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Research Article

Various Fixed Point Theorems in Complex Valued b-Metric Spaces

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We prove some common fixed point results for a pair of mappings which satisfy generalized contractive conditions with rational expressions having point-dependent control functions as coefficients in complex valued b-metric spaces. The results of this paper generalize and extend the several known results in complex valued b-metric spaces. Finally, examples are provided to verify the effectiveness and to usability of our main results.

1. Introduction and Preliminaries

The concept of complex valued metric space was introduced by Azam et al. [1], proving some fixed point results for mappings satisfying a rational inequality in complex valued metric spaces. Since then, several papers have dealt with fixed point theory in complex valued metric spaces (see [2–10] and references therein). Rao et al. [11] initiated the studying of fixed point results on complex valued b-metric spaces, which was more general than the complex valued metric spaces [1]. Following this paper, a number of authors have proved several fixed point results for various mapping satisfying a rational inequalities in the context of complex valued b-metric spaces (see [12–15]) and the related references therein.

Recently, Sintunavarat et al. [8, 9], Sitthikul and Saejung [10], and Singh et al. [7] obtained common fixed point results by replacing the constant of contractive condition to control functions in complex valued metric spaces. In a continuation of [7, 10, 14, 16], in this paper, we establish some common fixed point results for a pair of mappings satisfying more general contractive conditions involving rational expressions having point-dependent control functions as coefficients in complex valued b-metric spaces.

Consistent with Rao et al. [11], the following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a) if $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$;

(b) if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$;

(c) if $a, b \in \mathbb{R}$ and $a \le b$, then $az \precsim bz$ for all $z \in \mathbb{C}_+$.

The following definition is recently introduced by Rao et al. [11].

Definition 1 (see [11]). Let X be a nonempty set and let $s \ge 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is

called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 2 (see [11]). If $X = [0, 1]$, define a mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then, (X, d) is complex valued b-metric space with $s = 2$.

Definition 3 (see [11]). Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called an open set whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

(v) The family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$ is a subbasis for a Hausdorff topology τ on X.

Definition 4 (see [11]). Let (X, d) be a complex valued b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent and converges to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent in X, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 5 (see [11]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 6 (see [11]). *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

2. Main Result

Throughout this paper, let (X, d) be a complete complex valued b-metric space and $S, T : X \to X$ be mappings. In our results, we will use the following family of functions.

Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings. Let Ψ be the family of all functions $\psi : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$ and for fixed $a \in X$,

(F1) $\psi(TSx, y, a) \leq \psi(x, y, a)$;

(F2) $\psi(x, STy, a) \leq \psi(x, y, a)$.

We start this section with the following observation.

Proposition 7. *Let (X, d) be a complex valued b-metric space and let $S, T : X \to X$ be mappings. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned} x_{2n+1} &= Sx_{2n}, \\ x_{2n+2} &= Tx_{2n+1}, \\ &\qquad \forall n = 0, 1, 2, \ldots. \end{aligned} \tag{1}$$

Assume that there exists a mapping $\alpha \in \Psi$ for all $x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2, \ldots$. Then, $\alpha(x_{2n}, y, a) \leq \alpha(x_0, y, a)$ and $\alpha(x, x_{2n+1}, a) \leq \alpha(x, x_1, a)$.

Proof. Let $x, y \in X$ and $n = 0, 1, 2, \ldots$. Then, we have

$$\begin{aligned} \alpha(x_{2n}, y, a) &= \alpha(Tx_{2n-1}, y, a) = \alpha(TSx_{2n-2}, y, a) \\ &\leq \alpha(x_{2n-2}, y, a) = \alpha(TSx_{2n-4}, y, a) \leq \cdots \\ &\leq \alpha(x_0, y, a). \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} \alpha(x, x_{2n+1}, a) &= \alpha(x, Sx_{2n}, a) = \alpha(x, STx_{2n-1}, a) \\ &\leq \alpha(x, x_{2n-1}, a) = \alpha(x, STx_{2n-3}, a) \\ &\leq \cdots \leq \alpha(x, x_1, a). \end{aligned} \tag{3}$$

□

Lemma 8 (see [10]). *Let $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies $a_n \leq ha_{n-1}$, for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Now, we proceed to establish common fixed point theorems for the general contraction conditions in complex valued b-metric space.

