Research Article

Hyers-Ulam Stability of Fractional Nabla Difference Equations

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We investigate the Hyers-Ulam stability, the generalized Hyers-Ulam stability, and the F_α -Hyers-Ulam stability of a linear fractional nabla difference equation using discrete Laplace transform. We provide a few examples to illustrate the applicability of established results.

1. Introduction

Ulam [1] posed the following problem on the stability of functional equations in 1940.

Ulam's Problem (see [1]). Given a group G_1 , a metric group (G_2, d) , and a positive number ε , does there exist $\delta > 0$ such that if a mapping $f : G_1 \rightarrow G_2$ satisfies the inequality

$$d(f(xy), f(x), f(y)) < \delta, \quad (1)$$

for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \rightarrow G_2$ such that

$$d(f(x), T(x)) < \delta, \quad (2)$$

for all $x, y \in G_1$?

Hyers [2] solved the problem for additive functions defined on Banach spaces in 1941 as follows.

Hyers's Theorem (see [2]). Let E_1 be a normed vector space and E_2 a Banach space and suppose that the mapping $f : E_1 \rightarrow E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad (3)$$

for all $x, y \in E_1$, where $\varepsilon > 0$ is a constant. Then the limit

$$g(x) = \lim_n 2^{-n} f(2^n x) \quad (4)$$

exists for each $x \in E_1$ and g is the unique additive mapping satisfying

$$\|f(x) - g(x)\| \leq \varepsilon, \quad (5)$$

for all $x \in E_1$.

Rassias [3] provided a generalization of the Hyers theorem for linear mappings. Later, many mathematicians have extended Ulam's problem in different directions. Recently, a generalization of Ulam's problem on the stability of differential equations was proposed.

Let X be a normed space and I be an open interval. The differential equation $F(t, u, u', u'', \dots, u^{(n)}) = 0$ is Hyers-Ulam stable, if, for given $\varepsilon > 0$ and a function $v : I \rightarrow X$ such that $\|F(t, v, v', v'', \dots, v^{(n)})\| \leq \varepsilon$, there exists a solution $u : I \rightarrow X$ of the differential equation such that $\|u(t) - v(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε only. If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\phi(t)$ and $\psi(t)$, where $\phi, \psi : I \rightarrow [0, \infty)$ are functions not depending on v and u explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. For a detailed discussion on the Hyers-Ulam stability, refer to [4, 5].

Recently, Rezaei et al. [6] obtained the Hyers-Ulam stability of a linear differential equation using Laplace transforms. Motivated by this article, Wang and Xu [7, 8] and Wang and Li [9] investigated the same for a class of linear fractional differential equations involving both Riemann-Liouville and

Caputo type fractional derivatives. In this article, we extend this study to linear fractional nabla difference equations.

2. Preliminaries

Throughout this article, we use the following notation, definitions, and known results of fractional nabla calculus [10]: Denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. Define $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for any $a \in \mathbb{R}$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 1 (rising factorial function). For any $\alpha \in \mathbb{R}$, $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ such that $(t + \alpha) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\bar{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \quad 0^{\bar{\alpha}} = 0. \quad (6)$$

Definition 2. Let $u : \mathbb{N}_a \rightarrow \mathbb{R}$ and $0 < \alpha < 1$.

- (1) (Fractional nabla sum) [11]: the α th-order nabla sum of u is given by

$$(\nabla_a^{-\alpha} u)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^t (t-s+1)^{\bar{\alpha-1}} u(s), \quad t \in \mathbb{N}_a. \quad (7)$$

- (2) (R-L fractional nabla difference) [11]: the α th-order nabla difference of u is given by

$$(\nabla_a^{\alpha} u)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^t (t-s+1)^{\bar{-\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+1}. \quad (8)$$

- (3) (Caputo fractional nabla difference) [11]: the α th-order nabla difference of u is given by

$$(\nabla_{a*}^{\alpha} u)(t) = (\nabla_a^{\alpha} u)(t) - \frac{(t-a+1)^{\bar{-\alpha}}}{\Gamma(1-\alpha)} u(a), \quad (9)$$

$$t \in \mathbb{N}_{a+1}.$$

Nagai [12] and Atici and Eloe [13] defined the one- and two-parameter Mittag-Leffler functions of fractional nabla calculus as follows.

