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Research Article

n-Tupled Coincidence Point Theorems for Probabilistic ψ-Contractions in Menger Spaces

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We introduced n-tupled coincidence point for a pair of maps $T : X^n \to X$ and $A : X \to X$ in Menger space. Utilizing the properties of the pseudometric and the triangular norm, we will establish n-tupled coincidence point theorems under weak compatibility as well as n-tupled fixed point theorems for hybrid probabilistic ψ-contractions with a gauge function. Our main results do not require the conditions of continuity and monotonicity of ψ. At the end of this paper, an example is given to support our main theorem.

Dedicated to late Professor V. Lakshmikantham

1. Introduction and Preliminaries

Probabilistic metric space was introduced by Menger [1] in the year 1942 by generalizing metric spaces in which a distribution function was used instead of nonnegative real number as value of the metric.

Now we present some basic concepts and results which will be used in this paper.

Throughout this paper we will denote R as the set of real numbers, R^+ as the nonnegative real numbers, and Z^+ as the set of all positive integers.

If $\psi : R^+ \to R^+$ is a function such that $\psi(0) = 0$, then ψ is called a gauge function. If $t \in R^+$, then $\psi^n(t)$ denotes the nth iteration of $\psi(t)$ and $\psi^{-1}(\{0\}) = \{t \in R^+ : \psi(t) = 0\}$.

A mapping $f : R \to R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in R^+} f(t) = 0$, $\sup_{t\in R^+} f(t) = 1$.

We will denote by D the set of all distribution functions and by H the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases} \tag{1}$$

Now, we are ready to recall the following definitions and lemmas for our main results in Section 2.

Definition 1 (see [2]). A function $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (in short, t-norm) if the following conditions are satisfied for any $a, b, c, d \in [0,1]$:

(1) $\Delta(a, 1) = a$;

(2) $\Delta(a, b) = \Delta(b, a)$;

(3) $\Delta(a, b) \ge \Delta(c, d)$, for $a \ge c, b \ge d$;

(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Examples of t-norms are $\Delta_m(a, b) = \min\{a, b\}$ and $\Delta_p(a, b) = ab$ for all $a, b \in [0,1]$ and $\Delta_p \le \Delta_m$ for each t-norm.

Definition 2 (see [2, 3]). A triplet (X, F, Δ) is called a Menger probabilistic metric space, if X is a nonempty set, Δ is a t-norm, and F is a mapping from $X \times X$ into D. We will denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ will represent the value of $F_{x,y}$ at $t \in R$ satisfying the following conditions:

(1) $F_{x,y}(0) = 0$;

(2) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;

(3) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in R$;

(4) $F_{x,z}(t+s) \geq \Delta\{F_{x,y}(t), F_{y,z}(s)\}$ for all $x, y, z \in X$ and $t, s \in R^+$.

(X, F, Δ) is called a non-Archimedean Menger PM-space if it is a Menger PM-space satisfying the following condition:

(1) $F_{x,z}(\max\{t, s\}) \geq \Delta\{F_{x,y}(t), F_{y,z}(s)\}$ for all $x, y, z \in X$ and $t, s \in R^+$.

Schweizer and Sklar [4, 5] pointed out that if the t-norm Δ of a Menger PM-space satisfies the condition $\sup_{0<a<1}\Delta(a, a) = 1$, then (X, F, Δ) is a first countable Hausdorff topological space in the (ϵ, λ)-topology τ; that is, the family of sets

$$\{U_x(\epsilon, \lambda) : \epsilon > 0,\ \lambda \in (0, 1]\} \quad (x \in X) \tag{2}$$

is the base of neighborhoods of a point x for τ, where

$$U_x(\epsilon, \lambda) = \left\{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\right\}. \tag{3}$$

By virtue of this topology τ, a sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent and converges to x (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is a Cauchy sequence in (X, F, Δ) if any given $\epsilon > 0$ and $\lambda \in (0, 1]$, and there exists $N = N(\epsilon, \lambda) \in Z^+$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \in N$. (X, F, Δ) is said to be complete, if every Cauchy sequence in X is a convergent sequence in X.

Lemma 3 (see [6, 7]). *Let (X, d) be a usual metric space. Define a mapping $F : X \times X \to D^+$ by*

$$F_{x,y}(t) = H(t - d(x, y)) \quad \text{for } x, y \in X,\ t > 0. \tag{4}$$

Then (X, F, Δ_m) is a Menger PM-space; it is called the induced Menger PM-space by (X, d) and it is complete, if (X, d) is complete.

An arbitrary t-norm can be extended [3, Definition 2.1] in a unique way to an n-array operation. For $(a_1, a_2, a_3, \ldots, a_n) \in [0, 1]^n$ $(n \in Z^+)$, the value $\Delta^n(a_1, a_2, a_3, \ldots, a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1, a_2, a_3, \ldots, a_n) = \Delta(\Delta^{n-1}((a_1, a_2, a_3, \ldots, a_{n-1}), a_n))$. For each $a \in [0, 1]$, the sequence $\{\Delta^n(a)\}_{n=1}^{\infty}$ is defined by $\Delta^1(a) = a$ and $\Delta^n(a) = \Delta(\Delta^{n-1}(a), a)$.

Definition 4 (see [8]). A t-norm Δ is said to be of H-type if the sequence of functions $\{\Delta^n(a)\}_{n=1}^{\infty}$ is equicontinuous at $a = 1$.

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$ (see [8]). It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{0<a<1}\Delta(a, a) = 1$.

Lemma 5 (see [7, 9]). *Let (X, F, Δ) be a Menger PM-space. For each $\lambda \in (0, 1]$, define a function $d_\lambda : X \times X \to R^+$ by*

$$d_\lambda(x, y) = \inf\left\{t > 0 : F_{x,y}(t) > 1 - \lambda\right\}. \tag{5}$$

Then the following statements hold:

(1) *$d_\lambda(x, y) < t$ if and only if $F_{x,y}(t) > 1 - \lambda$;*

(2) *$d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$ and $\lambda \in (0, 1]$;*

(3) *$d_\lambda(x, y) = 0$ for all $\lambda \in (0, 1]$ if and only if $x = y$.*

Lemma 6 (see [7]). *Let (X, F, Δ) be a Menger PM-space and let $\{d_\lambda\}_{\lambda\in(0,1]}$ be a family of pseudometrics on X defined by (13). If Δ is a t-norm of H-type, then, for each $\lambda \in (0, 1]$, there exist $\mu \in (0, \lambda]$ such that, for each $m \in Z^+$,*

$$d_\lambda(x_0, x_m) \leq \sum_{t=0}^{m-1} d_\mu(x_i, x_{i+1}), \quad \forall x_0, x_1, \ldots, x_m \in X. \tag{6}$$

Lemma 7 (see [10]). *Let (X, F, Δ) be a non-Archimedean Menger PM-space and let $\{d_\lambda\}_{\lambda\in(0,1]}$ be a family of pseudometrics on X defined by (13). If Δ is a t-norm of H-type, then, for each $\lambda \in (0, 1]$, there exist $\mu \in (0, \lambda]$ such that, for each $m \in Z^+$,*

$$d_\lambda(x_0, x_m) \leq \max_{0\leq i\leq m-1} d_\mu(x_i, x_{i+1}), \quad \forall x_0, x_1, \ldots, x_m \in X. \tag{7}$$

Lemma 8 (see [11]). *Suppose that $F \in D^+$. For each $n \in Z^+$, let $F_n : R \to [0, 1]$ be nondecreasing and $g_n : (0, \infty) \to (0, \infty)$ satisfy $\lim_{n\to\infty} g_n(t) = 0$ for any $t > 0$. If*

$$F_n(g_n(t)) \geq F(t) \quad \text{for any } t > 0, \quad \text{then } \lim_{n\to\infty} F_n(t) = 1 \quad \text{for any } t > 0. \tag{8}$$

In this paper we used the new definitions of n-tupled coincidence point given by Imdad et al. [12] and n-tupled fixed point given by Samet and Vetro [13].

The following definitions are also needed for our main results.

Definition 9 (see [12]). An element $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}$ is called n-tupled common fixed point of the mapping $T : X^n \to X$ if $T(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = x^{(1)}$, $T(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}) = x^{(2)}$, $T(x^{(3)}, \ldots, x^{(n)}, x^{(1)}, x^{(2)}) = x^{(3)}, \ldots$, $T(x^{(n)}, x^{(1)}, \ldots, x^{(n-2)}, x^{(n-1)}) = x^{(n)}$.

Definition 10 (see [12]). An element $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}$ is called an n-tupled coincidence point of the mappings $T : X^n \to X$ and $A : X \to X$ if $T(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = Ax^{(1)}$, $T(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}) = Ax^{(2)}$, $T(x^{(3)}, \ldots, x^{(n)}, x^{(1)}, x^{(2)}) = Ax^{(3)}, \ldots, T(x^{(n)}, x^{(1)}, \ldots, x^{(n-2)}, x^{(n-1)}) = Ax^{(n)}$.

