Research Article

On Inclusion Relations between Some Sequence Spaces

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We determine the relations between the classes $\hat{S}_\lambda$ of almost $\lambda$-statistically convergent sequences and the relations between the classes $[\hat{V}, \lambda]$ of strongly almost $(V, \lambda)$-summable sequences for various sequences $\lambda, \mu \in \Lambda$. Furthermore we also give the relations between the classes $\hat{S}_\lambda$ of almost $\lambda$-statistically convergent sequences and the classes $[\hat{V}, \lambda]$ of strongly almost $(V, \lambda)$-summable sequences for various sequences $\lambda, \mu \in \Lambda$.

1. Introduction

A sequence $x = (x_k)$ of real (or complex) numbers is said to be statistically convergent to the number $L$ if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| k \in [L - \varepsilon, L + \varepsilon] \right| = 0.$$  (1)

In this case, we write $S\lim x = L$ or $x_k \to L(S)$ and $S$ denotes the set of all statistically convergent sequences.

A sequence $x = (x_k)$ of real (or complex) numbers is said to be almost statistically convergent to the number $L$ if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| k \in [L - \varepsilon, L + \varepsilon] \right| = 0 \text{ uniformly in } m.$$  (2)

In this case, we write $\hat{S}\lim x = L$ or $x_k \to L(\hat{S})$ and $\hat{S}$ denotes the set of all almost statistically convergent sequences [1].

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive real numbers tending to $\infty$ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$  (3)

The set of all such sequences will be denoted by $\Lambda$.

The generalized de la Vallée-Poussin mean is defined by

$$t_n (x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$  (4)

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be $(V, \lambda)$-summable to a number $L$ (see [2]) if

$$t_n (x) \to L \quad \text{as } n \to \infty.$$  (5)

If $\lambda_n = n$ for each $n \in \mathbb{N}$, then $(V, \lambda)$-summability reduces to $(C, 1)$-summability.

We write

$$[C, 1] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \text{ for some } L \right\},$$

$$[V, \lambda] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \text{ for some } L \right\}$$

for the sets of sequences $x = (x_k)$ which are strongly Cesàro summable and strongly $(V, \lambda)$-summable to $L$; that is, $x_k \to L[C, 1]$ and $x_k \to L[V, \lambda]$, respectively.
Savaş [1] defined the following sequence space:

\[ [\hat{V}, \lambda] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{|I_n|} \sum_{k \in I_n} |x_{k+m} - L| = 0 \text{ for some } L, \text{ uniformly in } m \right\} \]

for the sets of sequences \( x = (x_k) \) which are strongly almost \((\mathbb{V}, \lambda)\)-summable to \( L \); that is, \( x_k \to L [\hat{V}, \lambda] \). We will write \( [\hat{V}, \lambda]_0 = [\hat{V}, \lambda] \cap \ell_{cc} \).

The \( \lambda \)-statistical convergence was introduced by Mursaleen in [3] as follows.

Let \( \lambda = (\lambda_n) \in \Lambda \). A sequence \( x = (x_k) \) is said to be \( \lambda \)-statistically convergent or \( S_1 \)-convergent to \( L \) if for every \( \varepsilon > 0 \)

\[ \lim_{n \to \infty} \frac{1}{\lambda_n} \left\| \left\{ k \in I_n : |x_k - L| \geq \varepsilon \right\} \right\| = 0, \tag{8} \]

where \( I_n = [n - \lambda_n + 1, n] \). In this case we write \( S_1 \)-lim \( x = L \). \( S_1 \)-lim \( x = L \) and \( S_0 \)-lim \( x = L \). In this case, \( \hat{S}_1 \)-lim \( x = L \) or \( x_k \to L (\hat{S}_1) \) and \( \bar{S}_1 \) denotes the set of all \( \lambda \)-almost statistically convergent sequences. If we choose \( \lambda_n = n \) for all \( n \), then \( \lambda \)-almost statistical convergence reduces to almost statistical convergence [1].

2. Main Results

Throughout the paper, unless stated otherwise, by “for all \( n \in \mathbb{N}_0 \),” we mean “for all \( n \in \mathbb{N} \) except finite numbers of positive integers” where \( \mathbb{N}_0 = \{n_0, n_0 + 1, n_0 + 2, \ldots \} \) for some \( n_0 \in \mathbb{N} = \{1, 2, 3, \ldots \} \).