Theorem 9. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings. If there exist mappings $\alpha, \beta, \gamma, \delta \in \Psi$ such that for all $x, y \in X$ and for fixed $a \in X$,*

$$\text{(a) } \alpha(x, y, a) + \beta(x, y, a) + 2\gamma(x, y, a) + 2s\delta(x, y, a) < 1; \tag{4}$$

(b) $d(Sx, Ty)$

$$\begin{aligned} &\precsim \alpha(x,y,a)\, d(x,y) \\ &\quad + \frac{\beta(x,y,a)\, d(y,Ty)\, d(x,Sx)}{1+d(x,y)} \\ &\quad + \gamma(x,y,a)\,[d(x,Sx)+d(y,Ty)] \\ &\quad + \delta(x,y,a)\,[d(x,Ty)+d(y,Sx)]. \end{aligned} \tag{5}$$

Then, S and T have a unique common fixed point.

Proof. Let $x, y \in X$, from (5) we have

$$\begin{aligned} d(Sx,TSx) &\precsim \alpha(x,Sx,a)\, d(x,Sx) \\ &\quad + \frac{\beta(x,Sx,a)\, d(Sx,TSx)\, d(x,Sx)}{1+d(x,Sx)} \\ &\quad + \gamma(x,Sx,a)\,[d(x,Sx)+d(Sx,TSx)] \\ &\quad + \delta(x,Sx,a)\,[d(x,TSx)+d(Sx,Sx)] \\ &\precsim \alpha(x,Sx,a)\, d(x,Sx) \\ &\quad + \frac{\beta(x,Sx,a)\, d(Sx,TSx)\, d(x,Sx)}{1+d(x,Sx)} \\ &\quad + \gamma(x,Sx,a)\,[d(x,Sx)+d(Sx,TSx)] \\ &\quad + s\delta(x,Sx,a)\,[d(x,Sx)+d(Sx,TSx)] \end{aligned} \tag{6}$$

which implies that

$$\begin{aligned} &|d(Sx,TSx)| \\ &\quad \le \alpha(x,Sx,a)\,|d(x,Sx)| \\ &\quad + \frac{\beta(x,Sx,a)\,|d(Sx,TSx)|\,|d(x,Sx)|}{|1+d(x,Sx)|} \\ &\quad + \gamma(x,Sx,a)\,|d(x,Sx)+d(Sx,TSx)| \\ &\quad + s\delta(x,Sx,a)\,|d(x,Sx)+d(Sx,TSx)|. \end{aligned} \tag{7}$$

Since $|1+d(x,Sx)| \ge |d(x,Sx)|$,

$$\begin{aligned} &|d(Sx,TSx)| \\ &\quad \le \alpha(x,Sx,a)\,|d(x,Sx)| + \beta(x,Sx,a)\,|d(Sx,TSx)| \\ &\quad + \gamma(x,Sx,a)\,|d(x,Sx)+d(Sx,TSx)| \\ &\quad + s\delta(x,Sx,a)\,|d(x,Sx)+d(Sx,TSx)|. \end{aligned} \tag{8}$$

