Research Article

Some New Ostrowski's Inequalities for Functions Whose n th Derivatives Are r -Convex

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We prove some new Ostrowski's inequalities for functions whose n th derivatives are r -convex.

1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, given by the following theorem.

Theorem 1 ([1]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° (interior of I), and let $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in (a, b)$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right], \quad (1)$$

$$\forall x \in [a, b].$$

Inequality (1) has attracted much interest due to its diversity of applications in numerical analysis, probability theory, and other areas. We note that a numerous variants, extensions, and generalizations of inequality (1) have been discovered.

In [2], Cerone et al. proved the following identity.

Lemma 2 ([2, Lemma 2.1]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ one has the identity

$$\int_a^b f(t) dt$$

$$= \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) + (-1)^n \int_a^b P_n(x, t) f^{(n)}(t) dt, \quad (2)$$

where the kernel $P_n : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$P_n(x, t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x, b], \end{cases} \quad x \in [a, b] \quad (3)$$

and n is natural number, $n \geq 1$.

We also recall that a positive function $f : I \rightarrow \mathbb{R}$ is said to be r -convex on I , if the following inequality

$$f(tx + (1-t)y) \leq \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{1/r}, & r \neq 0 \\ [f(x)]^{1-t} [f(y)]^t, & r = 0 \end{cases} \quad (4)$$

holds for all $x, y \in I$ and $t \in [0, 1]$; see [3].

In this paper we establish some new Ostrowski's inequalities for functions whose n th derivatives are r -convex.

2. Main Results

In order to establish our results, we need these lemmas.

Lemma 3 ([4]). For $\alpha > 0$ and $k > 0, z > 0$,

$$J(\alpha, k) = \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty \quad (5)$$

$$H(\alpha, k, z) = \int_0^1 t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln k)^{i-1}}{(\alpha)_i} < \infty, \quad (6)$$

where $(\alpha)_i = \prod_{j=0}^{i-1} (\alpha + j)$.

Lemma 4 ([5]). For $a \geq 0$ and $b \geq 0$, the following algebraic inequalities are true:

$$(a+b)^\lambda \leq 2^{\lambda-1} (a^\lambda + b^\lambda), \quad \text{for } \lambda \geq 1, \\ (a+b)^\lambda \leq a^\lambda + b^\lambda, \quad \text{for } 0 \leq \lambda \leq 1. \quad (7)$$

Theorem 5. For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable on $[a, b] \subset I$ such that $f^{(n)} \in L([a, b])$. If $|f^{(n)}|$ is r -convex, then the following inequality

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ \leq \begin{cases} c_r \left(\frac{(x-a)^{n+1}}{n!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{n!} |f^{(n)}(b)| \right) B\left(\frac{1}{r} + 1, n+1\right) + \left(\frac{(x-a)^{n+1}}{n!} + \frac{(b-x)^{n+1}}{n!} \right) \frac{r c_r}{1 + (n+1)r} |f^{(n)}(x)| & \text{if } r \neq 0 \\ \frac{(x-a)^{n+1}}{(n+1)!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)| & \text{if } r = 0, A = B = 1 \\ \frac{(x-a)^{n+1}}{(n+1)!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{n!} |f^{(n)}(x)| \sum_{i=1}^{\infty} \frac{(\ln B)^{i-1}}{(n+1)_i} & \text{if } r = 0, A = 1 \neq B \\ \frac{(b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)| + \frac{(x-a)^{n+1}}{n!} |f^{(n)}(x)| \sum_{i=1}^{\infty} \frac{(-\ln A)^{i-1}}{(n+1)_i} & \text{if } r = 0, A \neq 1 = B \\ |f^{(n)}(x)| \left(\frac{(x-a)^{n+1}}{n!} \sum_{i=1}^{\infty} \frac{(-\ln A)^{i-1}}{(n+1)_i} + \frac{(b-x)^{n+1}}{n!} \sum_{i=1}^{\infty} \frac{(\ln B)^{i-1}}{(n+1)_i} \right) & \text{if } r = 0, A \neq 1, B \neq 1 \end{cases} \quad (8)$$

holds for all $x \in [a, b]$, where $A = |f^{(n)}(x)|/|f^{(n)}(a)|$, $B = |f^{(n)}(b)|/|f^{(n)}(x)|$, and

$$c_r = \begin{cases} 2^{1/r} & \text{if } 0 < r < 1 \\ 1 & \text{if } r \geq 1. \end{cases} \quad (9)$$