Definition 11 (see [12]). An element $(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)})$ is called an n-tupled common fixed point of the mappings $T : X^n \to X$ and $A : X \to X$ if $T(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = Ax^{(1)} = x^{(1)}$, $T(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}) = Ax^{(2)} = x^{(2)}$, $T(x^{(3)}, \ldots, x^{(n)}, x^{(1)}, x^{(2)}) = Ax^{(3)} = x^{(3)}, \ldots, T(x^{(n)}, x^{(1)}, \ldots, x^{(n-2)}, x^{(n-1)}) = Ax^{(n)} = x^{(n)}$.

Now, we are ready to introduce the concept of commutativity, compatibility, and weak compatibility in Menger PM-spaces for n-dimensions.

Definition 12. Let X be a nonempty set. Let $T : X^n \to X$ and $A : X \to X$ be two mappings. A is said to be commutative with T if $AT(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) = T(Ax^{(1)}, Ax^{(2)}, Ax^{(3)}, \ldots, Ax^{(n)})$ for all $(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}) \in X$. A point $u \in X$ is called a common fixed point of T and A if $u = Au = T(u, u, \ldots, u)$.

Definition 13. Let $T : X^n \to X$ and $A : X \to X$ be two mappings. Then T is said to be A-compatible if

$$\begin{gathered}\lim_{m\to\infty} F_{AT(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)}),T(Ax_m^{(1)},Ax_m^{(2)},\ldots,Ax_m^{(n)})}(t) = 1,\\ \lim_{m\to\infty} F_{AT(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)}),T(Ax_m^{(2)},\ldots,Ax_m^{(n)},Ax_m^{(1)})}(t) = 1,\\ \vdots\\ \lim_{m\to\infty} F_{AT(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)}),T(Ax_m^{(n)},Ax_m^{(1)},\ldots,Ax_m^{(n-1)})}(t) = 1,\end{gathered} \tag{9}$$

where $x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)}$ are the sequences in X such that

$$\begin{gathered}\lim_{m\to\infty} T\left(x_m^{(1)}, x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}\right) = \lim_{m\to\infty} Ax_m^{(1)} = x^{(1)},\\ \lim_{m\to\infty} T\left(x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}\right) = \lim_{m\to\infty} Ax_m^{(2)} = x^{(2)},\\ \lim_{m\to\infty} T\left(x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}, x_m^{(2)}\right) = \lim_{m\to\infty} Ax_m^{(3)} = x^{(3)},\\ \vdots\\ \lim_{m\to\infty} T\left(x_m^{(n)}, \ldots, x_m^{(n-2)}, x_m^{(n-1)}\right) = \lim_{m\to\infty} Ax_m^{(n)} = x^{(n)}\end{gathered} \tag{10}$$

for some $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ are satisfied.

Definition 14. The mappings $T : X^n \to X$ and $A : X \to X$ are called weakly compatible maps if

$$\begin{gathered}T\left(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}\right) = A\left(x^{(1)}\right),\\ T\left(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}\right) = A\left(x^{(2)}\right),\\ \vdots\\ T\left(x^{(n)}, x^{(1)}, x^{(3)}, \ldots, x^{(n-1)}\right) = A\left(x^{(n)}\right)\end{gathered} \tag{11}$$

imply

$$\begin{aligned}&A\left(T\left(x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}\right)\right)\\ &\quad = T\left(A\left(x^{(1)}\right), A\left(x^{(2)}\right), A\left(x^{(3)}\right), \ldots, A\left(x^{(n)}\right)\right),\\ &A\left(T\left(x^{(2)}, x^{(3)}, \ldots, x^{(n)}, x^{(1)}\right)\right)\\ &\quad = T\left(A\left(x^{(2)}\right), A\left(x^{(3)}\right), \ldots, A\left(x^{(n)}\right), A\left(x^{(1)}\right)\right),\\ &\quad\vdots\\ &A\left(T\left(x^{(n)}, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}\right)\right)\\ &\quad = T\left(A\left(x^{(n)}\right), A\left(x^{(1)}\right), A\left(x^{(2)}\right), \ldots, A\left(x^{(n-1)}\right)\right)\end{aligned} \tag{12}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$.

In this paper, we will introduce n-tupled coincidence point, n-tupled fixed point, commutativity, compatibility, and weak compatibility in Menger space for function of higher dimension. Utilizing the properties of the pseudometric and the triangular norm, we will establish n-tupled coincidence point results as well as n-tuple fixed point results using weak compatibility of mappings for hybrid probabilistic ψ-contractions with a gauge function in Menger spaces.

2. Main Results

Lemma 15. *Let X ba a nonempty set. Let $T : X^n \to X$ and $A : X \to X$ be two mappings. If $T(X^n) \subset A(X)$, then there exist n sequences $\{x_m^{(1)}\}_{m=1}^{\infty}, \{x_m^{(2)}\}_{m=1}^{\infty}, \ldots, \{x_m^{(n)}\}_{m=1}^{\infty}$ in X such that $Ax_{m+1}^{(1)} = T(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)})$, $Ax_{m+1}^{(2)} = T(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)}), \ldots, Ax_{m+1}^{(n)} = T(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)})$.*

Proof. Let $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}$ be n arbitrary points in X, since $T(X^n) \subseteq A(X)$.

We define $x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)} \in X$ such that

$$\begin{gathered}T\left(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}\right) = Ax_1^{(1)},\\ T\left(x_0^{(2)}, x_0^{(3)}, \ldots, x_0^{(n)}, x_0^{(1)}\right) = Ax_1^{(2)},\\ \vdots\\ T\left(x_0^{(n)}, x_0^{(1)}, \ldots, x_0^{(n-2)}, x_0^{(n-1)}\right) = ax_1^{(n)}.\end{gathered} \tag{13}$$

Again, for $T(X^n) \subseteq A(X)$ we can choose $x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)} \in X$ such that

$$\begin{gathered}T\left(x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \ldots, x_1^{(n)}\right) = Ax_2^{(1)},\\ T\left(x_1^{(2)}, x_1^{(3)}, \ldots, x_1^{(n)}, x_1^{(1)}\right) = Ax_2^{(2)},\\ \vdots\\ T\left(x_1^{(n)}, x_1^{(1)}, \ldots, x_1^{(n-2)}, x_1^{(n-1)}\right) = Ax_2^{(n)}.\end{gathered} \tag{14}$$

Continuing this process, we can construct n sequence $\{x_m^{(1)}\}_{m=1}^{\infty}, \{x_m^{(2)}\}_{m=1}^{\infty}, \ldots, \{x_m^{(n)}\}_{m=1}^{\infty}$ in X such that

$$\begin{gathered}Ax_{m+1}^{(1)} = T\left(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)}\right),\\ Ax_{m+1}^{(2)} = T\left(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)}\right),\\ \vdots\\ Ax_{m+1}^{(n)} = T\left(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)}\right).\end{gathered} \tag{15}$$

□

Now we will establish the following theorem by using Lemma 15.

Theorem 16. *Let (X, F, Δ) be a Menger PM-space such that Δ is a t-norm of H-type. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$, $\psi(t) < t$, and $\lim_{m\to\infty}\psi^m(t) = 0$ for any $t > 0$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned} &F_{T(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}),T(y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)})}(\psi(t)) \\ &\quad \geq \left[F_{Ax^{(1)},Ay^{(1)}}(t)\, F_{Ax^{(2)},Ay^{(2)}}(t)\cdots F_{Ax^{(n)},Ay^{(n)}}(t)\right]^{1/n} \end{aligned} \tag{16}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A, T) is weakly compatible.*

Then there exists $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

Proof. By Lemma 15, we can construct n sequences $\{x_m^{(1)}\}_{m=1}^{\infty}$, $\{x_m^{(2)}\}_{m=1}^{\infty}, \ldots, \{x_m^{(n)}\}_{m=1}^{\infty}$ in X such that $Ax_{m+1}^{(1)} = T(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(n)})$, $Ax_{m+1}^{(2)} = T(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)}), \ldots, Ax_{m+1}^{(n)} = T(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)})$.