Definition 3 (see [12, 13]). The one- and two-parameter nabla Mittag-Leffler functions are defined by

$$F_{\alpha}(\lambda, (t-a)^{\bar{\alpha}}) = \sum_{k=0}^{\infty} \frac{\lambda^k (t-a)^{\bar{k\alpha}}}{\Gamma(k\alpha + 1)}, \quad (10)$$

$$F_{\alpha, \beta}(\lambda, (t-a)^{\bar{\alpha}}) = \sum_{k=0}^{\infty} \frac{\lambda^k (t-a)^{\bar{k\alpha}}}{\Gamma(k\alpha + \beta)},$$

where $\alpha, \beta > 0$, $|\lambda| < 1$, and $t \in \mathbb{N}_a$.

Estimates of nabla Mittag-Leffler functions are provided in Lemma 4.

Lemma 4. Let $0 < \alpha, \beta < 1$. The functions F_{α} and $F_{\alpha, \alpha}$ are nonnegative and for any $0 < \lambda < 1$ and $t \in \mathbb{N}_0$,

$$F_{\alpha}(-\lambda, t^{\bar{\alpha}}) \leq 1, \quad (11)$$

$$F_{\alpha, \alpha}(-\lambda, t^{\bar{\alpha}}) \leq \frac{1}{\Gamma(\alpha)}.$$

Definition 5 (see [13]). Let $u : \mathbb{N}_a \rightarrow \mathbb{R}$. The N -transform of u is defined by

$$N_a[u(t)] = \sum_{j=a}^{\infty} u(j) (1-z)^{j-1}, \quad (12)$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 6 (see [13]). Let $u, v : \mathbb{N}_a \rightarrow \mathbb{R}$. The convolution of u and v is defined by

$$(u *_a v)(t) = \sum_{s=a}^t u(t+a-\rho(s)) v(s). \quad (13)$$

Atici and Eloe [13] developed the following properties of N -transforms.

Theorem 7 (see [13]). Assume that the following functions are well defined:

- (1) $N_a[(u *_a v)(t)] = N_1[u(t+a)] N_a[v(t)]$.
- (2) $N_a[(t-a+1)^{\bar{\alpha}}] = (1-z)^{a-1} (\Gamma(\alpha+1)/z^{\alpha+1})$, $\alpha \in \mathbb{R} \setminus \{\dots, -3, -2, -1\}$.
- (3) $N_a[(\nabla_a^{\alpha} u)(t)] = z^{\alpha} N_a[u(t)]$, $0 < \alpha < 1$.
- (4) $N_{a+1}[(\nabla_a^{\alpha} u)(t)] = z^{\alpha} N_a[u(t)] - (1-z)^{a-1} u(a)$, $0 < \alpha < 1$.
- (5) $N_a[(t-a+1)^{\bar{\alpha-1}} F_{\alpha, \alpha}(\lambda, (t-a+\alpha)^{\bar{\alpha}})] = (1-z)^{a-1} / (z^{\alpha} - \lambda)$.

3. Main Results

The main purpose of this section is to discuss the Hyers-Ulam stability of the following difference equation:

$$(\nabla_0^{\alpha} u)(t) + \lambda u(t) = f(t), \quad 0 < \alpha < 1, \quad \lambda \geq 0, \quad (14)$$

where $u, f : \mathbb{N}_0 \rightarrow \mathbb{R}$ and λ is a constant.