Proof. From Lemma 2 and properties of modulus, we have

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ \leq \int_a^x \frac{(u-a)^n}{n!} |f^{(n)}(u)| du \\ + \int_x^b \frac{(b-u)^n}{n!} |f^{(n)}(u)| du = \frac{(x-a)^{n+1}}{n!} \\ \cdot \int_0^1 t^n |f^{(n)}((1-t)a + tx)| dt + \frac{(b-x)^{n+1}}{n!} \\ \cdot \int_0^1 (1-t)^n |f^{(n)}((1-t)x + tb)| dt. \quad (10)$$

Since $|f^{(n)}|$ is r -convex function, for $r \neq 0$, the use of Lemma 4 gives

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$\leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 t^n \left((1-t) |f^{(n)}(a)|^r + t |f^{(n)}(x)|^r \right)^{1/r} dt \right) + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^n \left((1-t) |f^{(n)}(x)|^r + t |f^{(n)}(b)|^r \right)^{1/r} dt \right) \\ \leq \frac{(x-a)^{n+1}}{n!} c_r \left(\int_0^1 t^n (1-t)^{1/r} |f^{(n)}(a)|^r + t^{n+1/r} |f^{(n)}(x)|^r dt \right) + \frac{(b-x)^{n+1}}{n!} \\ \cdot c_r \left(\int_0^1 ((1-t)^{n+1/r} |f^{(n)}(x)|^r + (1-t)^n t^{1/r} |f^{(n)}(b)|^r) dt \right) = \frac{(x-a)^{n+1}}{n!} \\ \cdot c_r \left(B\left(n+1, \frac{1}{r} + 1\right) |f^{(n)}(a)|^r + \frac{r}{1 + (n+1)r} |f^{(n)}(x)|^r \right) + \frac{(b-x)^{n+1}}{n!} \\ \cdot c_r \left(\frac{r}{1 + (n+1)r} |f^{(n)}(x)|^r + B\left(\frac{1}{r} + 1, n+1\right) |f^{(n)}(b)|^r \right)$$

$$\begin{aligned}
& \cdot \left| f^{(n)}(b) \right| \Big) = c_r \left(\frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(a) \right| \right. \\
& + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(b) \right| \Big) B \left(\frac{1}{r} + 1, n+1 \right) \\
& + \left(\frac{(x-a)^{n+1}}{n!} + \frac{(b-x)^{n+1}}{n!} \right) \\
& \cdot \frac{r c_r}{1 + (n+1)r} \left| f^{(n)}(x) \right|,
\end{aligned} \tag{11}$$

where c_r is defined as in (9).

In the case where $r = 0$, (10) becomes

$$\begin{aligned}
& \left| \int_a^b f(t) dt \right. \\
& - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \Big| \\
& \leq \frac{(x-a)^{n+1}}{n!} \int_0^1 t^n \left| f^{(n)}(a) \right|^{1-t} \left| f^{(n)}(x) \right|^t dt \\
& + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(x) \right|^{1-t} \left| f^{(n)}(b) \right|^t dt \\
& = \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(a) \right| \int_0^1 t^n A^t dt \\
& + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \int_0^1 (1-t)^n B^t dt,
\end{aligned} \tag{12}$$

and we distinguish 4 cases.

If $A = B = 1$, then (12) gives

$$\begin{aligned}
& \left| \int_a^b f(t) dt \right. \\
& - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \Big| \\
& \leq \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(a) \right| \int_0^1 t^n dt \\
& + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \int_0^1 (1-t)^n dt \\
& = \frac{(x-a)^{n+1}}{(n+1)!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(n+1)!} \left| f^{(n)}(x) \right|.
\end{aligned} \tag{13}$$

If $A = 1 \neq B$, then (12) becomes

$$\begin{aligned}
& \left| \int_a^b f(t) dt \right. \\
& - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \Big| \\
& \leq \frac{(x-a)^{n+1}}{(n+1)!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \\
& \cdot \sum_{i=1}^{\infty} \frac{(\ln B)^{i-1}}{(n+1)_i},
\end{aligned} \tag{14}$$

where we have used (5).

If $A \neq 1 = B$, then (12) becomes

$$\begin{aligned}
& \left| \int_a^b f(t) dt \right. \\
& - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \Big| \\
& \leq \frac{(b-x)^{n+1}}{(n+1)!} \left| f^{(n)}(x) \right| + \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(x) \right| \\
& \cdot \sum_{i=1}^{\infty} \frac{(-\ln A)^{i-1}}{(n+1)_i},
\end{aligned} \tag{15}$$

where we have used (6) with $z = 1$.