Let $t > 0$. From (16), we have

$$\begin{aligned} &F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)) \\ &\quad = F_{T(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)}),T(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})}(\psi(t)) \\ &\quad \geq \Big[F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t) \\ &\qquad \cdot F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t)\cdots F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t)\Big]^{1/n} \\ &F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi(t)) \\ &\quad = F_{T(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)}),T(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)})}(\psi(t)) \\ &\quad \geq \Big[F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t)\cdots F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t) \\ &\qquad \cdot F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t)\Big]^{1/n} \\ &\qquad\qquad \vdots \\ &F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi(t)) \\ &\quad = F_{T(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}),T(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})}(\psi(t)) \\ &\quad \geq \Big[F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t) \\ &\qquad \cdot F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t)\cdots F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)\Big]^{1/n}. \end{aligned} \tag{17}$$

Suppose that $G_m(t) = [F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t)F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t)\cdots F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)]^{1/n}$. Then from the above inequalities, we obtain

$$\begin{aligned} G_{m+1}(\psi(t)) &= \Big[F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)) \\ &\quad \cdot F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi(t))\cdots F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi(t))\Big]^{1/n} \\ &\geq [G_m(t)\, G_m(t)\, G_m(t)\cdots G_m(t)]^{1/n} = G_m(t). \end{aligned} \tag{18}$$

This implies that

$$\begin{aligned} &F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\ &F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\ &\qquad\qquad \vdots \\ &F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t). \end{aligned} \tag{19}$$

Since $G_1(t) = [F_{Ax_0^{(n)},Ax_1^{(n)}}(t)F_{Ax_0^{(1)},Ax_1^{(1)}}(t)\cdots F_{Ax_0^{(n-1)},Ax_1^{(n-1)}}(t)]^{1/n}$ and $\lim_{n\to\infty}\psi^m(t) = 0$ for each $t > 0$, using Lemma 8, we have

$$\begin{aligned} &\lim_{n\to\infty} F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) = 1, \\ &\lim_{n\to\infty} F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t) = 1, \\ &\qquad\qquad \vdots \\ &\lim_{n\to\infty} F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t) = 1. \end{aligned} \tag{20}$$

Therefore, we get

$$\lim_{n\to\infty} G_m(t) = 1 \quad \forall t > 0. \tag{21}$$

Now we claim that, for any $k \in Z^+$,

$$\begin{aligned} &F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(t) \geq \Delta^k[G_m(t-\psi(t))], \\ &F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(t) \geq \Delta^k[G_m(t-\psi(t))], \\ &\qquad\qquad \vdots \\ &F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(t) \geq \Delta^k[G_m(t-\psi(t))]. \end{aligned} \tag{22}$$

Assume that (22) holds for some k. Since $\psi(t) < t$, we have $F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) \geq F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)) \geq G_m(t)$. By (16) and (22), we have

$$\begin{aligned} F_{Ax_{m+1}^{(1)},Ax_{m+k+1}^{(1)}}(\psi(t)) &\geq \Big[F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(t) \\ &\quad \cdot F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(t)\cdots F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(t)\Big]^{1/n} \\ &\geq \Big[\Delta^k[G_m(t-\psi(t))] \\ &\quad \cdot \Delta^k[G_m(t-\psi(t))]\cdots\Delta^k[G_m(t-\psi(t))]\Big]^{1/n} \\ &= \Delta^k[G_m(t-\psi(t))]. \end{aligned} \tag{23}$$

Hence, by the monotonicity of Δ, we have

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k+1}^{(1)}}(t) &= F_{Ax_m^{(1)},Ax_{m+k+1}^{(1)}}(t-\psi(t)+\psi(t)) \\ &\geq \Delta\Big[F_{Ax_{m+1}^{(1)},Ax_{m+1}^{(1)}}(t-\psi(t)), F_{Ax_{m+1}^{(1)},Ax_{m+k+1}^{(1)}}(\psi(t))\Big] \\ &\geq \Delta\Big[\Delta G_m(t-\psi(t)), \Delta^k G_m(t-\psi(t))\Big] \\ &\geq \Delta\Big[G_m(t-\psi(t)), \Delta^k G_m(t-\psi(t))\Big] \\ &= \Delta^{k+1}G_m(t-\psi(t)). \end{aligned} \tag{24}$$

Similarly, we obtain

$$\begin{aligned} F_{Ax_{m+1}^{(1)},Ax_{m+k+1}^{(1)}}(t) &\geq \Delta^{k+1}G_m(t-\psi(t)), \\ F_{Ax_{m+1}^{(2)},Ax_{m+k+1}^{(2)}}(t) &\geq \Delta^{k+1}G_m(t-\psi(t)), \\ &\vdots \\ F_{Ax_{m+1}^{(n)},Ax_{m+k+1}^{(n)}}(t) &\geq \Delta^{k+1}G_m(t-\psi(t)). \end{aligned} \tag{25}$$

Therefore by the induction, (22) holds for all $k \in Z^+$. Suppose that $\epsilon > 0$ and $\lambda \in (0,1]$ are given. By hypothesis, Δ is a t-norm of H-type. There exists $\delta > 0$ such that

$$\Delta^k(s) > 1-\lambda \quad \forall s \in (1-\delta, 1],\ k \in Z^+. \tag{26}$$

By using (21), there exist $N \in Z^+$ such that $G_m(\epsilon-\psi(\epsilon)) > 1-\delta$ for all $m \geq N$. Hence from (22) and (26) we get

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(\epsilon) &> 1-\lambda, \\ F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(\epsilon) &> 1-\lambda, \\ &\vdots \\ F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(\epsilon) &> 1-\lambda, \end{aligned} \tag{27}$$

for all $n \geq N$ and $k \in Z^+$.

Therefore $\{Ax_m^{(1)}\}, \{Ax_m^{(2)}\}, \ldots, \{Ax_m^{(n)}\}$ are all Cauchy sequence. Since $A(X)$ is complete, there exist $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in A(X)$ such that $\lim_{n\to\infty} Ax_{m+1}^{(1)} = x^{(1)}, \lim_{n\to\infty} Ax_{m+1}^{(2)} = x^{(2)}, \ldots, \lim_{n\to\infty} Ax_{m+1}^{(n)} = x^{(n)}$. Again $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in A(X)$ implies that there exists $p^{(1)}, p^{(2)}, \ldots, p^{(n)} \in X$ such that

$$A(p^{(1)}) = x^{(1)}, A(p^{(2)}) = x^{(2)}, \ldots, A(p^{(n)}) = x^{(n)}. \tag{28}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} T(x_m^{(1)}, x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}) &= \lim_{m\to\infty} Ax_{m+1}^{(1)} \\ &= A(p^{(1)}) = x^{(1)}, \\ \lim_{m\to\infty} T(x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}) &= \lim_{m\to\infty} Ax_{m+1}^{(2)} \\ &= A(p^{(2)}) = x^{(2)}, \\ &\vdots \\ \lim_{m\to\infty} T(x_m^{(n)}, \ldots, x_m^{(n-2)}, x_m^{(n-1)}) &= \lim_{m\to\infty} Ax_{m+1}^{(n)} \\ &= A(p^{(n)}) = x^{(n)}. \end{aligned} \tag{29}$$

From (16) and $\psi(t) < t$, we have

$$\begin{aligned} &F_{T(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(2)}),T(p^{(1)},p^{(2)},\ldots,p^{(n)})}(t) \\ &\quad \geq F_{T(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(2)}),T(p^{(1)},p^{(2)},\ldots,p^{(n)})}(\psi(t)) \\ &\quad \geq \Big[F_{Ax_m^{(1)},A_{(p(1))}}(t), F_{Ax_m^{(2)},A_{(p(2))}}(t), \ldots, \\ &\quad F_{Ax_m^{(n)},A_{(p(n))}}(t)\Big]^{1/n}. \end{aligned} \tag{30}$$

Taking the lim $m \to \infty$, we get

$$\lim_{m\to\infty} F_{Ap^{(1)},T(p^{(1)},p^{(2)},\ldots,p^{(n)})}(t) = 1. \tag{31}$$

Now again, we have

$$\begin{aligned} &F_{T(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)},),T(p^{(2)},\ldots,p^{(n)},p^{(1)})}(t) \\ &\quad \geq F_{T(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)},),T(p^{(2)},\ldots,p^{(n)},p^{(1)})}(\psi(t)) \\ &\quad \geq \Big[F_{Ax_m^{(2)},A(p(2))}(t), \ldots, F_{Ax_m^{(n)},Ap(n)}(t) \\ &\quad \cdot F_{Ax_m^{(1)},Ap(1)}(t)\Big]^{1/n}. \end{aligned} \tag{32}$$

Taking the lim $m \to \infty$, we get

$$\begin{aligned} \lim_{m\to\infty} F_{Ap^{(2)},T(p^{(2)},\ldots,p^{(n)},p^{(1)})}(t) &= 1, \\ &\vdots \end{aligned} \tag{33}$$

Similarly we have

$$\begin{aligned} &F_{T(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)},),T(p^{(n)},p^{(1)},\ldots,p^{(n-1)})}(t) \\ &\quad \geq F_{T(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)},),T(p^{(n)},p^{(1)},\ldots,p^{(n-1)})}(\psi(t)) \\ &\quad \geq \left[F_{Ax_m^{(n)},A(p(n))}(t), F_{Ax_m^{(1)},Ap(1)}(t), \ldots, \right. \\ &\quad \left. F_{Ax_m^{(n-1)},Ap(n-1)}(t)\right]^{1/n}. \end{aligned} \tag{34}$$

Taking the lim $m \to \infty$, we get

$$\lim_{m\to\infty} F_{Ap^{(n)},T(p^{(n)},p^{(1)},\ldots,p^{(n-1)})}(t) = 1. \tag{35}$$