In a similar way, by setting $x = Ty$ in (5), we have

$$\begin{aligned} &|d(STy,Ty)| \\ &\quad \le \alpha(Ty,y,a)\,|d(Ty,y)| \\ &\quad + \beta(Ty,y,a)\,|d(Ty,STy)| \\ &\quad + \gamma(Ty,y,a)\,|d(y,Ty)+d(Ty,STy)| \\ &\quad + s\delta(Ty,y,a)\,|d(y,Ty)+d(Ty,STy)|. \end{aligned} \tag{9}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (1). We show that $\{x_n\}$ is a Cauchy sequence. From Proposition 7 and for all $K = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned} &|d(x_{2K+1},x_{2K})| = |d(STx_{2K-1},Tx_{2K-1})| \\ &\quad \le \alpha(x_{2K},x_{2K-1},a)\,|d(x_{2K},x_{2K-1})| \\ &\quad + \beta(x_{2K},x_{2K-1},a)\,|d(x_{2K+1},x_{2K})| \\ &\quad + \gamma(x_{2K},x_{2K-1},a) \\ &\quad \cdot [|d(x_{2K+1},x_{2K})+d(x_{2K},x_{2K-1})|] \\ &\quad + s\delta(x_{2K},x_{2K-1},a) \\ &\quad \cdot [|d(x_{2K+1},x_{2K})+d(x_{2K},x_{2K-1})|] \\ &\quad \le \alpha(x_0,x_1,a)\,|d(x_{2K-1},x_{2K})| + \beta(x_0,x_1,a) \\ &\quad \cdot |d(x_{2K},x_{2K+1})| + \gamma(x_0,x_1,a) \\ &\quad \cdot [|d(x_{2K-1},x_{2K})+d(x_{2K},x_{2K+1})|] \\ &\quad + s\delta(x_0,x_1,a)\,[|d(x_{2K-1},x_{2K})+d(x_{2K},x_{2K+1})|] \end{aligned} \tag{10}$$

which yields that

$$\begin{aligned} &|d(x_{2K+1},x_{2K})| \\ &\quad \le \frac{\alpha(x_0,x_1,a)+\gamma(x_0,x_1,a)+s\delta(x_0,x_1,a)}{1-\beta(x_0,x_1,a)-\gamma(x_0,x_1,a)-s\delta(x_0,x_1,a)}\,|d(x_{2K-1}, \\ &\quad x_{2K})|. \end{aligned} \tag{11}$$

Similarly, one can obtain

$$\begin{aligned} &|d(x_{2K+2},x_{2K+1})| \\ &\quad \le \frac{\alpha(x_0,x_1,a)+\gamma(x_0,x_1,a)+s\delta(x_0,x_1,a)}{1-\beta(x_0,x_1,a)-\gamma(x_0,x_1,a)-s\delta(x_0,x_1,a)}\,|d(x_{2K}, \\ &\quad x_{2K+1})|. \end{aligned} \tag{12}$$

Let $\mu = (\alpha(x_0,x_1,a) + \gamma(x_0,x_1,a) + s\delta(x_0,x_1,a))/(1 - \beta(x_0,x_1,a) - \gamma(x_0,x_1,a) - s\delta(x_0,x_1,a)) < 1$.

Since $\alpha(x_0,x_1,a) + \beta(x_0,x_1,a) + 2\gamma(x_0,x_1,a) + 2s\delta(x_0,x_1,a) < 1$, thus we have $|d(x_{2K+2},x_{2K+1})| \le \mu|d(x_{2K},x_{2K+1})|$ and $|d(x_{2K+1},x_{2K})| \le \mu|d(x_{2K-1},x_{2K})|$, or in fact

$$|d(x_{n+1},x_{n+2})| \le \mu\,|d(x_n,x_{n+1})|. \tag{13}$$

Thus, by Lemma 8 we get that this sequence is Cauchy sequence in (X,d). Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Let, on contrary, $u \ne Su$; then

