Research Article

Strong h -Convexity and Separation Theorems

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Jensen inequality for strongly h -convex functions and a characterization of pairs of functions that can be separated by a strongly h -convex function are presented. As a consequence, a stability result of the Hyers-Ulam type is obtained.

1. Introduction

Let X be a normed space, let D be a convex subset of X , and let $c > 0$. A function $f : D \rightarrow \mathbb{R}$ is called *strongly convex with modulus c* (see, e.g., [1, 2]) if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2 \quad (1)$$

for all $x, y \in D$ and $t \in [0, 1]$. Recall also that the usual notion of convex functions corresponds to the case $c = 0$. Strongly convex functions, introduced by Polyak [3], play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, e.g., [2, 4–7] and the references therein).

In [8] Varošanec introduced the notion of h -convexity. Let $h : [0, 1] \rightarrow \mathbb{R}_+$ be a given function. A function $f : D \rightarrow \mathbb{R}$ is said to be h -convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (2)$$

for all $x, y \in D$ and $t \in [0, 1]$. This notion unifies and generalizes the known classes of convex functions, s -convex functions, Godunova-Levin functions, and P -functions, which are obtained by putting in (2) $h(t) = t$, $h(t) = t^s$ (where $s \in (0, 1)$), $h(t) = 1/t$ (with $h(0) = 0$), and $h(t) = 1$, respectively. Some properties of them can be found, for example, in [8–13].

Combining the above two ideas we say that a function $f : D \rightarrow \mathbb{R}$ is *strongly h -convex with modulus c* (cf. [14]) if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) - ct(1-t)\|x - y\|^2 \quad (3)$$

for all $x, y \in D$ and $t \in [0, 1]$.

In this note we present a Jensen-type inequality for such functions and give a characterization of pairs of functions that can be separated by a strongly h -convex one. Separation (or sandwich) theorems, that is, theorems providing conditions under which two given functions can be separated by a function from some special class, play an important role in many fields of mathematics and have various applications. In the literature one can find numerous results of this type (see, e.g., [12, 15–24]).

2. Jensen-Type Inequality

In the whole paper we assume that X is a real inner product space (i.e., the norm $\|\cdot\|$ in X is induced by an inner product: $\|x\| = \sqrt{\langle x | x \rangle}$). D is a convex nonempty subset of X and c is a positive constant.

A function $h : [0, 1] \rightarrow \mathbb{R}$ is said to be multiplicative if

$$h(st) = h(s)h(t), \quad s, t \in [0, 1]. \quad (4)$$

Note that if h is multiplicative, then it is nonnegative and either $h = 0$ or $h(1) = 1$. In what follows we assume that $h \neq 0$.

The following result is a counterpart of the classical Jensen inequality for strongly h -convex functions. It generalizes the Jensen-type inequality for strongly convex functions obtained in [4]. Similar results for h -convex functions are proved in [8, 12].

Theorem 1. Let $h : [0, 1] \rightarrow \mathbb{R}$ be a multiplicative function such that $h(t) \geq t$ for all $t \in [0, 1]$. If a function $f : D \rightarrow \mathbb{R}$ is strongly h -convex with modulus c , then

$$f\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n h(t_i) f(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2, \quad (5)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$ and $\bar{x} = t_1 x_1 + \dots + t_n x_n$.

Proof. For $n = 1$ inequality (5) is trivial and for $n = 2$ it follows from the definition of strong h -convexity (note that $t_1 \|x_1 - \bar{x}\|^2 + t_2 \|x_2 - \bar{x}\|^2 = t_1 t_2 \|x_1 - x_2\|^2$). Now, assuming (2) holds for some n , we will prove it for $n+1$. By the definition of strong h -convexity we get

$$\begin{aligned} & f\left(\sum_{i=1}^{n+1} t_i x_i\right) \\ &= f\left((1-t_{n+1}) \sum_{i=1}^n \frac{t_i}{1-t_{n+1}} x_i + t_{n+1} x_{n+1}\right) \\ &\leq h(1-t_{n+1}) f\left(\sum_{i=1}^n \frac{t_i}{1-t_{n+1}} x_i\right) \\ &\quad + h(t_{n+1}) f(x_{n+1}) \\ &\quad - c((1-t_{n+1}) \|s - \bar{x}\|^2 + t_{n+1} \|x_{n+1} - \bar{x}\|^2), \end{aligned} \quad (6)$$

where

$$\begin{aligned} s &= \sum_{i=1}^n \frac{t_i}{1-t_{n+1}} x_i, \\ \bar{x} &= (1-t_{n+1}) s + t_{n+1} x_{n+1} = \sum_{i=1}^{n+1} t_i x_i. \end{aligned} \quad (7)$$