**Theorem 1.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_n) \) be two sequences in \( \Lambda \) such that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \). Consider the following:

(i) If

\[ \liminf_{n \to \infty} \frac{\lambda_n}{\mu_n} > 0 \tag{10} \]

then \( S_1 \subseteq S_\mu \).

(ii) If

\[ \lim_{n \to \infty} \frac{\lambda_n}{\mu_n} = 1 \tag{11} \]

then \( S_1 \subseteq S_\mu \).

**Proof.** (i) Suppose that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \) and let (10) be satisfied. Then \( I_n \subset J_n \) so that for \( \varepsilon > 0 \) we may write

\[ \frac{1}{\mu_n} \left\| \left\{ k \in J_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \geq \frac{1}{\mu_n} \frac{\lambda_n}{\mu_n} \left\| \left\{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \]

and therefore we have

\[ \frac{1}{\mu_n} \left\| \left\{ k \in J_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \geq \frac{\lambda_n}{\mu_n} \left\| \left\{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \tag{12} \]

for all \( n \in \mathbb{N}_0 \), where \( J_n = [n - \lambda_n + 1, n] \). Now taking the limit as \( n \to \infty \) uniformly in \( m \) in the last inequality and using (10) we get \( x_k \to L(\hat{S}_\mu) \) so that \( \hat{S}_\mu \subseteq \bar{S}_1 \).

(ii) Let \( (x_k) \in \hat{S}_1 \) and (11) be satisfied. Since \( I_n \subset J_n \), for \( \varepsilon > 0 \), we may write

\[ \frac{1}{\mu_n} \left\| \left\{ k \in J_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \]

unifomly in \( m \).

In this case, we write \( S_1 \)-lim \( x = L \) or \( x_k \to L (\hat{S}_1) \) and \( \bar{S}_1 \) denotes the set of all \( \lambda \)-almost statistically convergent sequences. If we choose \( \lambda_n = n \) for all \( n \), then \( \lambda \)-almost statistical convergence reduces to almost statistical convergence [1].

From Theorem 1 we have the following result.

**Corollary 2.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_n) \) be two sequences in \( \Lambda \) such that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \). If (11) holds then \( \bar{S}_1 \subseteq \hat{S}_\mu \).

If we take \( \mu = (\mu_n) = (n) \) in Corollary 2 we have the following result.

**Corollary 3.** Let \( \lambda = (\lambda_n) \in \Lambda \). If \( \lim_{n \to \infty} (\lambda_n/n) = 1 \) then we have \( \bar{S}_1 = \hat{S}_\mu \).

**Theorem 4.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_n) \in \Lambda \) and suppose that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \). Consider the following:

(i) If (11) holds then \( S_1 \subseteq S_\mu \).

(ii) If (12) holds then \( S_1 \subseteq S_\mu \).

Proof. (i) Suppose that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \) and let (10) be satisfied. Then \( I_n \subset J_n \) so that for \( \varepsilon > 0 \) we may write

\[ \frac{1}{\mu_n} \left\| \left\{ k \in J_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \geq \frac{1}{\mu_n} \frac{\lambda_n}{\mu_n} \left\| \left\{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \]

and therefore we have

\[ \frac{1}{\mu_n} \left\| \left\{ k \in J_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \geq \frac{\lambda_n}{\mu_n} \left\| \left\{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \right\} \right\| \tag{13} \]

for all \( n \in \mathbb{N}_0 \), where \( J_n = [n - \mu_n + 1, n] \). Now taking the limit as \( n \to \infty \) uniformly in \( m \) in the last inequality and using (10) we get \( x_k \to L(\hat{S}_\mu) \) so that \( \hat{S}_\mu \subseteq \bar{S}_1 \).

From Theorem 1 we have the following result.

**Corollary 2.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_n) \) be two sequences in \( \Lambda \) such that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \). If (11) holds then \( \bar{S}_1 \subseteq \hat{S}_\mu \).

If we take \( \mu = (\mu_n) = (n) \) in Corollary 2 we have the following result.