$$|d(u,Su)| > 0. \tag{14}$$

So by using the triangular inequality and (5), we get

$$\begin{aligned} d(u,Su) &\precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Su) \\ &= sd(u,x_{2n+2}) + sd(Tx_{2n+1},Su) \\ &\precsim sd(u,x_{2n+2}) + s\alpha(u,x_{2n+1},a)\,d(u,x_{2n+1}) \\ &\quad + \frac{s\beta(u,x_{2n+1},a)\,d(x_{2n+1},Tx_{2n+1})\,d(u,Su)}{1+d(u,x_{2n+1})} \\ &\quad + s\gamma(u,x_{2n+1},a)\left[d(u,Su)+d(x_{2n+1},Tx_{2n+1})\right] \\ &\quad + s\delta(u,x_{2n+1},a)\left[d(u,Tx_{2n+1})+d(x_{2n+1},Su)\right] \\ &\precsim sd(u,x_{2n+2}) + s\alpha(u,x_1,a)\,d(u,x_{2n+1}) \\ &\quad + \frac{s\beta(u,x_1,a)\,d(x_{2n+1},x_{2n+2})\,d(u,Su)}{1+d(u,x_{2n+1})} \\ &\quad + s\gamma(u,x_1,a)\left[d(u,Su)+d(x_{2n+1},x_{2n+2})\right] \\ &\quad + s\delta(u,x_1,a)\left[d(u,x_{2n+2})+d(x_{2n+1},Su)\right]. \end{aligned} \tag{15}$$

This implies that

$$\begin{aligned} &|d(u,Su)| \\ &\quad \le s\left|d(u,x_{2n+2})\right| + s\alpha(u,x_1,a)\left|d(u,x_{2n+1})\right| \\ &\qquad + \frac{s\beta(u,x_1,a)\left|d(x_{2n+1},x_{2n+2})\right|\,|d(u,Su)|}{\left|1+d(u,x_{2n+1})\right|} \\ &\qquad + s\gamma(u,x_1,a)\left|d(u,Su)+d(x_{2n+1},x_{2n+2})\right| \\ &\qquad + s\delta(u,x_1,a)\left|d(u,x_{2n+2})+d(u,Su)\right|. \end{aligned} \tag{16}$$

Letting $n \to \infty$, it follows that

$$\begin{aligned} |d(u,Su)| &\le s\gamma(u,x_1,a)\,|d(u,Su)| + s\delta(u,x_1,a) \\ &\quad \cdot |d(u,Su)| = s\left[\gamma(u,x_1,a)+\delta(u,x_1,a)\right] \\ &\quad \cdot |d(u,Su)| \le s\left[\alpha(u,x_1,a)+\beta(u,x_1,a)\right. \\ &\quad \left. + 2\gamma(u,x_1,a)+2\delta(u,x_1,a)\right]|d(u,Su)| \\ &\quad < |d(u,Su)|, \end{aligned} \tag{17}$$

a contradiction, and so $|d(u,Su)| = 0$; that is, $u = Su$. It follows similarly that $u = Tu$. This implies that u is a common fixed point of S and T.

We now prove that this u is unique:

$$\begin{aligned} d(u,u^*) &= d(Su,Tu^*) \\ &\precsim \alpha(u,u^*,a)\,d(u,u^*) \\ &\quad + \frac{\beta(u,u^*,a)\,d(u^*,Tu^*)\,d(u,Su)}{1+d(u,u^*)} \\ &\quad + \gamma(u,u^*a)\left[d(u,Su)+d(u^*,Tu^*)\right] \\ &\quad + \delta(u,u^*,a)\left[d(u,Tu^*)+d(u^*,Su)\right] \\ &\precsim \left[\alpha(u,u^*,a)+2\delta(u,u^*a)\right]d(u,u^*). \end{aligned} \tag{18}$$

Therefore, we have

$$\left|d(u,u^*)\right| \le \left[\alpha(u,u^*,a)+2\delta(u,u^*,a)\right]\left|d(u,u^*)\right|. \tag{19}$$

Since $\alpha(u,u^*,a)+2\delta(u,u^*,a) < 1$, we have $|d(u,u^*)| = 0$.