So, we have

$$\begin{aligned} T\left(p^{(1)}, p^{(2)}, \ldots, p^{(n)}\right) &= A\left(p^{(1)}\right) = x^{(1)}, \\ T\left(p^{(2)}, \ldots, p^{(n)}, p^{(1)}\right) &= A\left(p^{(2)}\right) = x^{(2)}, \\ &\vdots \\ T\left(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)}\right) &= A\left(p^{(n)}\right) = x^{(n)}. \end{aligned} \tag{36}$$

Now we suppose that T and A are weakly compatible maps, so (36) implies that

$$\begin{aligned} &A\left(T\left(p^{(1)}, p^{(2)}, \ldots, p^{(n)}\right)\right) \\ &\quad = T\left(A\left(p^{(1)}\right), A\left(p^{(2)}\right), \ldots, A\left(p^{(n)}\right)\right) \Longrightarrow \\ &A\left(x^{(1)}\right) = T\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right), \\ &A\left(T\left(p^{(2)}, \ldots, p^{(n)}, p^{(1)}\right)\right) \\ &\quad = T\left(A\left(p^{(2)}\right), \ldots, A\left(p^{(n)}\right), A\left(p^{(1)}\right)\right) \Longrightarrow \\ &A\left(x^{(2)}\right) = T\left(x^{(2)}, \ldots, x^{(n)}, x^{(1)}\right), \\ &\qquad \vdots \\ &A\left(T\left(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)}\right)\right) \\ &\quad = T\left(A\left(p^{(n)}\right), A\left(p^{(1)}\right), \ldots, A\left(p^{(n-1)}\right)\right) \Longrightarrow \\ &A\left(x^{(n)}\right) = T\left(x^{(n)}, x^{(1)}, \ldots, x^{(n-1)}\right). \end{aligned} \tag{37}$$

Hence T and A have n-tuple coincidence point. □

By replacing inequality (16) in Theorem 16 by (38), we have the following theorem.

Theorem 17. *Let (X, F, Δ) be a Menger PM-space such that Δ is a t-norm of H-type. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$, $\psi(t) > t$, and $\lim_{m\to\infty}\psi^m(t) = \infty$ for any $t > 0$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned} &F_{T(x^{(1)},x^{(2)},\ldots,x^{(n)}),T(y^{(1)},y^{(2)},\ldots,y^{(n)})}(t) \\ &\quad \geq \min\left[F_{Ax^{(1)},Ay^{(1)}}\psi(t), F_{Ax^{(2)},Ay^{(2)}}\psi(t), \ldots, \right. \\ &\quad \left. F_{Ax^{(n)},Ay^{(n)}}\psi(t)\right] \end{aligned} \tag{38}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A, T) is weakly compatible.*

Then there exists $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

Proof. Suppose $t > 0$. From (38), we have

$$\begin{aligned} &F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) = F_{T(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)}),T(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})}(t) \\ &\quad \geq \min\left[F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}\psi(t), F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}\psi(t), \ldots, \right. \\ &\quad \left. F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}\psi(t)\right], \\ &F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t) = F_{T(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)}),T(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)})}(t) \\ &\quad \geq \min\left[F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}\psi(t), \ldots, F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}\psi(t), \right. \\ &\quad \left. F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}\psi(t)\right], \\ &\qquad \vdots \\ &F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t) = F_{T(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}),T(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})}(t) \\ &\quad \geq \min\left[F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}\psi(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}\psi(t), \ldots, \right. \\ &\quad \left. F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}\psi(t)\right]. \end{aligned} \tag{39}$$

Suppose that $E_m(t) = \min\{F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t), \ldots, F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)\}$. Then from the above inequalities, we obtain

$$E_{m+1}(t) \geq E_{m+1}\psi(t). \tag{40}$$

This implies that

$$\begin{aligned} E_{m+1}(t) &\geq E_m(\psi(t)) \geq E_{m-1}\psi^2(t) \geq \cdots \\ &\geq E_1(\psi^m(t)). \end{aligned} \tag{41}$$

Since

$$\begin{aligned} &\lim_{t\to\infty} E_1(t) \\ &\quad = \lim_{t\to\infty} \min\left\{F_{Ax_0^{(1)},Ax_1^{(1)}}, F_{Ax_0^{(2)},Ax_1^{(2)}}, \ldots, F_{Ax_0^{(n)},Ax_1^{(n)}}\right\} \\ &\quad = 1 \end{aligned} \tag{42}$$

and $\lim_{m\to\infty}\psi^m(t) = \infty$ for any $t > 0$, we have $\lim_{m\to\infty}E_1(\psi^m(t)) = 1$.

Moreover, we have

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) &\geq E_1(\psi^m(t)), \\ F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t) &\geq E_1(\psi^m(t)), \\ &\vdots \\ F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t) &\geq E_1(\psi^m(t)). \end{aligned} \tag{43}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) &= 1, \\ \lim_{m\to\infty} F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t) &= 1, \\ &\vdots \\ \lim_{m\to\infty} F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t) &= 1. \end{aligned} \tag{44}$$

This implies that

$$\lim_{n\to\infty} E_m(t) = 1 \quad \forall t > 0. \tag{45}$$

In the next step we show that, for any $k \in Z^+$,

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(t) &\geq \Delta^k[E_m(\psi(t)-t)], \\ F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(t) &\geq \Delta^k[E_m(\psi(t)-t)], \\ &\vdots \\ F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(t) &\geq \Delta^k[E_m(\psi(t)-t)]. \end{aligned} \tag{46}$$

We will prove the above by induction method; for $k = 1$, it is obvious. Assume that (46) holds for some k. Since $\psi(t) > t$, we have

$$F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) \geq E_m(\psi(t)) \geq E_m(t). \tag{47}$$

Now we have

$$\begin{aligned} F_{Ax_{m+1}^{(1)},Ax_{m+k+1}^{(1)}}(t) &\geq \min\Big\{F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}\psi(t), \\ &\quad F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}\psi(t), \ldots, F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}\psi(t),\Big\} \\ &\geq \Delta^k[E_m(\psi(t)-t)]. \end{aligned} \tag{48}$$

Thus, by the monotonicity of Δ, we have

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k+1}^{(1)}}\psi(t) &= F_{Ax_m^{(1)},Ax_{m+k+1}^{(1)}}(\psi(t)-t+t) \\ &\geq \Delta\left[F_{Ax_{m+1}^{(1)},Ax_{m+1}^{(1)}}(\psi(t)-t), F_{Ax_{m+1}^{(1)},Ax_{m+k+1}^{(1)}}(t)\right] \\ &\geq \Delta\left[\Delta E_m(\psi(t)-t), \Delta^k E_m(\psi(t)-t)\right] \\ &\geq \Delta\left[E_m(\psi(t)-t), \Delta^k E_m(\psi(t)-t)\right] \\ &= \Delta^{k+1} E_m(\psi(t)-t). \end{aligned} \tag{49}$$

Similarly we get

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k+1}^{(1)}}\psi(t) &\geq \Delta^{k+1} E_m(\psi(t)-t), \\ F_{Ax_{m+1}^{(2)},Ax_{m+k+1}^{(2)}}\psi(t) &\geq \Delta^{k+1} E_m(\psi(t)-t), \\ &\vdots \\ F_{Ax_{m+1}^{(n)},Ax_{m+k+1}^{(n)}}\psi(t) &\geq \Delta^{k+1} E_m(\psi(t)-t). \end{aligned} \tag{50}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0, 1]$ are given. Since Δ is t-norm of H-type, there exist $\delta > 0$ such that

$$\Delta^k(s) > 1-\lambda \quad \forall s \in (1-\delta, 1], \ k \in Z^+. \tag{51}$$

By (45), there exist $N \in Z^+$ such that $E_m(\psi(\epsilon)-\epsilon) > 1-\delta$ for all $n \geq N$. Hence from (50) and (51), we get

$$\begin{aligned} F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(\epsilon) &> 1-\lambda, \\ F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(\epsilon) &> 1-\lambda, \\ &\vdots \\ F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(\epsilon) &> 1-\lambda, \end{aligned} \tag{52}$$

for all $n \geq N$ and $k \in Z^+$.

Therefore $\{Ax_m^{(1)}\}, \{Ax_m^{(2)}\}, \ldots, \{Ax_m^{(n)}\}$ are all Cauchy sequence.

Arguing as in Theorem 16, we have T and A having n-tuple coincidence point. □

Again replacing inequality (38) by (53), we have the following result.