In the case where $A \neq 1$ and $B \neq 1$, using Lemma 3, (12) gives

$$\begin{aligned}
& \left| \int_a^b f(t) dt \right. \\
& - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \Big| \\
& \leq \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(x) \right| \sum_{i=1}^{\infty} \frac{(-\ln A)^{i-1}}{(n+1)_i} \\
& + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \sum_{i=1}^{\infty} \frac{(\ln B)^{i-1}}{(n+1)_i}.
\end{aligned} \tag{16}$$

The desired result follows from (11) and (13)–(16). $\square$

Theorem 6. For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable on $[a, b] \subset I$ such that $f^{(n)} \in L([a, b])$ and let $q > 1$. If $|f^{(n)}|^q$ is r -convex, then the following inequality

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$\leq \begin{cases} \left(\frac{rc_r}{1+r} \right)^{1/q} \left[\frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} (|f^{(n)}(a)|^q + |f^{(n)}(x)|^q)^{1/q} + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} (|f^{(n)}(x)|^q + |f^{(n)}(b)|^q)^{1/q} \right] & \text{if } r \neq 0 \\ \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| & \text{if } r = 0, A = B = 1 \\ \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{B^q - 1}{q \ln B} \right)^{1/q} |f^{(n)}(x)| & \text{if } r = 0, A = 1 \neq B \\ \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| + \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{A^q - 1}{q \ln A} \right)^{1/q} |f^{(n)}(a)| & \text{if } r = 0, A \neq 1 = B \\ \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{A^q - 1}{q \ln A} \right)^{1/q} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{B^q - 1}{q \ln B} \right)^{1/q} |f^{(n)}(x)| & \text{if } r = 0, A \neq 1, B \neq 1 \end{cases} \quad (17)$$

holds for all $x \in [a, b]$, where $A = |f^{(n)}(x)|/|f^{(n)}(a)|$, $B = |f^{(n)}(b)|/|f^{(n)}(x)|$, $1/p + 1/q = 1$, and c_r is defined as in (9).

Proof. From Lemma 2, properties of modulus, and Hölder's inequality, we have

$$\left| \int_a^b f(t) dt \right|$$

$$\begin{aligned} & - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\ & \leq \int_a^x \frac{(u-a)^n}{n!} |f^{(n)}(u)| du \\ & + \int_x^b \frac{(b-u)^n}{n!} |f^{(n)}(u)| du = \frac{(x-a)^{n+1}}{n!} \\ & \cdot \int_0^1 t^n |f^{(n)}((1-t)a + tx)| dt + \frac{(b-x)^{n+1}}{n!} \\ & \cdot \int_0^1 (1-t)^n |f^{(n)}((1-t)x + tb)| dt \\ & \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 t^{np} dt \right)^{1/p} \\ & \cdot \left(\int_0^1 |f^{(n)}((1-t)a + tx)|^q dt \right)^{1/q} \\ & + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^{np} dt \right)^{1/p} \\ & \cdot \left(\int_0^1 |f^{(n)}((1-t)x + tb)|^q dt \right)^{1/q} \\ & = \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\int_0^1 |f^{(n)}((1-t)a + tx)|^q dt \right)^{1/q} \\ & + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\int_0^1 |f^{(n)}((1-t)x + tb)|^q dt \right)^{1/q} \end{aligned}$$

$$+ \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\int_0^1 |f^{(n)}((1-t)x + tb)|^q dt \right)^{1/q}.$$

(18)

In the case where $r \neq 0$, the use of r -convexity of $|f^{(n)}|^q$ and Lemma 4 gives

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] \cdot f^{(k)}(x) \right| \\ & \leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\int_0^1 ((1-t) |f^{(n)}(a)|^{qr} \right. \\ & \quad \left. + t |f^{(n)}(x)|^{qr})^{1/r} dt \right)^{1/q} \\ & + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\int_0^1 ((1-t) |f^{(n)}(x)|^{qr} \right. \\ & \quad \left. + t |f^{(n)}(b)|^{qr})^{1/r} dt \right)^{1/q} \\ & \leq \frac{(x-a)^{n+1} c_r^{1/q}}{(np+1)^{1/p} n!} \left(\int_0^1 ((1-t)^{1/r} |f^{(n)}(a)|^q \right. \\ & \quad \left. + t^{1/r} |f^{(n)}(x)|^q) dt \right)^{1/q} \\ & + \frac{(b-x)^{n+1} c_r^{1/q}}{(np+1)^{1/p} n!} \left(\int_0^1 ((1-t)^{1/r} |f^{(n)}(x)|^q \right. \\ & \quad \left. + t^{1/r} |f^{(n)}(b)|^q) dt \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&= \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{rc_r}{1+r} \right)^{1/q} \left(|f^{(n)}(a)|^q \right. \\
&\quad \left. + |f^{(n)}(x)|^q \right)^{1/q} + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{rc_r}{1+r} \right)^{1/q} \\
&\quad \cdot \left(|f^{(n)}(x)|^q + |f^{(n)}(b)|^q \right)^{1/q}.
\end{aligned} \tag{19}$$