Thus, $u = u^*$, which proves the uniqueness of common fixed point in X. This concludes the theorem. □

Remark 10. If we replace $\alpha, \beta : X \times X \times X \to [0,1)$ by $\Lambda, E : X \to [0,1)$, with $\alpha(x,y,a) = \Lambda(x)$ and $\beta(x,y,a) = E(x)$ for all $x, y \in X$ and so $s\Lambda(x) + E(x) < 1$, then we get the result of Theorem 3.1 of Sintunavarat and Kumam [8] (complex valued b-metric space version).

Remark 11. If we set mappings $\alpha, \beta : X \times X \times X \to [0,1)$ as $\alpha(x,y,a) = \alpha'$ and $\beta(x,y,a) = \beta'$, where $\alpha', \beta' \in [0,1)$ such that $s\alpha' + \beta' < 1$ and for all $x, y \in X$, we get Theorem 4 of Azam et al. [1] (complex valued b-metric space version).

Next theorem is presented for single mapping satisfying slightly different conditions.

Theorem 12. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping. If there exist mappings $\alpha, \beta, \gamma, \delta \in \Psi$ such that for all $x, y \in X$ and for fixed $a \in X$,*

$$\text{(a) } \alpha(x,y,a) + \beta(x,y,a) + 2\gamma(x,y,a) + 2s\delta(x,y,a) < 1; \tag{20}$$

$$\begin{aligned} \text{(b) } d(Tx,Ty) &\precsim \alpha(x,y,a)\,d(x,y) \\ &\quad + \beta(x,y,a)\frac{[1+d(x,Tx)]\,d(y,Ty)}{1+d(x,y)} \\ &\quad + \gamma(x,y,a)\left[d(x,Tx)+d(y,Ty)\right] \\ &\quad + \delta(x,y,a)\left[d(x,Ty)+d(y,Tx)\right], \end{aligned} \tag{21}$$

then T has a unique fixed point.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by $x_{n+1} = Tx_n$, where $n = 0, 1, 2, \ldots$. Now we show that $\{x_n\}$ is a Cauchy sequence. From condition (21), we have

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(Tx_n,Tx_{n+1}) \\ &\precsim \alpha(x_n,x_{n+1},a)\,d(x_n,x_{n+1}) \\ &\quad + \beta(x_n,x_{n+1},a)\frac{\left[1+d(x_n,Tx_n)\right]d(x_{n+1},Tx_{n+1})}{1+d(x_n,x_{n+1})} \\ &\quad + \gamma(x_n,x_{n+1},a)\left[d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})\right] \\ &\quad + \delta(x_n,x_{n+1},a)\left[d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)\right] \\ &\precsim \alpha(x_0,x_0,a)\,d(x_n,x_{n+1}) \\ &\quad + \beta(x_0,x_0,a)\,d(x_{n+1},x_{n+2}) \\ &\quad + \gamma(x_0,x_0,a)\left[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})\right] \\ &\quad + s\delta(x_0,x_0,a)\left[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})\right]. \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}
&\left|d\left(x_{n+1},x_{n+2}\right)\right| \leq \alpha\left(x_0,x_0,a\right)\left|d\left(x_n,x_{n+1}\right)\right| + \beta\left(x_0,x_0,a\right) \\
&\quad \cdot \left|d\left(x_{n+1},x_{n+2}\right)\right| + \gamma\left(x_0,x_0,a\right)\left[\left|d\left(x_n,x_{n+1}\right)\right.\right. \\
&\quad \left.\left. + d\left(x_{n+1},x_{n+2}\right)\right|\right] + s\delta\left(x_0,x_0,a\right)\left[\left|d\left(x_n,x_{n+1}\right)\right.\right. \\
&\quad \left.\left. + d\left(x_{n+1},x_{n+2}\right)\right|\right] \\
&\quad \leq \frac{\alpha\left(x_0,x_0,a\right)+\gamma\left(x_0,x_0,a\right)+s\delta\left(x_0,x_0,a\right)}{1-\beta\left(x_0,x_0,a\right)-\gamma\left(x_0,x_0,a\right)-s\delta\left(x_0,x_0,a\right)}\left|d\left(x_n,\right.\right. \\
&\quad \left.\left. x_{n+1}\right)\right|.
\end{aligned} \tag{23}$$