Theorem 18. *Let (X, F, Δ) be a Menger PM-space such that Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \psi^m(t) < +\infty$ for any $t > 0$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned} &F_{T(x^{(1)},x^{(2)},x^{(3)},\ldots,x^{(n)}),T(y^{(1)},y^{(2)},y^{(3)},\ldots,y^{(n)})}(\psi(t)) \\ &\quad \geq \Big[\Delta\Big(F_{Ax^{(1)},Ay^{(1)}}(t), F_{Ax^{(2)},Ay^{(2)}}(t), \ldots, \\ &\quad F_{Ax^{(n)},Ay^{(n)}}(t)\Big)\Big]^{1/n} \end{aligned} \tag{53}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A, T) is weakly compatible.*

Then there exists $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

Proof. Suppose $t > 0$. From (53), we have

$$\begin{aligned}
&F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(1)},x_{m-1}^{(2)},\dots,x_{m-1}^{(n)}),T(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)})}(\psi(t)) \\
&\quad \geq \Big[\Delta\Big(F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t), F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t),\dots, \\
&\qquad F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t)\Big)\Big]^{1/n}, \\
&F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(2)},\dots,x_{m-1}^{(n)},x_{m-1}^{(1)}),T(x_m^{(2)},\dots,x_m^{(n)},x_m^{(1)})}(\psi(t)) \\
&\quad \geq \Big[\Delta\Big(F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t),\dots,F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), \\
&\qquad F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t)\Big)\Big]^{1/n}, \\
&\qquad\qquad \vdots \\
&F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(n)},x_{m-1}^{(1)},\dots,x_{m-1}^{(n-1)}),T(x_m^{(n)},x_m^{(1)},\dots,x_m^{(n-1)})}(\psi(t)) \\
&\quad \geq \Big[\Delta\Big(F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t),\dots, \\
&\qquad F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)\Big)\Big]^{1/n}.
\end{aligned} \tag{54}$$

Suppose that $G_m(t) = [\Delta(F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t), \dots, F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t))]^{1/n}$. Then, operating by t-norm Δ on the above equations, from the condition $\Delta \geq \Delta_p$, we obtain

$$\begin{aligned}
G_{m+1}(\psi(t)) &= \Big[\Delta\Big(F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)), \\
&\quad F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi(t)),\dots,F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi(t))\Big)\Big]^{1/n} \\
&\geq [\Delta(G_m(t), G_m(t), G_m(t),\dots,G_m(t))]^{1/n} \\
&\geq [G_m(t)\cdot G_m(t)\cdot G_m(t)\cdots G_m(t)]^{1/n} \geq G_m(t).
\end{aligned} \tag{55}$$

This implies that

$$\begin{aligned}
&F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\
&F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\
&\qquad\qquad \vdots \\
&F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t).
\end{aligned} \tag{56}$$

In the next step we show that $\{Ax_m^{(1)}\}$ is Cauchy sequence. For each $\lambda \in (0,1]$, suppose that $D_\lambda + 1 = \inf\{t > 0; G_1(t) > 1-\lambda\}$. Then, $G_1(D_\lambda) > 1-\lambda$. From (56) we see that $F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi^m(D_\lambda+1)) > 1-\lambda$. By Lemma 5, we have

$$d_\lambda\left(Ax_m^{(1)}, Ax_{m+1}^{(1)}\right) < \psi^n(D_\lambda + 1), \quad \text{for each } \lambda \in (0,1]. \tag{57}$$

By Lemma 6, for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that

$$d_\lambda\left(Ax_m^{(1)}, Ax_q^{(1)}\right) \leq \sum_{i=m}^{q-1} d_\mu\left(Ax_i^{(1)}, Ax_{i+1}^{(1)}\right) \quad \text{for each } m,q \in Z^+ \text{ with } q > m. \tag{58}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0,1]$ are given. Since $\sum_{m=1}^{\infty}\psi^n(D_\mu+1) < +\infty$, there exists $N \in Z^+$ such that $\sum_{i=m}^{q-1}\psi^i(D_\mu+1) < \epsilon$ for all $q > m \geq N$. Thus, by (57) and (58), we have $d_\lambda(Ax_m^{(1)}, Ax_q^{(1)}) < \epsilon$. Using Lemma 5, we obtain that $F_{Ax_m^{(1)},Ax_q^{(1)}} > 1-\lambda$ for all $p > m \geq N$; that is, $\{Ax_m^{(1)}\}$ is a Cauchy sequence. Similarly, we can show that $\{Ax_m^{(2)}\},\dots,\{Ax_m^{(n)}\}$ are also Cauchy sequence.

In a similar manner of Theorem 16, we can find T and A having n-tuple coincidence point. □

3. n-Tupled Coincidence Point Results in Non-Archimedean Menger Spaces

In this section, we are going to prove two coincidence point theorems in non-Archimedean Menger space.

Theorem 19. *Let (X,F,Δ) be a non-Archimedean Menger PM-space such that $\sup_{0<a<1}\Delta(a,a) = 1$ and $\Delta \geq \Delta_p$. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{m\to\infty}\psi^m(t) = +\infty$ for any $t > 0$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned}
&F_{T(x^{(1)},x^{(2)},\dots,x^{(n)}),T(y^{(1)},y^{(2)},\dots,y^{(n)})}(t) \\
&\quad \geq \Big[\Delta\Big(F_{Ax^{(1)},Ay^{(1)}}(\psi(t)), F_{Ax^{(2)},Ay^{(2)}}(\psi(t)),\dots, \\
&\qquad F_{Ax^{(n)},Ay^{(n)}}(\psi(t))\Big)\Big]^{1/n}
\end{aligned} \tag{59}$$

for all $x^{(1)}, x^{(2)}, x^{(3)},\dots,x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)},\dots,y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A,T) is weakly compatible.*

If there exist $z_1, z_2, \ldots, z_n \in X$ such that for any $t > 0$

$$\begin{gathered}\lim_{m\to\infty}\prod_{i=m}^{\infty} F_{Az^{(1)},T(z^{(1)},z^{(2)},z^{(3)},\ldots,z^{(n)})}\left(\psi^i(t)\right) = 1,\\ \lim_{m\to\infty}\prod_{i=m}^{\infty} F_{Az^{(2)},T(z^{(2)},z^{(3)},\ldots,z^{(n)},z^{(1)})}\left(\psi^i(t)\right) = 1,\\ \vdots\\ \lim_{m\to\infty}\prod_{i=m}^{\infty} F_{Az^{(n)},T(z^{(n)},z^{(1)},z^{(2)},\ldots,z^{(n-1)})}\left(\psi^i(t)\right) = 1,\end{gathered} \tag{60}$$

then there exists $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

Proof. From (59), we have supposed that $t > 0$,

$$\begin{aligned}F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) &= F_{T(x_{m-1}^{(1)},x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)}),T(x_m^{(1)},x_m^{(2)},\ldots,x_m^{(n)})}(t)\\ &\geq \Big[\Delta\Big(F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(\psi(t)), F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(\psi(t)),\ldots,\\ &\quad F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(\psi(t))\Big)\Big]^{1/n},\\ F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t) &= F_{T(x_{m-1}^{(2)},\ldots,x_{m-1}^{(n)},x_{m-1}^{(1)}),T(x_m^{(2)},\ldots,x_m^{(n)},x_m^{(1)})}(t)\\ &\geq \Big[\Delta\Big(F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(\psi(t)),\ldots,F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(\psi(t)),\\ &\quad F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(\psi(t))\Big)\Big]^{1/n},\\ &\vdots\\ F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t) &= F_{T(x_{m-1}^{(n)},x_{m-1}^{(1)},\ldots,x_{m-1}^{(n-1)}),T(x_m^{(n)},x_m^{(1)},\ldots,x_m^{(n-1)})}(t)\\ &\geq \Big[\Delta\Big(F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(\psi(t)), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(\psi(t)),\ldots,\\ &\quad F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(\psi(t))\Big)\Big]^{1/n}.\end{aligned} \tag{61}$$

Suppose that $G_m(t) = [\Delta(F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t), \ldots, F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t))]^{1/n}$. Then operating t-norm Δ above with $\Delta \geq \Delta_p$, we have

$$\begin{aligned}G_{m+1}(t) &\geq \Big[\Delta\Big(F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t), F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t),\ldots,\\ &\quad F_{Ax_m^{(n-1)},Ax_{m+1}^{(n-1)}}(t)\Big)\Big]^{1/n} \geq [\Delta(G_m(\psi(t)), G_m(\psi(t)),\\ &\quad \ldots, G_m(\psi(t)))]^{1/n} \geq [G_m(\psi(t))\, G_m(\psi(t))\\ &\quad \cdots G_m(\psi(t))]^{1/n} = G_m(\psi(t)).\end{aligned} \tag{62}$$

This implies that

$$\begin{aligned}G_{m+1}(t) &\geq G_m(\psi(t)) \geq G_{m-1}(\psi^2(t)) \geq \cdots\\ &\geq G_1(\psi^m(t)).\end{aligned} \tag{63}$$