For $r = 0$, using the r -convexity of $|f^{(n)}|^q$, (18) gives

$$\begin{aligned}
&\left| \int_a^b f(t) dt \right. \\
&\quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| \left(\int_0^1 A^{qt} dt \right)^{1/q} \\
&\quad + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| \left(\int_0^1 B^{qt} dt \right)^{1/q}.
\end{aligned} \tag{20}$$

Analogously to Theorem 5, we will treat the 4 cases.

If $A = B = 1$, then (20) gives

$$\begin{aligned}
&\left| \int_a^b f(t) dt \right. \\
&\quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| \\
&\quad + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)|.
\end{aligned} \tag{21}$$

If $A = 1 \neq B$, then (20) becomes

$$\begin{aligned}
&\left| \int_a^b f(t) dt \right. \\
&\quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)|
\end{aligned}$$

$$\begin{aligned}
&+ \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(|f^{(n)}(x)|^q \int_0^1 B^{qt} dt \right)^{1/q} \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| \\
&\quad + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{B^q - 1}{q \ln B} \right)^{1/q} |f^{(n)}(x)|.
\end{aligned} \tag{22}$$

If $A \neq 1 = B$, then (20) becomes

$$\begin{aligned}
&\left| \int_a^b f(t) dt \right. \\
&\quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| \\
&\quad + \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(|f^{(n)}(a)|^q \int_0^1 A^{qt} dt \right)^{1/q} \\
&\leq \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| \\
&\quad + \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{A^q - 1}{q \ln A} \right)^{1/q} |f^{(n)}(a)|.
\end{aligned} \tag{23}$$

In the case where $A \neq 1$ and $B \neq 1$, (20) gives

$$\begin{aligned}
&\left| \int_a^b f(t) dt \right. \\
&\quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{A^q - 1}{q \ln A} \right)^{1/q} |f^{(n)}(a)| \\
&\quad + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{B^q - 1}{q \ln B} \right)^{1/q} |f^{(n)}(x)|.
\end{aligned} \tag{24}$$

The desired result follows from (19) and (21)–(24). $\square$

Theorem 7. For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable on $[a, b] \subset I$ such that $f^{(n)} \in L([a, b])$ and let $q > 1$. If $|f^{(n)}|^q$ is convex, then the following inequality

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
& \leq \begin{cases} \left(\frac{(n+1)^{1/q}}{(n+1)!} c_r^{1/q} \left[(x-a)^{n+1} \left(B \left(n+1, \frac{1}{r} + 1 \right) |f^{(n)}(a)|^q + \frac{r}{1+(n+1)r} |f^{(n)}(x)|^q + (b-x)^{n+1} \left(\frac{r}{1+(n+1)r} |f^{(n)}(x)|^q + B \left(\frac{1}{r} + 1, n+1 \right) |f^{(n)}(b)|^q \right) \right] \right)^{1/q} & \text{if } r \neq 0 \\ \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| & \text{if } r=0, A=B=1 \\ \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{B^q-1}{q \ln B} \right)^{1/q} |f^{(n)}(x)| & \text{if } r=0, A=1 \neq B \\ \frac{(b-x)^{n+1}}{(np+1)^{1/p} n!} |f^{(n)}(x)| + \frac{(x-a)^{n+1}}{(np+1)^{1/p} n!} \left(\frac{A^q-1}{q \ln A} \right)^{1/q} |f^{(n)}(a)| & \text{if } r=0, A \neq 1 = B \\ |f^{(n)}(x)| \left(\frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} \left(\sum_{i=1}^{\infty} \frac{(-\ln A^q)^{i-1}}{(n+1)_i} \right)^{1/q} + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} \left(\sum_{i=1}^{\infty} \frac{(\ln B^q)^{i-1}}{(n+1)_i} \right)^{1/q} \right) & \text{if } r=0, A \neq 1, B \neq 1 \end{cases}
\end{aligned} \tag{25}$$

holds for all $x \in [a, b]$, where $A = |f^{(n)}(x)|/|f^{(n)}(a)|$, $B = |f^{(n)}(b)|/|f^{(n)}(x)|$, and c_r is defined as in (9).