By the inductive assumption we have

$$\begin{aligned} f\left(\sum_{i=1}^n \frac{t_i}{1-t_{n+1}} x_i\right) &\leq \sum_{i=1}^n h\left(\frac{t_i}{1-t_{n+1}}\right) f(x_i) \\ &\quad - c \sum_{i=1}^n \frac{t_i}{1-t_{n+1}} \|x_i - s\|^2. \end{aligned} \quad (8)$$

Now, using the above inequalities, the multiplicativity of h , and the assumption $h(t) \geq t$, we obtain

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} t_i x_i\right) &\leq \sum_{i=1}^{n+1} h(t_i) f(x_i) \\ &\quad - c \left(h(1-t_{n+1}) \sum_{i=1}^n \frac{t_i}{1-t_{n+1}} \|x_i - s\|^2 \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. + (1-t_{n+1}) \|s - \bar{x}\|^2 + t_{n+1} \|x_{n+1} - \bar{x}\|^2 \right) \\ &\leq \sum_{i=1}^{n+1} h(t_i) f(x_i) - c \left(\sum_{i=1}^n t_i \|x_i - s\|^2 \right. \\ &\quad \left. + (1-t_{n+1}) \|s - \bar{x}\|^2 + t_{n+1} \|x_{n+1} - \bar{x}\|^2 \right). \end{aligned} \quad (9)$$

To finish the proof it is enough to show that

$$\begin{aligned} & \sum_{i=1}^n t_i \|x_i - s\|^2 + (1-t_{n+1}) \|s - \bar{x}\|^2 + t_{n+1} \|x_{n+1} - \bar{x}\|^2 \\ &= \sum_{i=1}^{n+1} t_i \|x_i - \bar{x}\|^2, \end{aligned} \quad (10)$$

or, equivalently,

$$\begin{aligned} L &:= \sum_{i=1}^n t_i (\|x_i - s\|^2 - \|x_i - \bar{x}\|^2) \\ &= -(1-t_{n+1}) \|s - \bar{x}\|^2. \end{aligned} \quad (11)$$

Since

$$\begin{aligned} \|x_i - s\|^2 - \|x_i - \bar{x}\|^2 &= \langle (s - \bar{x}) | (s + \bar{x} - 2x_i) \rangle, \\ \bar{x} - t_{n+1} x_{n+1} &= (1-t_{n+1}) s, \end{aligned} \quad (12)$$

we have

$$\begin{aligned} L &= \sum_{i=1}^n \langle (s - \bar{x}) | (s + \bar{x} - 2x_i) \rangle \\ &= \left\langle (s - \bar{x}) \left| \sum_{i=1}^n t_i (s + \bar{x} - 2x_i) \right. \right\rangle \\ &= \langle (s - \bar{x}) | ((1-t_{n+1})(s + \bar{x}) - 2(\bar{x} - t_{n+1} x_{n+1})) \rangle \\ &= \langle (s - \bar{x}) | ((1-t_{n+1})(s + \bar{x}) - 2(1-t_{n+1})s) \rangle \\ &= \langle (s - \bar{x}) | (1-t_{n+1})(\bar{x} - s) \rangle \\ &= -(1-t_{n+1}) \|s - \bar{x}\|^2, \end{aligned} \quad (13)$$

which finishes the proof. $\square$

3. Separation by Strongly h -Convex Functions

It is proved in [15] that two functions $f, g : D \rightarrow \mathbb{R}$ defined on a convex subset D of a vector space can be separated by a convex function if and only if

$$f\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n t_i g(x_i), \quad (14)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$.

In this section we present counterparts of that result related to strong h -convexity.

Theorem 2. Let $f, g : D \rightarrow \mathbb{R}$ be given functions and $h : [0, 1] \rightarrow \mathbb{R}$ be a multiplicative function such that $h(t) \geq t$ for all $t \in [0, 1]$. If there exists a function $\varphi : D \rightarrow \mathbb{R}$ strongly h -convex with modulus c such that

$$f \leq \varphi \leq g \quad \text{on } D, \quad (15)$$

then

$$f\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n h(t_i) g(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2, \quad (16)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$ and $\bar{x} = t_1 x_1 + \dots + t_n x_n$.