Let $v : \mathbb{N}_0 \rightarrow \mathbb{R}$ and $w(t) = (\nabla_0^{\alpha} v)(t) + \lambda v(t) - f(t)$ for $t \in \mathbb{N}_0$. Using Theorem 7, we have

$$N_1[w(t)]$$

$$= \left[(z^{\alpha} + \lambda) N_0[v(t)] - (1 + \lambda) \left(\frac{1}{1-z} \right) v(0) \right]$$

$$- \left[N_0[f(t)] - \left(\frac{1}{1-z} \right) f(0) \right] \quad (15)$$

$$= (z^{\alpha} + \lambda) N_0[v(t)] - N_0[f(t)]$$

$$- \left(\frac{1}{1-z} \right) [(1 + \lambda) v(0) - f(0)],$$

which implies

$$N_0[v(t)] = (z^\alpha + \lambda)^{-1} \left[N_1[w(t)] + N_0[f(t)] + \left(\frac{1}{1-z} \right) [(1+\lambda)v(0) - f(0)] \right]. \quad (16)$$

Set

$$\begin{aligned} u(t) &= v(0)(1+\lambda)(t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t+\alpha)^{\overline{\alpha}}) \\ &\quad + \sum_{s=1}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) f(s). \end{aligned} \quad (17)$$

Clearly $u(0) = v(0)$. Applying the N_0 transform on both sides of (17), we get

$$\begin{aligned} N_0[u(t)] &= [(1+\lambda)v(0) - f(0)] N_0[(t+1)^{\overline{\alpha-1}} \\ &\quad \cdot F_{\alpha,\alpha}(-\lambda, (t+\alpha)^{\overline{\alpha}})] + N_0 \left[\sum_{s=0}^t (t-s+1)^{\overline{\alpha-1}} \right. \\ &\quad \cdot F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) f(s) \Big] = [(1+\lambda)v(0) \\ &\quad - f(0)] N_0[(t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t+\alpha)^{\overline{\alpha}})] \\ &\quad + N_0[t^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-1+\alpha)^{\overline{\alpha}}) *_0 f(t)] \\ &= \left(\frac{(1+\lambda)v(0) - f(0)}{1-z} \right) (z^\alpha + \lambda)^{-1} \\ &\quad + N_1[t^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-1+\alpha)^{\overline{\alpha}})] N_0[f(t)] \\ &= \left(\frac{(1+\lambda)v(0) - f(0)}{1-z} \right) (z^\alpha + \lambda)^{-1} + (z^\alpha + \lambda)^{-1} \\ &\quad \cdot N_0[f(t)], \end{aligned} \quad (18)$$

which implies

$$\begin{aligned} (z^\alpha + \lambda) N_0[u(t)] &= \left(\frac{(1+\lambda)v(0) - f(0)}{1-z} \right) \\ &\quad + N_0[f(t)] \\ \text{or } \left[z^\alpha N_0[u(t)] - \left(\frac{1}{1-z} \right) u(0) \right] \\ &\quad + \lambda \left[N_0[u(t)] - \left(\frac{1}{1-z} \right) u(0) \right] \\ &= \left[N_0[f(t)] - \left(\frac{1}{1-z} \right) f(0) \right] \\ \text{or } N_1[(\nabla_0^\alpha u)(t) + \lambda u(t)] &= N_1[f(t)]. \end{aligned} \quad (19)$$

Since N_1 is one-to-one, it follows that $(\nabla_0^\alpha u)(t) + \lambda u(t) = f(t)$, so $u(t)$ is a solution of (14). From (16) and (18), we get

$$\begin{aligned} N_0[u(t)] - N_0[v(t)] &= (z^\alpha + \lambda)^{-1} N_1[w(t)] = (z^\alpha \\ &\quad + \lambda)^{-1} \left[N_0[w(t)] - \left(\frac{1}{1-z} \right) w(0) \right] \\ &= N_1 \left[t^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-1+\alpha)^{\overline{\alpha}}) \right] N_0[w(t)] \\ &\quad - w(0) N_0 \left[(t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t+\alpha)^{\overline{\alpha}}) \right] \\ &= N_0 \left[t^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-1+\alpha)^{\overline{\alpha}}) *_0 w(t) \right] - w(0) \\ &\quad \cdot N_0 \left[(t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t+\alpha)^{\overline{\alpha}}) \right] \\ &= N_0 \left[\sum_{s=1}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) w(s) \right]. \end{aligned} \quad (20)$$

Since N_0 is one-to-one, it follows that

$$\begin{aligned} u(t) - v(t) &= \sum_{s=1}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) w(s). \end{aligned} \quad (21)$$

First, we establish the generalized Hyers-Ulam stability of (14) as follows.

