Research Article

Sectional Category of the Ganea Fibrations
and Higher Relative Category

Jean-Paul Doeraene

Département de Mathématiques, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

Correspondence should be addressed to Jean-Paul Doeraene; doeraene@math.univ-lille1.fr

Received 7 April 2016; Accepted 29 June 2016

Academic Editor: Dan Burghelea

Copyright © 2016 Jean-Paul Doeraene. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We first compute James’ sectional category (secat) of the Ganea map $g_0$ of any map $t_X$; we show that $\text{secat}_X g_0$ is the integer part of $\text{secat}_X t_X / (k + 1)$. Next we compute the relative category (relcat) of $g_0$. In order to do this, we introduce the relative category of order $k$ (relcat$_k$) of a map and show that $\text{relcat}_k g_0$ is the integer part of $\text{relcat}_k t_X / (k + 1)$. Then we establish some inequalities linking secat and relcat of any order: we show that $\text{secat}_X t_X \leq \text{relcat}_k t_X \leq \text{secat}_X + k + 1$ and $\text{relcat}_k t_X \leq \text{relcat}_{k+1} t_X \leq \text{relcat}_{k+2} t_X + 1$. We give examples that show that these inequalities may be strict.

1. Introduction

The “Lusternik-Schnirelmann category” $\text{cat} X$ of a topological space $X$ is the least integer $n \geq 0$ such that $X$ can be covered by $n + 1$ open subsets $U_i$, $0 \leq i \leq n$, such that each inclusion $U_i \hookrightarrow X$ is nullhomotopic; that is, the based path-space fibration $PX \rightarrow X$ has a partial section on $U_i$. More generally, the “sectional category” $\text{secat} p$ of a fibration $p : E \rightarrow X$, originally defined by Schwarz [1], is the least integer $n \geq 0$ such that $X$ can be covered by $n + 1$ open subsets with a partial section of $p$ on each of these sets. This notion extends to any continuous map $t_X : A \rightarrow X$ by taking the standard homotopy replacement of $t_X$ by a fibration $p : E \rightarrow X$ and setting $\text{secat} t_X = \text{secat} p$. So $\text{cat} X = \text{secat}(\ast \rightarrow X)$. Sectional category earned its renown recently as Farber’s notion of “topological complexity” [2] of a space $A$, which measures the difficulty of solving the motion planning problem: the topological complexity of $A$ is the sectional category of the diagonal $\Delta : A \rightarrow A \times A$ or equivalently of the (unbased) fibration $\pi : A^I \rightarrow A \times A : \alpha \mapsto (\alpha(0), \alpha(1))$.

For a given space $X$, Ganea [3] defined a sequence of fibrations $p_k : E_k \rightarrow X$ for $k \geq 0$, starting with $p_0 : PX \rightarrow X$. The fundamental property of the sequence is that it gives another criterion for detecting the category: $\text{cat} X$ is the least $n$ such that $p_n$ has a section (at least for a sufficiently nice space: normal, well pointed). This construction can be generalized for any map $t_X : A \rightarrow X$; that is, there is a sequence of maps $g_k(t_X) : G_k(t_X) \rightarrow X$, starting with $g_0(t_X) = t_X$, and $\text{secat} t_X$ is the least $n$ such that $g_n(t_X)$ has a homotopy section; see Definition 3. We recover the Ganea construction when $A = \ast$; in this case we write $g_k(X)$ instead of $g_k(t_X)$.

In this paper, we first show that the sectional category of $k$th Ganea map $g_k(X)$ of $X$ is the integer part of $\text{cat} X / (k + 1)$. More generally, the sectional category of the Ganea map $g_k(X)$ associated with any map $t_X$ is the integer part of $\text{secat} t_X / (k + 1)$.