Let $\mu = (\alpha(x_0,x_0,a) + \gamma(x_0,x_0,a) + s\delta(x_0,x_0,a))/(1 - \beta(x_0,x_0,a) - \gamma(x_0,x_0,a) - s\delta(x_0,x_0,a)) < 1$; then,

$$\left|d\left(x_{n+1},x_{n+2}\right)\right| \leq \mu\left|d\left(x_n,x_{n+1}\right)\right| \quad \forall n = 0,1,2,\ldots. \tag{24}$$

By Lemma 8, this sequence is Cauchy sequence in (X,d). Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Next, we show that u is a fixed point of T.

From (21), we have

$$\begin{aligned}
&d\left(u,Tu\right) \precsim sd\left(u,Tx_n\right) + sd\left(Tx_n,Tu\right) \\
&\quad \precsim sd\left(u,Tx_n\right) + s\alpha\left(x_n,u,a\right)d\left(x_n,u\right) \\
&\qquad + s\beta\left(x_n,u,a\right)\frac{\left[1+d\left(x_n,Tx_n\right)\right]d\left(u,Tu\right)}{1+d\left(x_n,u\right)} \\
&\qquad + s\gamma\left(x_n,u,a\right)\left[d\left(x_n,Tx_n\right)+d\left(u,Tu\right)\right] \\
&\qquad + s\delta\left(x_n,u,a\right)\left[d\left(x_n,Tu\right)+d\left(u,Tx_n\right)\right] \\
&\quad \precsim sd\left(u,x_{n+1}\right) + s\alpha\left(x_0,u,a\right)d\left(x_n,u\right) \\
&\qquad + s\beta\left(x_0,u,a\right)\frac{\left[1+d\left(x_n,x_{n+1}\right)\right]d\left(u,Tu\right)}{1+d\left(x_n,u\right)} \\
&\qquad + s\gamma\left(x_0,u,a\right)\left[d\left(x_n,x_{n+1}\right)+d\left(u,Tu\right)\right] \\
&\qquad + s\delta\left(x_0,u,a\right)\left[d\left(x_n,Tu\right)+d\left(u,x_{n+1}\right)\right].
\end{aligned} \tag{25}$$

This implies that

$$\begin{aligned}
&\left|d\left(u,Tu\right)\right| \\
&\quad \leq s\left|d\left(u,x_{n+1}\right)\right| + s\alpha\left(x_0,u,a\right)\left|d\left(x_n,u\right)\right| \\
&\qquad + s\beta\left(x_0,u,a\right)\frac{\left|1+d\left(x_n,x_{n+1}\right)\right|\left|d\left(u,Tu\right)\right|}{\left|1+d\left(x_n,u\right)\right|} \\
&\qquad + s\gamma\left(x_0,u,a\right)\left|\left[d\left(x_n,x_{n+1}\right)+d\left(u,Tu\right)\right]\right| \\
&\qquad + s\delta\left(x_0,u,a\right)\left|\left[d\left(x_n,Tu\right)+d\left(u,x_{n+1}\right)\right]\right|
\end{aligned} \tag{26}$$