Theorem 8. Let $v : \mathbb{N}_0 \rightarrow \mathbb{R}$ and $\phi : \mathbb{N}_0 \rightarrow [0, \infty)$. If

$$|(\nabla_0^\alpha v)(t) + \lambda v(t) - f(t)| \leq \phi(t), \quad t \in \mathbb{N}_0, \quad (22)$$

then, there exists a solution $u : \mathbb{N}_0 \rightarrow \mathbb{R}$ of (14) and $\psi : \mathbb{N}_0 \rightarrow [0, \infty)$ such that

$$|u(t) - v(t)| \leq \psi(t), \quad t \in \mathbb{N}_0. \quad (23)$$

Proof. Using (21) and Lemma 4, we have

$$\begin{aligned} |u(t) - v(t)| &\leq \sum_{s=1}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) |w(s)| \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^t (t-s+1)^{\overline{\alpha-1}} \phi(s) = (\nabla_1^{-\alpha} \phi)(t) \\ &= \psi(t). \end{aligned} \quad (24)$$

□

Now, consider a particular case of Theorem 8 which we define as the F_α -Hyers-Ulam stability of (14).

Corollary 9. Let $\varepsilon > 0$, $0 < \eta < 1$, and $v : \mathbb{N}_0 \rightarrow \mathbb{R}$. If

$$|(\nabla_0^\alpha v)(t) + \lambda v(t) - f(t)| \leq \varepsilon F_\alpha(\eta, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0, \quad (25)$$

then, there exists a solution $u : \mathbb{N}_0 \rightarrow \mathbb{R}$ of (14) such that

$$|u(t) - v(t)| \leq K(\varepsilon) F_\alpha(\eta, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0, \quad (26)$$

where

$$K(\varepsilon) = \frac{\varepsilon}{\eta}. \quad (27)$$

Next, we investigate the Hyers-Ulam stability of (14).

Theorem 10. Let $\varepsilon > 0$ and $v : \mathbb{N}_0 \rightarrow \mathbb{R}$. If

$$|(\nabla_0^\alpha v)(t) + \lambda v(t) - f(t)| \leq \varepsilon, \quad t \in \mathbb{N}_0, \quad (28)$$

there exists a solution $u : \mathbb{N}_0 \rightarrow \mathbb{R}$ of (14) such that

$$|u(t) - v(t)| \leq K(\varepsilon), \quad t \in \mathbb{N}_0, \quad (29)$$

where

$$K(\varepsilon) = \frac{\varepsilon}{\lambda}. \quad (30)$$

Proof. Using (21) and Lemma 4, we have

$$\begin{aligned} |u(t) - v(t)| &\leq \sum_{s=0}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) |w(s)| \\ &\leq \varepsilon \sum_{s=0}^t (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(-\lambda, (t-s+\alpha)^{\overline{\alpha}}) \\ &= \varepsilon \sum_{s=0}^t (t-s+1)^{\overline{\alpha-1}} \sum_{k=0}^{\infty} \frac{(-\lambda)^k (t-s+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k + \alpha)} \\ &= \varepsilon \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k + \alpha)} \sum_{s=0}^t (t-s+1)^{\overline{\alpha-1}} (t-s+\alpha)^{\overline{\alpha k}} \\ &= \varepsilon \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k + \alpha)} \sum_{s=0}^t (t-s+1)^{\overline{\alpha k + \alpha - 1}} \\ &= \varepsilon \sum_{k=0}^{\infty} \frac{(-\lambda)^k (t+1)^{\overline{\alpha k + \alpha}}}{\Gamma(\alpha k + \alpha + 1)} \\ &= \frac{\varepsilon}{\lambda} [1 - F_{\alpha}(-\lambda, (t+1)^{\overline{\alpha}})] \leq \frac{\varepsilon}{\lambda}. \end{aligned} \quad (31)$$