Proof. By the Jensen inequality for strongly h -convex functions (Theorem 1) we have

$$\begin{aligned} f\left(\sum_{i=1}^n t_i x_i\right) &\leq \varphi\left(\sum_{i=1}^n t_i x_i\right) \\ &\leq \sum_{i=1}^n h(t_i) \varphi(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2 \\ &\leq \sum_{i=1}^n h(t_i) g(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2. \end{aligned} \quad (17)$$

□

Theorem 3. Let $f, g : D \rightarrow \mathbb{R}$ be given functions and $h : [0, 1] \rightarrow \mathbb{R}$ be a multiplicative function such that $h(t) \leq t$ for all $t \in [0, 1]$. If

$$f\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n h(t_i) g(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2, \quad (18)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$ and $\bar{x} = t_1 x_1 + \dots + t_n x_n$, then there exists a function $\varphi : D \rightarrow \mathbb{R}$ strongly h -convex with modulus c such that

$$f \leq \varphi \leq g \quad \text{on } D. \quad (19)$$

Proof. Fix $x \in D$ and define a function $\varphi : D \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(x) &= \inf \left\{ \sum_{i=1}^n h(t_i) g(x_i) - \sum_{i=1}^n t_i \|x_i - x\|^2 : n \right. \\ &\in \mathbb{N}, x_1, \dots, x_n \in D, t_1, \dots, t_n \\ &\in [0, 1] \text{ such that } t_1 + \dots + t_n = 1, x = t_1 x_1 + \dots \\ &\left. + t_n x_n \right\}. \end{aligned} \quad (20)$$

By (18) the definition is correct and $f(x) \leq \varphi(x)$ for all $x \in D$. On the other hand, taking $n = 1$ in the above definition (and, consequently, $t_1 = 1$, $x = x_1$) and using the fact that $h(1) = 1$, we get $\varphi(x) \leq g(x)$ for all $x \in D$.

To prove that φ is strongly h -convex with modulus c , fix $x, y \in D$ and $t \in [0, 1]$. Take arbitrary $u_1, \dots, u_n \in D$,

$\alpha_1, \dots, \alpha_n \in [0, 1]$ and $v_1, \dots, v_m \in D$, $\beta_1, \dots, \beta_m \in [0, 1]$ such that $\alpha_1 + \dots + \alpha_n = 1$, $\beta_1 + \dots + \beta_m = 1$ and $x = \alpha_1 u_1 + \dots + \alpha_n u_n$, $y = \beta_1 v_1 + \dots + \beta_m v_m$. Since $\sum_{i=1}^n t \alpha_i + \sum_{j=1}^m (1-t) \beta_j = 1$, the point $tx + (1-t)y$ is a convex combination of $u_1, \dots, u_n, v_1, \dots, v_m$, and

$$tx + (1-t)y = \sum_{i=1}^n t \alpha_i u_i + \sum_{j=1}^m (1-t) \beta_j v_j. \quad (21)$$

Therefore, by the definition of φ we have

$$\begin{aligned} \varphi(tx + (1-t)y) &\leq \sum_{i=1}^n h(t \alpha_i) g(u_i) + \sum_{j=1}^m h((1-t) \beta_j) g(v_j) \\ &\quad - c \left(t \sum_{i=1}^n \alpha_i \|u_i - m\|^2 + (1-t) \sum_{j=1}^m \beta_j \|v_j - m\|^2 \right), \end{aligned} \quad (22)$$

where $m = tx + (1-t)y$. By the multiplicativity of h we have

$$\begin{aligned} h(t \alpha_i) &= h(t) h(\alpha_i), \\ h((1-t) \beta_j) &= h(1-t) h(\beta_j). \end{aligned} \quad (23)$$

Note also that

$$\begin{aligned} \sum_{i=1}^n \alpha_i \|u_i - m\|^2 &= \sum_{i=1}^n \alpha_i (\|u_i\|^2 - 2 \langle u_i | m \rangle + \|m\|^2) \\ &= \sum_{i=1}^n \alpha_i \|u_i\|^2 - 2 \langle x | m \rangle + \|m\|^2 \\ &= \sum_{i=1}^n \alpha_i \|u_i - x\|^2 + \|x - m\|^2, \end{aligned} \quad (24)$$

and, similarly,

$$\sum_{j=1}^m \beta_j \|v_j - m\|^2 = \sum_{j=1}^m \beta_j \|v_j - y\|^2 + \|y - m\|^2. \quad (25)$$