which on making $n \to \infty$ reduces to

$$\begin{aligned}
&\left|d\left(u,Tu\right)\right| \leq s\beta\left(x_0,u,a\right)\left|d\left(u,Tu\right)\right| + s\gamma\left(x_0,u,a\right) \\
&\quad \cdot \left|d\left(u,Tu\right)\right| + s\delta\left(x_0,u,a\right)\left|d\left(u,Tu\right)\right| \\
&\quad = \left[s\beta\left(x_0,u,a\right)+s\gamma\left(x_0,u,a\right)+s\delta\left(x_0,u,a\right)\right] \\
&\quad \cdot \left|d\left(u,Tu\right)\right| \leq s\left[\alpha\left(x_0,u,a\right)+\beta\left(x_0,u,a\right)\right. \\
&\quad \left. + 2\gamma\left(x_0,u,a\right)+2\delta\left(x_0,u,a\right)\right]\left|d\left(u,Tu\right)\right|,
\end{aligned} \tag{27}$$

a contradiction, and so $|d(u,Tu)| = 0$; that is, $u = Tu$. This implies that u is a fixed point of T.

Uniqueness of fixed point is an easy consequence of condition (22). This completes the proof. □

Corollary 13. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping. If there exist mappings $\alpha,\beta,\gamma,\delta \in \Psi$ such that for all $x,y \in X$ and for some fixed n,*

$$\begin{aligned}
&\text{(a) } \alpha\left(x,y,a\right)+\beta\left(x,y,a\right)+2\gamma\left(x,y,a\right) \\
&\quad + 2s\delta\left(x,y,a\right) < 1;
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\text{(b) } d\left(T^n x,T^n y\right) \precsim \alpha\left(x,y,a\right)d\left(x,y\right) \\
&\quad + \beta\left(x,y,a\right)\frac{\left[1+d\left(x,T^n x\right)\right]d\left(y,T^n y\right)}{1+d\left(x,y\right)} \\
&\quad + \gamma\left(x,y,a\right)\left[d\left(x,T^n x\right)+d\left(y,T^n y\right)\right] \\
&\quad + \delta\left(x,y,a\right)\left[d\left(x,T^n y\right)+d\left(y,T^n x\right)\right],
\end{aligned} \tag{29}$$

then T has a unique fixed point.

Proof. By Theorem 12, there exists $v \in X$ such that $T^n v = v$. Then,

$$\begin{aligned}
&d\left(Tv,v\right) = d\left(TT^n v,T^n v\right) = d\left(T^n Tv,T^n v\right) \\
&\quad \precsim \alpha\left(Tv,v,a\right)d\left(Tv,v\right) \\
&\qquad + \beta\left(Tv,v,a\right)\frac{\left[1+d\left(x,T^n Tv\right)\right]d\left(v,T^n v\right)}{1+d\left(Tv\cdot v\right)} \\
&\qquad + \gamma\left(Tv,v,a\right)\left[d\left(Tv,T^n Tv\right)+d\left(v,T^n v\right)\right] \\
&\qquad + \delta\left(Tv,v,a\right)\left[d\left(Tv,T^n v\right)+d\left(v,T^n Tv\right)\right] \\
&\quad \precsim \alpha\left(Tv,v,a\right)d\left(Tv,v\right) \\
&\qquad + \beta\left(Tv,v,a\right)\frac{\left[1+d\left(x,TT^n v\right)\right]d\left(v,v\right)}{1+d\left(Tv\cdot v\right)} \\
&\qquad + \gamma\left(Tv,v,a\right)\left[d\left(Tv,TT^n v\right)+d\left(v,v\right)\right] \\
&\qquad + \delta\left(Tv,v,a\right)\left[d\left(Tv,v\right)+d\left(v,TT^n v\right)\right] \\
&\quad = \alpha\left(Tv,v,a\right)d\left(Tv,v\right) + 2\delta\left(Tv,v,a\right)d\left(Tv,v\right) \\
&\quad = \left(\alpha+2\delta\right)\left(Tv,v,a\right)d\left(Tv,v\right)
\end{aligned} \tag{30}$$