**Corollary 3.** Let \( \lambda = (\lambda_n) \in \Lambda \). If \( \lim_{n \to \infty} (\lambda_n/n) = 1 \) then we have \( \bar{S}_1 = \hat{S}_\mu \).

**Theorem 4.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_n) \in \Lambda \) and suppose that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_0 \). Consider the following:

(i) If (11) holds then \( S_1 \subseteq S_\mu \).

(ii) If (12) holds then \( S_1 \subseteq S_\mu \).
Proof. (i) Suppose that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_n \). Then \( I_n \subseteq J_n \) so that we may write

\[
\frac{1}{\mu_n} \sum_{k \in J_n} |x_{k+m} - L| \geq \frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L|
\]

(15)

for all \( n \in \mathbb{N}_n \). This gives that

\[
\frac{1}{\mu_n} \sum_{k \in J_n} |x_{k+m} - L| \geq \frac{\lambda_n}{\mu_n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L|.
\]

(16)

Then taking limit as \( n \to \infty \), uniformly in \( m \) in the last inequality, and using (10) we obtain \( x_k \to L[\bar{V}, \mu] \Rightarrow x_k \to L[\bar{V}, \lambda] \).

(ii) Let \( x = (x_n) \in [\bar{V}, \lambda]_{\infty} \) be any sequence. Suppose that \( x_k \to L[\bar{V}, \mu] \) and that (11) holds. Since \( x = (x_n) \in I_n \), then there exists some \( M > 0 \) such that \( |x_{k+m} - L| \leq M \) for all \( k \) and \( m \). Since \( \lambda_n \leq \mu_n \) so that \( 1/\mu_n \leq 1/\lambda_n \), and \( I_n \subseteq J_n \) for all \( n \in \mathbb{N}_n \), we may write

\[
\frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L| = \frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L| + \frac{1}{\mu_n} \sum_{k \in J_n} |x_{k+m} - L|
\]

\[
\leq \frac{\mu_n - \lambda_n}{\mu_n} M + \frac{1}{\mu_n} \sum_{k \in J_n} |x_{k+m} - L|
\]

\[
\leq \left( 1 - \frac{\lambda_n}{\mu_n} \right) M + \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L|
\]

(17)

for every \( n \in \mathbb{N}_n \). Since \( \lim_{n} (\lambda_n/\mu_n) = 1 \) by (11) and since \( x_k \to L[\bar{V}, \lambda] \) the first term and the second term of right hand side of above inequality tend to 0 as \( n \to \infty \), uniformly in \( m \) (note that \( 1 - \lambda_n/\mu_n \geq 0 \) for all \( n \in \mathbb{N}_n \)). Then we get \( x_k \to L[\bar{V}, \lambda] \Rightarrow x_k \to L[\bar{V}, \mu] \). Since \( x = (x_n) \in [\bar{V}, \lambda]_{\infty} \) is an arbitrary sequence we obtain \( [\bar{V}, \lambda]_{\infty} \subseteq [\bar{V}, \mu] \).

Since clearly (11) implies (10) from Theorem 4 we have the following result.

Corollary 5. Let \( \lambda, \mu \in \Lambda \) such that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_n \). If (11) holds then \( [\bar{V}, \lambda]_{\infty} \subseteq [\bar{V}, \mu]_{\infty} \).

Theorem 6. Let \( \lambda, \mu \in \Lambda \) such that \( \lambda_n \leq \mu_n \) for all \( n \in \mathbb{N}_n \). Consider the following:

(i) If (10) holds then

\[
x_k \to L[\bar{V}, \mu] \Rightarrow x_k \to L(\bar{S}_\lambda)
\]

(18)

and the inclusion \( [\bar{V}, \mu] \subseteq \bar{S}_\lambda \) holds for some \( \lambda, \mu \in \Lambda \).

(ii) If \( (x_n) \in \ell_{\infty} \) and \( x_k \to L(\bar{S}_\lambda) \) then \( x_k \to L[\bar{V}, \mu] \), whenever (11) holds.

(iii) If (11) holds then \( \ell_{\infty} \cap \bar{S}_\lambda = [\bar{V}, \mu]_{\infty} \).