Thus, we have

$$\begin{aligned}F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(t) &\geq G_1(\psi^m(t)), F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(t)\\ &\geq G_1(\psi^m(t)),\ldots,F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(t)\\ &\geq G_1(\psi^m(t)).\end{aligned} \tag{64}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0,1]$ are given. By (60), there exists $N \in Z^+$ such that

$$\begin{gathered}\prod_{i=m}^{m+k} F_{Ax_0^{(1)},Ax_1^{(1)}}\left(\psi^i(\epsilon)\right) > 1-\lambda,\\ \prod_{i=m}^{m+k} F_{Ax_0^{(2)},Ax_1^{(2)}}\left(\psi^i(\epsilon)\right) > 1-\lambda,\\ \vdots\\ \prod_{i=m}^{m+k} F_{Ax_0^{(n)},Ax_1^{(n)}}\left(\psi^i(\epsilon)\right) > 1-\lambda,\end{gathered} \tag{65}$$

for all $n \geq N$ and $k \in Z^+$. Hence, from (64) it follows that

$$\begin{aligned}F_{Ax_m^{(1)},Ax_{m+k}^{(1)}}(\epsilon) &\geq \Delta^k\Big(F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\epsilon), F_{Ax_{m+1}^{(1)},Ax_{m+2}^{(1)}}(\epsilon),\\ &\quad \ldots, F_{Ax_{m+k-1}^{(1)},Ax_{m+k}^{(1)}}(\epsilon)\Big) \geq \Delta^k\Big(G_1(\psi^m(\epsilon)),\\ &\quad G_1(\psi^{m+1}(\epsilon)),\ldots,G_1(\psi^{m+k}(\epsilon))\Big) \geq \prod_{i=m}^{m+k} G_1(\psi^i(\epsilon))\\ &\geq \prod_{i=m}^{m+k}\Big[\Delta\Big(F_{Ax_0^{(1)},Ax_1^{(1)}}(\psi^i(\epsilon)), F_{Ax_0^{(2)},Ax_1^{(2)}}(\psi^i(\epsilon)),\\ &\quad \ldots, F_{Ax_0^{(n)},Ax_1^{(n)}}(\psi^i(\epsilon))\Big)\Big]^{1/n} > 1-\lambda.\end{aligned} \tag{66}$$

This shows that $\{Ax_m^{(1)}\}$ is a Cauchy sequence. Similarly we get the following:

$$\begin{gathered}F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(\epsilon) > 1-\lambda,\\ F_{Ax_m^{(2)},Ax_{m+k}^{(2)}}(\epsilon) > 1-\lambda,\\ \vdots\\ F_{Ax_m^{(n)},Ax_{m+k}^{(n)}}(\epsilon) > 1-\lambda.\end{gathered} \tag{67}$$

Hence $\{Ax_m^{(2)}\}, \{Ax_m^{(3)}\}, \ldots, \{Ax_m^{(n)}\}$ are all Cauchy sequence. Since $A(X)$ is complete, there exist $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in A(X)$ such that $\lim_{n\to\infty} Ax_{m+1}^{(1)} = x^{(1)}$, $\lim_{n\to\infty} Ax_{m+1}^{(2)} = x^{(2)}, \ldots, \lim_{n\to\infty} Ax_{m+1}^{(n)} = x^{(n)}$. Again $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in A(X)$ implies the existence of $p^{(1)}, p^{(2)}, \ldots, p^{(n)} \in X$ so that

$$A\left(p^{(1)}\right) = x^{(1)}, A\left(p^{(2)}\right) = x^{(2)}, \ldots, A\left(p^{(n)}\right) = x^{(n)}. \tag{68}$$

Hence

$$\begin{aligned}
&\lim_{m\to\infty} T\left(x_m^{(1)}, x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}\right) = \lim_{m\to\infty} Ax_{m+1}^{(1)} \\
&\quad = A\left(p^{(1)}\right) = x^{(1)}, \\
&\lim_{m\to\infty} T\left(x_m^{(2)}, x_m^{(3)}, \ldots, x_m^{(n)}, x_m^{(1)}\right) = \lim_{m\to\infty} Ax_{m+1}^{(2)} \\
&\quad = A\left(p^{(2)}\right) = x^{(2)}, \\
&\qquad \vdots \\
&\lim_{m\to\infty} T\left(x_m^{(n)}, \ldots, x_m^{(n-2)}, x_m^{(n-1)}\right) = \lim_{m\to\infty} Ax_{m+1}^{(n)} \\
&\quad = A\left(p^{(n)}\right) = x^{(n)}.
\end{aligned} \tag{69}$$

We have

$$\begin{aligned}
&F_{T(x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(2)},), T(p^{(1)}, p^{(2)}, \ldots, p^{(n)})}(t) \\
&\quad \geq \Delta\left[F_{Ax_m^{(1)}, Ap^{(1)}}(\psi(t)), F_{Ax_m^{(2)}, Ap^{(2)}}(\psi(t)), \ldots,\right. \\
&\quad \left. F_{Ax_m^{(n)}, Ap^{(n)}}(\psi(t))\right]^{1/n}.
\end{aligned} \tag{70}$$

Taking the lim $m \to \infty$, we get

$$\lim_{m\to\infty} F_{Ap^{(1)}, T(p^{(1)}, p^{(2)}, \ldots, p^{(n)})}(t) = 1. \tag{71}$$

Now again, we have

$$\begin{aligned}
&F_{T(x_m^{(2)}, \ldots, x_m^{(n)}, x_m^{(1)},), T(p^{(2)}, \ldots, p^{(n)}, p^{(1)})}(t) \\
&\quad \geq \Delta\left[F_{Ax_m^{(2)}, Ap^{(2)}}(\psi(t)), \ldots, F_{Ax_m^{(n)}, Ap^{(n)}}(\psi(t)),\right. \\
&\quad \left. F_{Ax_m^{(1)}, Ap^{(1)}}(\psi(t))\right]^{1/n}.
\end{aligned} \tag{72}$$

Taking lim $m \to \infty$, we get

$$\begin{aligned}
&\lim_{m\to\infty} F_{Ap^{(2)}, T(p^{(2)}, \ldots, p^{(n)}, p^{(1)})}(t) = 1, \\
&\qquad \vdots
\end{aligned} \tag{73}$$

Similarly, we have

$$\begin{aligned}
&F_{T(x_m^{(n)}, x_m^{(1)}, \ldots, x_m^{(n-1)},), T(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)})}(t) \\
&\quad \geq \Delta\left[F_{Ax_m^{(n)}, Ap^{(n)}}\psi(t), F_{Ax_m^{(1)}, Ap^{(1)}}\psi(t), \ldots,\right. \\
&\quad \left. F_{Ax_m^{(n-1)}, Ap^{(n-1)}}\psi(t)\right]^{1/n}.
\end{aligned} \tag{74}$$

Taking the lim $m \to \infty$, we get

$$\lim_{m\to\infty} F_{Ap^{(n)}, T(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)})}(t) = 1. \tag{75}$$

The above implies that

$$\begin{aligned}
&T\left(p^{(1)}, p^{(2)}, \ldots, p^{(n)}\right) = A\left(p^{(1)}\right) = x^{(1)}, \\
&T\left(p^{(2)}, \ldots, p^{(n)}, p^{(1)}\right) = A\left(p^{(2)}\right) = x^{(2)}, \\
&\qquad \vdots \\
&T\left(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)}\right) = A\left(p^{(n)}\right) = x^{(n)},
\end{aligned} \tag{76}$$

but T and A are weakly compatible, so that (76) implies that

$$\begin{aligned}
&A\left(T\left(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)}\right)\right) \\
&\quad = T\left(A\left(p^{(1)}\right), A\left(p^{(2)}\right), \ldots, A\left(p^{(n)}\right)\right) \Longrightarrow \\
&A\left(x^{(1)}\right) = T\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right), \\
&A\left(T\left(p^{(2)}, \ldots, p^{(n)}, p^{(1)}\right)\right) \\
&\quad = T\left(A\left(p^{(2)}\right), \ldots, A\left(p^{(n)}\right), A\left(p^{(1)}\right)\right) \Longrightarrow \\
&A\left(x^{(2)}\right) = T\left(x^{(2)}, \ldots, x^{(n)}, x^{(1)}\right), \\
&\qquad \vdots \\
&A\left(T\left(p^{(n)}, p^{(1)}, \ldots, p^{(n-1)}\right)\right) \\
&\quad = T\left(A\left(p^{(n)}\right), A\left(p^{(1)}\right), \ldots, A\left(p^{(n-1)}\right)\right) \Longrightarrow \\
&A\left(x^{(n)}\right) = T\left(x^{(n)}, x^{(1)}, \ldots, x^{(n-1)}\right).
\end{aligned} \tag{77}$$

Hence T and A have n-tuple coincidence point. □

In the following theorem we are going to replace inequality (59) by (78).