and so $d(Tv,v) = 0$. So $Tv = v$. Therefore, the fixed point of T is unique. □

Example 14. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{C}$ be defined by $d(x, y) = i|x - y|^2$ for all $x, y \in X$. Then, (X, d) is a complex valued b-metric space with the coefficient $s = 2$. Now we define self-mappings $S, T : X \to X$ by $S(x) = x/4$ and $T(y) = y/4$. Further, for all $x, y \in X$ and for fixed $a = 1/3 \in X$, we define the functions $\alpha, \beta, \gamma, \delta : X \times X \times X \to [0, 1)$ by

$$\begin{aligned} \alpha(x, y, a) &= \left(\frac{x}{4} + \frac{y}{5} + a\right), \\ \beta(x, y, a) &= \frac{xya}{10}, \\ \gamma(x, y, a) &= \frac{x^2 y^2 a^2}{10}, \\ \delta(x, y, a) &= \frac{x^3 y^3 a^3}{10}. \end{aligned} \tag{31}$$

Clearly $\alpha(x, y, a) + \beta(x, y, a) + 2\gamma(x, y, a) + 2s\delta(x, y, a) < 1$ for all $x, y \in X$ and for a fixed $a = 1/3 \in X$.

Now consider

$$\begin{aligned} \alpha(TSx, y, a) &= \alpha\left(T\left(\frac{x}{4}\right), y, a\right) = \alpha\left(\frac{x}{16}, y, a\right) \\ &= \frac{x}{64} + \frac{y}{5} + a \le \frac{x}{4} + \frac{y}{5} + a \\ &= \alpha(x, y, a). \end{aligned} \tag{32}$$

That is, $\alpha(TSx, y, a) \le \alpha(x, y, a)$ for all $x, y \in X$ and for a fixed $a = 1/3 \in X$. Now

$$\begin{aligned} \alpha(x, STy, a) &= \alpha\left(x, S\left(\frac{y}{4}\right), a\right) = \alpha\left(x, \frac{y}{16}, a\right) \\ &= \frac{x}{4} + \frac{y}{80} + a \le \frac{x}{4} + \frac{y}{5} + a \\ &= \alpha(x, y, a). \end{aligned} \tag{33}$$

That is, $\alpha(x, STy, a) \le \alpha(x, y, a)$ for all $x, y \in X$ and for a fixed $a = 1/3 \in X$. Similarly, we can show that

$$\begin{aligned} &\beta(TSx, y, a) \le \beta(x, y, a), \\ &\beta(x, STy, a) \le \beta(x, y, a), \\ &\gamma(TSx, y, a) \le \gamma(x, y, a), \\ &\gamma(x, STy, a) \le \gamma(x, y, a), \\ &\delta(TSx, y, a) \le \delta(x, y, a), \\ &\delta(x, STy, a) \le \delta(x, y, a). \end{aligned} \tag{34}$$

Now for the verification of inequality (5), it is sufficient to show that $(Sx, Ty) \precsim \alpha(x, y, a)d(x, y)$.

Consider

$$\begin{aligned} d(Sx, Ty) &= d\left(\frac{x}{4}, \frac{y}{4}\right) = \left|\frac{x}{4} - \frac{y}{4}\right|^2 i = \frac{1}{16}|x - y|^2 i \\ &\precsim \frac{1}{3}|x - y|^2 i \precsim \left(\frac{x}{4} + \frac{y}{5} + \frac{1}{3}\right)|x - y|^2 i \\ &= \alpha(x, y, a)\, d(x, y), \\ &\qquad \forall x, y \in X, \text{ for } a = \frac{1}{3} \in X. \end{aligned} \tag{35}$$

That is, $d(Sx, Ty) \precsim \alpha(x, y, a)d(x, y)$ for all $x, y \in X$ and for fixed $a = 1/3 \in X$. Therefore, all the conditions of Theorem 9 are satisfied, also $x = 0$ remains fixed under S and T and is indeed unique.

Competing Interests

The authors declare that they have no competing interests.

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