Proof. (i) Let \( \epsilon > 0 \) be given and let \( x_k \to L[\bar{V}, \mu] \). Now for every \( \epsilon > 0 \) we may write

\[
\sum_{k \in J_n} |x_{k+m} - L| \geq \sum_{k \in I_n} |x_{k+m} - L|
\]

\[
\geq \sum_{k \in I_n} |x_{k+m} - L| \geq \epsilon \left| \{ k \in I_n : |x_{k+m} - L| \geq \epsilon \} \right|
\]

(19)

so that

\[
\frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L| \geq \left( 1 - \frac{\lambda_n}{\mu_n} \right) M + \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L| \geq \frac{\lambda_n}{\mu_n} \frac{1}{\lambda_n} \left| \{ k \in I_n : |x_{k+m} - L| \geq \epsilon \} \right| \epsilon
\]

(20)

for all \( n \in \mathbb{N}_n \). Then taking limit as \( n \to \infty \), uniformly in \( m \) in the last inequality, and using (10) we obtain \( x_k \to L(\bar{S}_\lambda) \) whenever \( x_k \to L[\bar{V}, \mu] \). Since \( x = (x_n) \in [\bar{V}, \mu] \) is an arbitrary sequence we obtain \( [\bar{V}, \mu] \subseteq \bar{S}_\lambda \).

(ii) Suppose that \( x_k \to L(\bar{S}_\lambda) \) and \( x = (x_n) \in \ell_{\infty} \). Then there exists some \( M > 0 \) such that \( |x_{k+m} - L| \leq M \) for all \( k \) and \( m \). Since \( 1/\mu_n \leq 1/\lambda_n \), then for every \( \epsilon > 0 \) we may write

\[
\frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L| = \frac{1}{\mu_n} \sum_{k \in I_n} |x_{k+m} - L|
\]

\[
\leq \frac{\mu_n - \lambda_n}{\mu_n} M + \frac{1}{\mu_n} \sum_{k \in J_n} |x_{k+m} - L|
\]

\[
\leq \left( 1 - \frac{\lambda_n}{\mu_n} \right) M + \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L|
\]

\[
\leq \left( 1 - \frac{\lambda_n}{\mu_n} \right) M + \frac{1}{\lambda_n} \sum_{k \in J_n} |x_{k+m} - L| + \frac{1}{\lambda_n} \sum_{k \in J_n} |x_{k+m} - L| \geq \epsilon \left| \{ k \in I_n : |x_{k+m} - L| \geq \epsilon \} \right|
\]

\[
+ \epsilon
\]

(21)
for all $n \in \mathbb{N}_0$. Using (11) we obtain that $x_k \rightarrow L[\hat{V}, \mu]$ whenever $x_k \rightarrow L(\hat{S}_\lambda)$. Since $x = (x_k) \in \ell_\infty \cap \hat{S}_\lambda$ is an arbitrary sequence we obtain $\ell_\infty \cap \hat{S}_\lambda \subseteq [\hat{V}, \mu]$.

(iii) The proof follows from (i) and (ii), so we omit it.

From Theorem 1(i) and Theorem 6(i) we obtain the following result.

**Corollary 7.** If $\liminf_{n \rightarrow \infty} (\lambda_n / \mu_n) > 0$ then $\hat{S}_\mu \cap [\hat{V}, \mu] \subseteq \hat{S}_\lambda$.

If we take $\mu_n = n$ for all $n$ in Theorem 6 then we have the following results, because $\lim_{n \rightarrow \infty} (\lambda_n / \mu_n) = 1$ implies that $\liminf_{n \rightarrow \infty} (\lambda_n / \mu_n) = 1 > 0$; that is, (11) $\Rightarrow$ (10).

**Corollary 8.** If $\lim_{n \rightarrow \infty} (\lambda_n / n) = 1$ then

(i) if $(x_k) \in \ell_\infty$ and $x_k \rightarrow L(\hat{S}_\lambda)$ then $x_k \rightarrow L[C, 1]$,

(ii) if $x_k \rightarrow L[C, 1]$ then $x_k \rightarrow L(\hat{S}_\lambda)$.

**Competing Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


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