Theorem 20. *Let (X, F, Δ) be a non-Archimedean Menger PM-space such that Δ is a t-norm of H-type. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{m\to\infty}\psi^m(t) = 0$ for any $t > 0$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned}
&F_{T(x^{(1)}, x^{(2)}, \ldots, x^{(n)}), T(y^{(1)}, y^{(2)}, \ldots, y^{(n)})}(\psi(t)) \\
&\quad \geq \min\left[F_{Ax^{(1)}, Ay^{(1)}}(t), F_{Ax^{(2)}, Ay^{(2)}}(t), \ldots,\right. \\
&\quad \left. F_{Ax^{(n)}, Ay^{(n)}}(t)\right]
\end{aligned} \tag{78}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A, T) is weakly compatible.*

Then there exists $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

Proof. Suppose that $G_m(t) = \min[F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t),\dots,F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)]$. From (78), we have

$$\begin{aligned}
&F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(1)},x_{m-1}^{(2)},\dots,x_{m-1}^{(n)}),T(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)})}(\psi(t)) \\
&\quad \geq \min\left[F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t), F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t),\dots,\right. \\
&\quad \left. F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t)\right] = G_m(t), \\
&F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(2)},\dots,x_{m-1}^{(n)},x_{m-1}^{(1)}),T(x_m^{(2)},\dots,x_m^{(n)},x_m^{(1)})}(\psi(t)) \\
&\quad \geq \min\left[F_{Ax_{m-1}^{(2)},Ax_m^{(2)}}(t),\dots,F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t),\right. \\
&\quad \left. F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t)\right] = G_m(t), \\
&\qquad\qquad \vdots \\
&F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi(t)) \\
&\quad = F_{T(x_{m-1}^{(n)},x_{m-1}^{(1)},\dots,x_{m-1}^{(n-1)}),T(x_m^{(n)},x_m^{(1)},\dots,x_m^{(n-1)})}(\psi(t)) \\
&\quad \geq \min\left[F_{Ax_{m-1}^{(n)},Ax_m^{(n)}}(t), F_{Ax_{m-1}^{(1)},Ax_m^{(1)}}(t),\dots,\right. \\
&\quad \left. F_{Ax_{m-1}^{(n-1)},Ax_m^{(n-1)}}(t)\right] = G_m(t).
\end{aligned} \tag{79}$$

Then, from the above it follows that $G_{m+1}(\psi(t)) \geq G_m(t)$. we have

$$\begin{aligned}
&F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\
&F_{Ax_m^{(2)},Ax_{m+1}^{(2)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t), \\
&\qquad\qquad \vdots \\
&F_{Ax_m^{(n)},Ax_{m+1}^{(n)}}(\psi^m(t)) \geq G_m(\psi^{m-1}(t)) \geq \cdots \geq G_1(t).
\end{aligned} \tag{80}$$

In the next step, we will show that $\{Ax_m^{(1)}\}, \{Ax_m^{(2)}\},\dots,\{Ax_m^{(n)}\}$ are Cauchy sequences. For this $\lambda \in (0,1]$, suppose that $B_\lambda = \inf\{t > 0 : E_1(t) > 1-\lambda\}$. Then, $E_1(B_\lambda+1) > 1-\lambda$. From (80) we see that $F_{Ax_m^{(1)},Ax_{m+1}^{(1)}}(\psi^m(B_\lambda+1)) > 1-\lambda$. It follows from Lemma 5 that

$$d_\lambda\left(Ax_m^{(1)}, Ax_{m+1}^{(1)}\right) < \psi^m(B_\lambda+1), \quad \text{for each } \lambda \in (0,1]. \tag{81}$$

Using Lemma 7 we obtain that, for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that

$$d_\lambda\left(Ax_m^{(1)}, Ax_{m+1}^{(1)}\right) \leq \max_{m\leq i\leq q-1} d_\mu(Ax_i, Ax_{i+1}), \quad \forall m,q \in Z^+ \text{ with } q > m. \tag{82}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0,1]$ are given. Since $\lim_{m\to\infty}\psi^m(B_\lambda+1) = 0$, there exists $N \in Z^+$ such that $\psi^m(B_\lambda+1) < \epsilon$ for all $q > m \geq N$. Thus, by (81) and (82), we have $d_\lambda(Ax_m^{(1)}, Ax_q^{(1)}) < \epsilon$. Furthermore, by Lemma 5 we have $F_{Ax_m^{(1)},Ax_q^{(1)}} > 1-\lambda$ for all $q > m \geq N$; that is, $Ax_m^{(1)}$ is a Cauchy sequence. Similarly, we can show that $\{Ax_m^{(2)}\},\dots,\{Ax_m^{(n)}\}$ are all Cauchy sequence.

In a similar manner of Theorem 19 we can find T and A having n-tuple coincidence point. □

4. Corollaries and Examples

Now we are ready to deduce some of the main theorems to obtain the following corollaries.

In this section, we will obtain some corollaries and an example of our main results proven in Sections 2 and 3.

By putting $A = I_x$ (identity map on X) in Theorem 16 and as an immediate consequence, we have the following.

Corollary 21. *Let (X,F,Δ) be a Menger PM-space such that Δ is a t-norm of H-type. Let $\psi : R^+ \to R^+$ be a gauge function such that $\psi^{-1}(\{0\}) = \{0\}$, $\psi(t) < t$, and $\lim_{m\to\infty}\psi^m(t) = 0$ for any $t > 0$. Let $T : X^n \to X$ be a mapping such that*

$$\begin{aligned}
&F_{T(x^{(1)},x^{(2)},x^{(3)},\dots,x^{(n)}),T(y^{(1)},y^{(2)},y^{(3)},\dots,y^{(n)})}(\psi(t)) \\
&\quad \geq \left[F_{x^{(1)},y^{(1)}}(t)\, F_{x^{(2)},y^{(2)}}(t) \cdots F_{x^{(n)},y^{(n)}}(t)\right]^{1/n}
\end{aligned} \tag{83}$$

for all $x^{(1)}, x^{(2)}, x^{(3)},\dots,x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)},\dots,y^{(n)} \in X$, and $t > 0$. Then there exists $x^{(1)}, x^{(2)},\dots,x^{(n)} \in X$ such that T has n-tupled fixed point in X.

Since each hybrid contraction with a gauge function ψ includes the case of linear contraction as a special case, if we set $\psi(t) = \alpha t$ or $\psi(t) = t/\alpha$, $\alpha \in (0,1)$, in theorems of Section 2, then we have the following n-tupled coincidence points for the mappings T and A corollaries as follows.

Corollary 22. *Let (X,F,Δ) be a Menger PM-space such that Δ is a t-norm of H-type and $\alpha \in (0,1)$. Let $T : X^n \to X$ and $A : X \to X$ be two mappings such that*

$$\begin{aligned}
&F_{T(x^{(1)},x^{(2)},x^{(3)},\dots,x^{(n)}),T(y^{(1)},y^{(2)},y^{(3)},\dots,y^{(n)})}(\alpha t) \\
&\quad \geq \left[F_{Ax^{(1)},Ay^{(1)}}(t)\, F_{Ax^{(2)},Ay^{(2)}}(t) \cdots F_{Ax^{(n)},Ay^{(n)}}(t)\right]^{1/n}
\end{aligned} \tag{84}$$

for all $x^{(1)}, x^{(2)}, x^{(3)},\dots,x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)},\dots,y^{(n)} \in X$, and $t > 0$, where

(1) *$T(X^n) \subset A(X)$;*

(2) *$A(X)$ is complete;*

(3) *the pair (A,T) is weakly compatible.*

Then there exists $x^{(1)}, x^{(2)}, x^{(3)},\dots,x^{(n)} \in X$ such that A and T have n-tupled coincidence point in X.

If we take the mapping A as the identity mapping on X in Corollary 22, we get the following n-tupled fixed point theorems for the mapping T.

Corollary 23. *Let (X, F, Δ) be a complete Menger PM-space such that Δ is a t-norm of H-type and $\alpha \in (0,1)$. Let $T : X^n \to X$ be a mapping such that*

$$F_{T(x^{(1)},x^{(2)},x^{(3)},\dots,x^{(n)}),T(y^{(1)},y^{(2)},y^{(3)},\dots,y^{(n)})}(\alpha t) \geq \left[F_{x^{(1)},y^{(1)}}(t) F_{x^{(2)},y^{(2)}}(t) \cdots F_{x^{(n)},y^{(n)}}(t)\right]^{1/n} \tag{85}$$

for all $x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)}, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(n)} \in X$, and $t > 0$. Then there exists $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in X$ such that T has n-tupled coincidence point in X.

Now we are ready to give an illustrative example to support our main Theorem 16. One can see further example of this nature in [12, 14].