Hence, using the fact that $h(t) \leq t$, we get

$$\begin{aligned} t \sum_{i=1}^n \alpha_i \|u_i - m\|^2 + (1-t) \sum_{j=1}^m \beta_j \|v_j - m\|^2 &\geq h(t) \sum_{i=1}^n \alpha_i \|u_i - x\|^2 + h(1-t) \sum_{j=1}^m \beta_j \|v_j - y\|^2 \\ &\quad + t \|x - m\|^2 + (1-t) \|y - m\|^2 \\ &= h(t) \sum_{i=1}^n \alpha_i \|u_i - x\|^2 + h(1-t) \sum_{j=1}^m \beta_j \|v_j - y\|^2 \\ &\quad + t (1-t) \|x - y\|^2. \end{aligned} \quad (26)$$

Substituting (23) and (26) into (22), we obtain

$$\begin{aligned} \varphi(tx + (1-t)y) &\leq h(t) \left(\sum_{i=1}^n h(\alpha_i) g(u_i) - c \sum_{i=1}^n \alpha_i \|u_i - x\|^2 \right) \end{aligned}$$

$$\begin{aligned}
& + h(1-t) \left(\sum_{j=1}^m h(\beta_j) g(v_j) - c \sum_{j=1}^m \beta_j \|v_j - y\|^2 \right) \\
& - ct(1-t) \|x - y\|^2.
\end{aligned} \tag{27}$$

Now, taking the infimum in the first term and next in the second term of the right hand side of (27) and using the definition of φ , we get

$$\begin{aligned}
\varphi(tx + (1-t)y) & \leq h(t)\varphi(x) + h(1-t)\varphi(y) \\
& - ct(1-t)\|x - y\|^2,
\end{aligned} \tag{28}$$

which shows that φ is strongly h -convex with modulus c and finishes the proof. $\square$

Remark 4. The method used in the proof of Theorem 3 is similar to that in [12, Theorem 3]. However, in our case we assume additionally that $h(t) \leq t$, $t \in [0, 1]$. The following example shows that this assumption is essential. Let $h(t) = 1$, $t \in [0, 1]$ and consider $f, g : [0, 1] \rightarrow \mathbb{R}$, $f = -1$ and $g = 0$. Then condition (18) is satisfied with $c = 1$, but there is no function φ strongly h -convex (with any modulus) satisfying $-1 = f \leq \varphi \leq g = 0$. Indeed, if φ is strongly h -convex with $h = 1$, then φ is nonnegative and some of its values are positive (putting $y = x$ in the definition of strong h -convexity, we get $\varphi \geq 0$, but $\varphi = 0$ is not strongly h -convex).

As a consequence of Theorem 3 we obtain the following Hyers-Ulam-type stability result for strongly h -convex functions.

Let ε be a positive constant. We say that a function $g : D \rightarrow \mathbb{R}$ is ε -strongly h -convex with modulus c if

$$g\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n h(t_i) g(x_i) - c \sum_{i=1}^n t_i \|x_i - \bar{x}\|^2 + \varepsilon, \tag{29}$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, and $t_1, \dots, t_n > 0$ with $t_1 + \dots + t_n = 1$ and $\bar{x} = t_1 x_1 + \dots + t_n x_n$.

Corollary 5. Let $h : [0, 1] \rightarrow \mathbb{R}$ be a multiplicative function such that $h(t) \leq t$ for all $t \in [0, 1]$. If a function $g : D \rightarrow \mathbb{R}$ is ε -strongly h -convex with modulus c , then there exists a function $\varphi : D \rightarrow \mathbb{R}$ strongly h -convex with modulus c such that

$$g(x) - \varepsilon \leq \varphi(x) \leq g(x), \quad x \in D. \tag{30}$$

Proof. Define $f(x) = g(x) - \varepsilon$, $x \in D$. Then f and g satisfy (18). Therefore, by Theorem 3, there exists $\varphi : D \rightarrow \mathbb{R}$, strongly h -convex with modulus c , such that $g - \varepsilon \leq \varphi \leq g$, on D . $\square$

Competing Interests

The authors declare that they have no competing interests.

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