Proof. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
& \leq \int_a^x \frac{(u-a)^n}{n!} |f^{(n)}(u)| du + \int_x^b \frac{(b-u)^n}{n!} |f^{(n)}(u)| du \\
& = \frac{(x-a)^{n+1}}{n!} \int_0^1 t^n |f^{(n)}((1-t)a+tx)| dt + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n |f^{(n)}((1-t)x+tb)| dt \\
& \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 t^n dt \right)^{1-1/q} \left(\int_0^1 t^n |f^{(n)}((1-t)a+tx)|^q dt \right)^{1/q} \\
& \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1-1/q} \left(\int_0^1 (1-t)^n |f^{(n)}((1-t)x+tb)|^q dt \right)^{1/q} \\
& = \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} \left(\int_0^1 t^n |f^{(n)}((1-t)a+tx)|^q dt \right)^{1/q} \\
& \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} \left(\int_0^1 (1-t)^n |f^{(n)}((1-t)x+tb)|^q dt \right)^{1/q}.
\end{aligned} \tag{26}$$

If $r \neq 0$, using r -convexity of $|f^{(n)}|^q$ and Lemma 4, we get

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
& \leq \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} \cdot c_r^{1/q} \left(\int_0^1 \left[t^n (1-t)^{1/r} |f^{(n)}(a)|^q \right. \right. \\
& \quad \left. \left. + t^{n+1/r} |f^{(n)}(x)|^q \right] dx \right)^{1/q} \\
& \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} \cdot c_r^{1/q} \left(\int_0^1 \left[t^n (1-t)^{1/r} |f^{(n)}(x)|^q \right. \right. \\
& \quad \left. \left. + t^{n+1/r} |f^{(n)}(b)|^q \right] dt \right)^{1/q} \\
& = \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} c_r^{1/q} \left(B \left(n+1, \frac{1}{r} + 1 \right) \cdot |f^{(n)}(a)|^q + \frac{r}{1+(n+1)r} |f^{(n)}(x)|^q \right)^{1/q} \\
& \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} c_r^{1/q} \left(\frac{r}{1+(n+1)r} |f^{(n)}(x)|^q + B \left(\frac{1}{r} + 1, n+1 \right) \cdot |f^{(n)}(b)|^q \right)^{1/q}.
\end{aligned} \tag{27}$$

Now, suppose that $r = 0$, and from the convexity of $|f^{(n)}|^q$, (25) becomes

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
& \leq \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} \left(|f^{(n)}(a)|^q \int_0^1 t^n A^{qt} dt \right)^{1/q} \\
& \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} \left(|f^{(n)}(x)|^q \int_0^1 (1-t)^n B^{qt} dt \right)^{1/q}.
\end{aligned} \tag{28}$$

If $A = B = 1$, then (27) gives

$$\begin{aligned} & \left| \int_a^b f(t) dt \right. \\ & \quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ & \leq \frac{(x-a)^{n+1}}{(n+1)!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)|. \end{aligned} \quad (29)$$

If $A = 1 \neq B$, then (27) becomes

$$\begin{aligned} & \left| \int_a^b f(t) dt \right. \\ & \quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ & \leq \frac{(x-a)^{n+1}}{(n+1)!} |f^{(n)}(a)| \\ & \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)| \\ & \quad \cdot \left(\sum_{i=1}^{\infty} \frac{(\ln B^q)^{i-1}}{(n+1)_i} \right)^{1/q}. \end{aligned} \quad (30)$$

If $A \neq 1 = B$, then (27) becomes

$$\begin{aligned} & \left| \int_a^b f(t) dt \right. \\ & \quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ & \leq \frac{(b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)| \\ & \quad + \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} |f^{(n)}(x)| \\ & \quad \cdot \left(\sum_{i=1}^{\infty} \frac{(-\ln A^q)^{i-1}}{(n+1)_i} \right)^{1/q}. \end{aligned} \quad (31)$$

In the case where $A \neq 1$ and $B \neq 1$, (27) gives

$$\begin{aligned} & \left| \int_a^b f(t) dt \right. \\ & \quad \left. - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ & \leq \frac{(n+1)^{1/q} (x-a)^{n+1}}{(n+1)!} |f^{(n)}(x)| \end{aligned}$$

$$\begin{aligned} & \cdot \left(\sum_{i=1}^{\infty} \frac{(-\ln A^q)^{i-1}}{(n+1)_i} \right)^{1/q} \\ & \quad + \frac{(n+1)^{1/q} (b-x)^{n+1}}{(n+1)!} |f^{(n)}(x)| \\ & \quad \cdot \left(\sum_{i=1}^{\infty} \frac{(\ln B^q)^{i-1}}{(n+1)_i} \right)^{1/q}, \end{aligned} \quad (32)$$

where we have used Lemma 3. The desired result follows from (26) and (28)–(31). $\square$

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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