Example 24. Suppose that $X = [-1, 1] \subset R = \Delta_m$. Then Δ_m is a t-norm of H-type and $\Delta_m \geq \Delta_p$. Define $F : X \times X \to D^+$ by

$$F(x, y)(t) = F_{x,y}(t) = \begin{cases} e^{-|x-y|/t}: & \text{if } t \leq 0 \\ 0: & \text{if } t \leq 0. \end{cases} \tag{86}$$

We claim that (X, F, Δ_m) is a Menger PM-space. Conditions (1), (2), and (3) of Theorem 16 are very easy to check. To prove condition (4), we assume that $s, t > 0$ for all $x, y, z \in X$ and

$$\Delta_m\left(F_{x,z}(t), F_{z,y}(t)\right) = \min\left\{e^{-|x-z|/t}, e^{-|z-y|/t}\right\} = e^{-|x-z|/t}. \tag{87}$$

Then we have $t|z - y| \leq s|x - z|$, and so $((t + s)/t)|x - z| = |x - z| + (s/t)|x - z| \geq |x - z| + |z - y| \geq |x - y|$. It follows that

$$F_{x,z}(t + s) = e^{-|x-y|/(t+s)} \geq e^{-|x-z|/t} = \Delta_m\left(F_{x,z}(t), F_{z,y}(t)\right). \tag{88}$$

Hence condition (4) holds. It is clear that (X, F, Δ_m) is complete.

Suppose that $\psi(t) = t/n$.

For $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in X$, define $T : X^n \to X$ and $A : X \to X$ as follows:

$$T\left(x^{(1)}, x^{(2)}, \dots, x^{(n)}\right) = \frac{\left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \cdots + \left(x^{(n)}\right)^2}{4n^3}, \tag{89}$$

$$A(x) = \frac{x}{2}, \tag{90}$$

respectively.

Note that $(0, 0, 0, \dots, 0)$ is the point of coincidence of T and A. It is clear that the air (T, A) is weakly compatible on X.

Also we will show that the pair (T, A) is not compatible. Let us consider the sequences

$\{x_m^{(1)}\} = \{1/2 + 1/m\}$, $\{x_m^{(2)}\} = \{1/2 - 1/m\}$, $\{x_m^{(3)}\} = \{1/2 + 1/m\}, \dots, \{x_m^{(n)}\} = \{1/2 - 1/m\}$ (when n is even); then

$$AT\left(x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n)}\right) = \frac{1}{8n^3}\left(\frac{n}{4} + \frac{n}{m^2}\right),$$
$$T\left(A\left(x_m^{(1)}\right), A\left(x_m^{(2)}\right), \dots, A\left(x_m^{(n)}\right)\right) = \frac{1}{16n^3}\left(\frac{n}{4} + \frac{n}{m^2}\right). \tag{91}$$

Now

$$\begin{aligned} &\lim_{m\to\infty} F_{A(T(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)})),A(T(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)}))}(t) \\ &= \lim_{m\to\infty} F_{(1/8n^3)(n/4+n/m^2),1/16n^3 n/4+n/m^2}(t) \\ &= \lim_{m\to\infty} e^{-\frac{\left|\left(1/8n^3\right)\left(n/4+n/m^2\right)-\left(1/16n^3\right)\left(n/4+n/m^2\right)\right|}{t}} \\ &= e^{-\frac{\left|1/32n^2-1/64n^2\right|}{t}} = e^{-\left|\frac{1}{64n^2 t}\right|} \nrightarrow 1 \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{92}$$

and $\lim_{m\to\infty} T(x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n)}) = 1/16n^2$, $\lim_{m\to\infty} g(x_m^1) = 1/4$, $\lim_{m\to\infty} x_m^1 = 1/2$.

Hence $\lim_{m\to\infty} T(x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n)}) \neq \lim_{m\to\infty} g(x_m^1) \neq \lim_{m\to\infty} x_m^1$.

Similarly, we can check when n is odd. For this we can consider the sequence $\{x_m^{(1)}\} = \{1/2 + 1/m\}$, $\{x_m^{(2)}\} = \{1/2 - 1/m\}$, $\{x_m^{(3)}\} = \{1/2 + 1/m\}, \dots, \{x_m^{(n)}\} = \{1/2 + 1/m\}$ (when n is odd); then

$$\lim_{m\to\infty} F_{AT(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)}),AT(x_m^{(1)},x_m^{(2)},\dots,x_m^{(n)})}(t) = e^{-\left|\frac{1}{64n^2 t}\right|} \nrightarrow 1 \quad \text{as } m \longrightarrow \infty. \tag{93}$$

And $\lim_{m\to\infty} T(x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n)}) \neq \lim_{m\to\infty} g(x_m^1) \neq \lim_{m\to\infty} x_m^1$.

Hence pair (T, A) is not compatible. Then, for each $t > 0$ and for each $x^{(1)}, x^{(2)}, \dots, x^{(n)}, y^{(1)}, y^{(2)}, \dots, y^{(n)} \in X$, we have

$$\begin{aligned} &\left|\left(x^{(1)}\right)^2 - \left(y^{(1)}\right)^2 + \left(\left(x^{(2)}\right)^2 - \left(y^{(2)}\right)^2\right) + \cdots + \left(\left(x^{(n)}\right)^2 - \left(y^{(n)}\right)^2\right)\right| \leq \left|\left(\left(x^{(1)}\right)^2 - \left(y^{(1)}\right)^2\right)\right| + \cdots + \left|\left(\left(x^{(n)}\right)^2 - \left(y^{(n)}\right)^2\right)\right| \\ &\leq \left|\left(x^{(1)} - y^{(1)}\right)\right|\left(\left|x^{(1)}\right| + \left|y^{(1)}\right|\right) + \left|x^{(2)} - y^{(2)}\right|\left(\left|x^{(2)}\right| + \left|y^{(2)}\right|\right) + \cdots + \left|x^{(n)} - y^{(n)}\right|\left(\left|x^{(n)}\right| + \left|y^{(n)}\right|\right) \\ &\leq 2n \max\left(\left|\left(x^{(1)} - y^{(1)}\right)\right|, \left|x^{(2)} - y^{(2)}\right|, \dots, \left|x^{(n)} - y^{(n)}\right|\right) \end{aligned}$$

$$\left|\frac{\left(x^{(1)}\right)^2-\left(y^{(1)}\right)^2+\left(\left(x^{(2)}\right)^2-\left(y^{(2)}\right)^2\right)+\cdots+\left(\left(x^{(n)}\right)^2-\left(y^{(n)}\right)^2\right)}{4n^2t}\right|$$
$$\leq \frac{2n}{4n^2t}\max\left(\left|\left(x^{(1)}-y^{(1)}\right)\right|,\left|x^{(2)}-y^{(2)}\right|,\ldots,\left|x^{(n)}-y^{(n)}\right|\right)\leq \max\left(\frac{\left|\left(x^{(1)}-y^{(1)}\right)\right|}{2nt},\frac{\left|x^{(2)}-y^{(2)}\right|}{2nt},\ldots,\frac{\left|x^{(n)}-y^{(n)}\right|}{2nt}\right). \tag{94}$$

Now

$$\begin{aligned}
F_{T(x^{(1)},x^{(2)},\ldots,x^{(n)}),T(y^{(1)},y^{(2)},\ldots,y^{(n)})}\left(\psi(t)\right) &= F_{T(x^{(1)},x^{(2)},\ldots,x^{(n)}),T(y^{(1)},y^{(2)},\ldots,y^{(n)})}\left(\frac{t}{n}\right)\\
&= e^{-\left|\frac{T\left(x^{(1)},x^{(2)},\ldots,x^{(n)}\right)-T\left(y^{(1)},y^{(2)},\ldots,y^{(n)}\right)}{t/n}\right|}\\
&= e^{-\left|\frac{\left(x^{(1)}\right)^2-\left(y^{(1)}\right)^2+\left(\left(x^{(2)}\right)^2-\left(y^{(2)}\right)^2\right)+\cdots+\left(\left(x^{(n)}\right)^2-\left(y^{(n)}\right)^2\right)/4n^3}{t/n}\right|}\\
&= e^{-\left|\frac{\left(x^{(1)}\right)^2-\left(y^{(1)}\right)^2+\left(\left(x^{(2)}\right)^2-\left(y^{(2)}\right)^2\right)+\cdots+\left(\left(x^{(n)}\right)^2-\left(y^{(n)}\right)^2\right)}{4n^2t}\right|}\\
&\geq \min\left\{e^{-\frac{\left|x^{(1)}-y^{(1)}\right|}{2nt}},e^{-\frac{\left|x^{(2)}-y^{(2)}\right|}{2nt}},\ldots,e^{-\frac{\left|x^{(n)}-y^{(n)}\right|}{2nt}}\right\}\\
&= \min\left\{e^{-\frac{\left|x^{(1)}-y^{(1)}\right|}{2t}},e^{-\frac{\left|x^{(2)}-y^{(2)}\right|}{2t}},\ldots,e^{-\frac{\left|x^{(n)}-y^{(n)}\right|}{2t}}\right\}^{1/n}\\
&= \Delta_m\left\{F_{Ax_1,Ay_1}(t),F_{Ax_2,Ay_2}(t),\ldots,F_{Ax_n,Ay_n}(t)\right\}^{1/n}.
\end{aligned} \tag{95}$$

Hence all the conditions of Theorem 18 are satisfied and $(0,0,\ldots,0)$ is the point of coincidence of T and A.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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