□

Finally, we discuss the Hyers-Ulam stability of the following Caputo type linear fractional nabla difference equation:

$$(\nabla_{0*}^\alpha u)(t) + \lambda u(t) = f(t), \quad 0 < \alpha < 1, \quad \lambda \geq 0. \quad (32)$$

Using Definition 2 in (32), we get

$$(\nabla_0^\alpha u)(t) + \lambda u(t) = g(t), \quad t \in \mathbb{N}_1, \quad (33)$$

which is similar to (14). Here

$$g(t) = f(t) + \frac{(t+1)^{-\alpha}}{\Gamma(1-\alpha)} u(0), \quad t \in \mathbb{N}_0. \quad (34)$$

Example 11. Consider the following fractional nabla difference equation:

$$(\nabla_0^{0.25} u)(t) + (0.09)u(t) = \frac{(t+1)^{-0.25}}{\Gamma(0.75)}, \quad t \in \mathbb{N}_1. \quad (35)$$

For $\varepsilon = 0.1$ and $v(t) = 1$ for all $t \in \mathbb{N}_1$, we have

$$\begin{aligned} &\left| (\nabla_0^{0.25} v)(t) + (0.09)v(t) - \frac{(t+1)^{-0.25}}{\Gamma(0.75)} \right| \\ &= \left| \frac{(t+1)^{-0.25}}{\Gamma(0.75)} + (0.09) - \frac{(t+1)^{-0.25}}{\Gamma(0.75)} \right| \leq \varepsilon. \end{aligned} \quad (36)$$

Let $u(0) = 1$. Then, the exact solution of (35) is given by

$$\begin{aligned} u(t) &= (1.1)(t+1)^{-0.75} F_{0.25,0.25}(-0.1, (t+0.25)^{\overline{0.25}}) \\ &+ \sum_{s=1}^t (t-s+1)^{-0.75} F_{0.25,0.25}(-0.1, (t-s+0.25)^{\overline{0.25}}) \\ &\cdot (s+1)^{-0.25}, \quad t \in \mathbb{N}_0. \end{aligned} \quad (37)$$

Consequently, for $t \in \mathbb{N}_0$, we get

$$|u(t) - v(t)| \leq K(\varepsilon), \quad (38)$$

where

$$K(\varepsilon) = \frac{\varepsilon}{\lambda} = 1.1111. \quad (39)$$

Thus, (35) is Hyers-Ulam stable on $\mathbb{N}_0$. Furthermore, we illustrate these concepts numerically in Table 1.

Example 12. Consider the following fractional nabla difference equation:

$$(\nabla_0^{0.75} u)(t) + (0.5)u(t) = (0.25) \left(\frac{1}{t+1} \right)^{0.5}, \quad t \in \mathbb{N}_1. \quad (40)$$

For $v(t) = (t+1)^{\overline{0.25}}$, we have

$$\begin{aligned} &\left| (\nabla_0^{0.75} v)(t) + (0.5)v(t) - (0.25) \left(\frac{1}{t+1} \right)^{0.5} \right| \\ &= \left| \frac{\Gamma(1.25)}{\Gamma(0.75)} (t+1)^{-0.5} + (0.5)(t+1)^{\overline{0.25}} \right. \\ &\quad \left. - (0.25) \left(\frac{1}{t+1} \right)^{0.5} \right| \leq \frac{\Gamma(1.25)}{\Gamma(0.75)} (t+1)^{-0.5} + (0.5)(t+1)^{\overline{0.25}} \\ &\quad + (0.25) \left(\frac{1}{t+1} \right)^{0.5} \leq 0.7397 (t+1)^{-0.5} \\ &\quad + (0.5)(t+1)^{\overline{0.25}} + (0.25)(t+1)^{-0.5} = (t+1)^{-0.5} \\ &\quad + (0.5)(t+1)^{\overline{0.25}} = \phi(t), \quad t \in \mathbb{N}_0. \end{aligned} \quad (41)$$

