lambda_{T_2 \circ T_1 \circ (T_2 \circ T_1)^{n-1}} \cong \Lambda_{T_1 \circ (T_2 \circ T_1)^{n-1} \circ T_2} \\ &= \Lambda_{(T_1 \circ T_2)^n}. \end{aligned} \quad (19)$$

Finally, if n is a negative integer, one may apply Theorem 5 to complete the proof. $\square$

Example 22. Let $X = \{1, 2, \dots, n\}$ and $\mathfrak{B} = \mathcal{P}(X)$. Define the set function $\mu : \mathfrak{B} \rightarrow [0, 1]$ by $\mu(A) := (1/n)\text{card}(A)$. Clearly μ is a probability measure on $\mathfrak{B}$ and therefore $(X, \mathfrak{B}, \mu)$ is a probability space. Note that

$$\begin{aligned} L^2(\mu) &= \left\{f : X \rightarrow \mathbb{C} : \int_X |f|^2 d\mu < \infty\right\} \\ &= \left\{f : X \rightarrow \mathbb{C} : \sum_{j=1}^n |f(j)|^2 < \infty\right\} = \mathbb{C}^X. \end{aligned} \quad (20)$$

So we may consider any $f \in L^2(\mu)$ as a vector in $\mathbb{C}^n$; say $(c_1, c_2, \dots, c_n)$, where $c_j = f(j)$. Let $T : X \rightarrow X$ be the permutation $T = \sigma_k = (12 \cdots k)$, where $1 \leq k \leq n$. In other words,

$$\begin{aligned} T(j) &= j+1 \quad \text{for } j = 1, \dots, k-1, \\ T(k) &= 1, \\ T(j) &= j \quad \text{for } j > k. \end{aligned} \quad (21)$$

Then T is a measure-preserving transformation on $(X, \mathfrak{B}, \mu)$.

Now let $f_0 = (c_1, c_2, \dots, c_n) \in L^2(\mu)$. For $f = (z_1, z_2, \dots, z_n) \in L^2(\mu)$, we have

$$f \sim_T f_0 \quad \text{iff } (f - f_0) \circ T = f - f_0. \quad (22)$$

Applying the previous relation on $j = 1, 2, \dots, k-1$ we will have

$$z_{j+1} - z_j = c_{j+1} - c_j \quad \text{for } j = 1, 2, \dots, k-1. \quad (23)$$

Therefore $f \sim_T f_0$ if and only if $z_{j+1} - z_j = c_{j+1} - c_j$ for $j = 1, 2, \dots, k-1$. Hence

$$\begin{aligned} [f_0]_T &= \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_{j+1} - z_j = c_{j+1} \\ &\quad - c_j \text{ for } j = 1, 2, \dots, k-1\}. \end{aligned} \quad (24)$$

Now define $\Psi : \Lambda_T \rightarrow \mathbb{C}^{k-1}$ by

$$\Psi([f]_T) := (z_2 - z_1, z_3 - z_2, \dots, z_k - z_{k-1}), \quad (25)$$

where $f = (z_1, z_2, \dots, z_n)$. Clearly Ψ is a well-defined linear transformation. Moreover, if $\Psi([f]_T) = (0, 0, \dots, 0)$, then $z_{j+1} - z_j = 0$ for $j = 1, 2, \dots, k-1$ so $[f]_T = (0, 0, \dots, 0)$ and hence Ψ is one-to-one.

On the other hand let $(c_1, c_2, \dots, c_{k-1}) \in \mathbb{C}^{k-1}$. If we put $f = (d_0, d_1, \dots, d_{k-1}, 0, \dots, 0)$, where $d_0 = 0$ and $d_j = \sum_{i=1}^j c_i$ ($j = 1, 2, \dots, k-1$), then $\Psi([f]_T) = (c_1, c_2, \dots, c_{k-1})$. So Ψ is onto. Therefore Ψ is an isomorphism between Λ_T and $\mathbb{C}^{k-1}$ and so $\Lambda_T \cong \mathbb{C}^{k-1}$.

Corollary 23. For any permutation $\sigma \in \mathcal{S}_n$, where $\sigma = (i_1 i_2 \dots i_k)$ and $X = \{1, 2, \dots, n\}$ is equipped with normalized counting measure, one has $\Lambda_\sigma \cong \mathbb{C}^{k-1}$.

Corollary 24. For any $n \in \mathbb{N}$ there is a measure-preserving transformation T on a probability space $(X, \mathfrak{B}, \mu)$ such that $\Lambda_T \cong \mathbb{C}^n$.

Example 25. Let $\mathbb{K} = \{z \in \mathbb{C} : |z| = 1\}$, let $\mathfrak{B}$ be the σ -algebra of Borel subsets of $\mathbb{K}$, and let m be the Haar measure on $\mathbb{K}$. Let $a \in \mathbb{K}$ not be a root of unity and define $T : \mathbb{K} \rightarrow \mathbb{K}$ by $T(z) = az$. Then T is an ergodic measure-preserving transformation and so by Theorem 1 $\Lambda_T = L_c^2(\mathbb{K}, m)$. On the other hand,

$$\begin{aligned} \dim(L_c^2(\mathbb{K}, m)) &= \dim(L^2(\mathbb{K}, m)) \\ &= \dim(L^2([0, 1], m)) = c. \end{aligned} \quad (26)$$

Therefore $\dim \Lambda_T = c$.

4. Concluding Remarks

In this paper we assigned a vector space Λ_T to a measure-preserving map T on a probability space $(X, \mathfrak{B}, \mu)$. It is an invariant space under conjugacy of dynamical systems and is an algebraic weight for ergodicity of T . The map T is ergodic if and only if Λ_T has its greatest possible size. The middle states give the weighted ergodicity. So, for any two measure-preserving maps T_1 and T_2 on a probability space $(X, \mathfrak{B}, \mu)$, we say that T_2 is more ergodic than T_1 if and only if $\Lambda_{T_1} \leq \Lambda_{T_2}$. We summarize the following results:

- (1) For any measure-preserving map $T : (X, \mathfrak{B}, \mu) \rightarrow (X, \mathfrak{B}, \mu)$, T is more ergodic than T^2 . This was expected since, exactly as in the case of Theorem 18, if T^2 is ergodic, then T is ergodic.
- (2) If $m \mid n$, then T^m is more ergodic than T^n . As in the previous part, this was expected since if $m \mid n$ and T^n are ergodic, then T^m is ergodic.
- (3) If T_1 and T_2 are invertible measure-preserving maps on a probability space $(X, \mathfrak{B}, \mu)$, then $T_1 \circ T_2$ and $T_2 \circ T_1$ have similar ergodicity treatment.
- (4) If T is invertible, then T and T^{-1} have similar ergodicity treatment.

Competing Interests

The authors declare that they have no competing interests.

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