TABLE 1

t	$v(t)$	$ w(t) $	ε	$u(t)$	$ u(t) - v(t) $	$K(\varepsilon)$
0	1	0.09	0.1	1.0000	0	1.1111
1	1	0.09	0.1	1.1364	0.1364	1.1111
2	1	0.09	0.1	1.1389	0.1389	1.1111
3	1	0.09	0.1	1.1346	0.1346	1.1111
4	1	0.09	0.1	1.1292	0.1292	1.1111
5	1	0.09	0.1	1.1238	0.1238	1.1111
6	1	0.09	0.1	1.1188	0.1188	1.1111
7	1	0.09	0.1	1.1140	0.1140	1.1111
8	1	0.09	0.1	1.1096	0.1096	1.1111
9	1	0.09	0.1	1.1054	0.1054	1.1111
10	1	0.09	0.1	1.1015	0.1015	1.1111
11	1	0.09	0.1	1.0979	0.0979	1.1111
12	1	0.09	0.1	1.0944	0.0944	1.1111

TABLE 2

t	$v(t)$	$ w(t) $	$\phi(t)$	$u(t)$	$ u(t) - v(t) $	$\psi(t)$
0	0.9064	1.5142	2.2257	0.9064	0	0.0362
1	1.1330	1.0452	1.4527	0.5711	0.5619	1.4980
2	1.2746	0.9846	1.3020	0.4384	0.8362	2.4424
3	1.3808	0.9751	1.2443	0.3618	1.0190	3.2293
4	1.4672	0.9803	1.2182	0.3110	1.1562	3.9384
5	1.5405	0.9908	1.2064	0.2746	1.2659	4.6007
6	1.6047	1.0036	1.2045	0.2471	1.3576	5.2317
7	1.6620	1.0172	1.2023	0.2257	1.4363	5.8401
8	1.7139	1.0311	1.2050	0.2084	1.5055	6.4313
9	1.7616	1.0449	1.2095	0.1942	1.5674	7.0092
10	1.8056	1.0584	1.2151	0.1824	1.6232	7.5762
11	1.8466	1.0716	1.2214	0.1723	1.6743	8.1343
12	1.8851	1.0845	1.2282	0.1636	1.7215	8.6849

Let $u(0) = \Gamma(1.25)$. Then, the exact solution of (40) is given by

$$\begin{aligned}
 u(t) &= (1.5) \Gamma(1.25) (t+1)^{-0.25} \\
 &\cdot F_{0.75,0.75} \left(-0.5, (t+0.75)^{0.75} \right) + (0.25) \\
 &\cdot \sum_{s=1}^t (t-s+1)^{-0.25} F_{0.75,0.75} \left(-0.5, (t-s+0.75)^{0.75} \right) \\
 &\cdot \left(\frac{1}{s+1} \right)^{0.5}, \quad t \in \mathbb{N}_0.
 \end{aligned} \quad (42)$$

Consequently, we get

$$|u(t) - v(t)| \leq \psi(t), \quad (43)$$

where

$$\begin{aligned}
 \psi(t) &= \nabla_1^{-0.75} \phi(t) = \nabla_0^{-0.75} \phi(t) - \frac{(t+1)^{-0.25}}{\Gamma(0.75)} \phi(0) \\
 &= \frac{\Gamma(0.5)}{\Gamma(1.25)} (t+1)^{0.25} + (0.5) \frac{\Gamma(1.25)}{\Gamma(2)} (t+1)^{\bar{1}}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{(t+1)^{-0.25}}{\Gamma(0.75)} \phi(0) \\
 &= (1.9954) (t+1)^{0.25} + (0.4532) (t+1) \\
 & - (1.8162) (t+1)^{-0.25}, \quad t \in \mathbb{N}_0.
 \end{aligned} \quad (44)$$

Thus, (40) is generalized Hyers-Ulam stable. Furthermore, we illustrate these concepts numerically in Table 2.

Competing Interests

The author declares that they have no competing interests.

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