As we may “think of” the sectional category as the degree of obstruction for a map to have a homotopy section, this shows us how this degree of obstruction decreases when we consider the successive Ganea maps. For instance, for a space $X$ with $\text{cat} X = 7$, the successive values of $\text{secat}(g_k(X))$ for $0 \leq k \leq 7$ are

$$7 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 \quad 0.$$ (1)

In [4], we used the same Ganea-type construction to define the “relative category” of a map (relcat for short). As a particular case, the relative category of the diagonal map $\Delta : X \rightarrow X \times X$ is the “monoidal topological complexity” of $X$ defined in [5]. It turns out that the relative category can differ...
from the sectional category by at most one. More precisely, we have
\[ \text{secat}_{iX} \leq \text{relcat}_{iX} \leq \text{secat}_{iX} + 1. \] (2)

This establishes a dichotomy between maps: those for which the sectional category equals the relative category and those for which they differ by 1.

In this paper we introduce the "relative category of order \( k \)" (relcat\(_k\)) and show that the relative category of \( k \)th Ganea map \( g_k(iX) \) associated with a map \( iX \) is the integer part of relcat\(_k\)\(iX/(k + 1)\). When \( iX : \ast \to X \), we write relcat\(_k\)\(iX = \text{cat}_kX\).

Warning. Despite cat\(_k\) is sometimes used in the literature for Fox's \( k \)-dimensional category, this is not the meaning of this notation in this paper.

We link all these invariants together by several inequalities:
\[ \text{secat}_{iX} \leq \text{relcat}_{iX} \leq \text{secat}_{iX} + k + 1, \]
\[ \text{relcat}_{iX} \leq \text{relcat}_{k+1}iX \leq \text{relcat}_{iX} + 1. \] (3)

Finally, we show that, with some hypothesis on the connectivity of \( iX \) and the homotopical dimension of the source of \( g_k(iX) \), relcat\(_k\)\(iX = \text{secat}_iX \) for all \( j \leq k \).

For a given space \( X \) (resp.: map \( iX \)), the set of integers \( k \) for which the equality \( \text{cat}_{iX}X = \text{cat}_iX \) (resp., relcat\(_k\)\(iX = \text{relcat}_{iX} \)) holds is an interesting datum about this space (resp., map). The maximum number of such integers is \( \text{cat}_X \) (resp., relcat\(_iX\)). For instance, for \( X = K(Q, 1) \), there is just one such \( k \), which is 0, namely,
\[ \text{cat}_0X = \text{cat}_1X = 2, \]
\[ \text{cat}_kX = k + 1 \quad \text{for} \quad k > 1. \] (4)

2. Sectional Category of the Ganea Maps

We use the symbol = both to mean that maps are homotopic and to mean that spaces are of the same homotopy type. We denote the integer part of a rational number \( q \) by \( [q] \).

We build all our spaces and maps with "homotopy commutative diagrams," especially "homotopy pullbacks" and "homotopy pushouts," in the spirit of [6].

Recall the following construction.

Definition 1. For any map \( iX : A \to X \), the Ganea construction of \( iX \) is the following sequence of homotopy commutative diagrams (\( i \geq 0 \)):

\[
\begin{array}{c}
F_i \\
\eta_i \\
G_i \\
\alpha_{i+1} \\
A \\
\end{array}
\begin{array}{c}
\beta_i \\
\gamma_i \\
g_i \\
g_{i+1} \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
G_{i+1} \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
X \\
\end{array}
\]

where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map \( g_{i+1} = (g_i, iX) : G_{i+1} \to X \) is the whisker map induced by this homotopy pushout. The iteration starts with \( g_0 = iX : A \to X \).

In other words, the map \( g_{i+1} \) is the join of \( g_i \) and \( iX \) over \( X \); namely, \( g_{i+1} = iX \cdot g_i \). When we need to be precise, we denote \( G_i \) by \( g_i(iX) \) and \( g_i \) by \( g_i(X) \). If \( A = \ast \), we also write \( G_i(X) \) and \( g_i(X) \), respectively.

Notice that, as the outside square is a homotopy pullback, \( g_i \) and \( \eta_i \) have a common homotopy fiber, so their connectivity is equal.

For coherence, let \( \alpha_0 = \text{id}_A \). For any \( i \geq 0 \), there is a whisker map \( \theta_i = (\text{id}_A, \alpha_i) : A \to F_i \) induced by the homotopy pullback. Thus, \( \theta_i \) is a homotopy section of \( \eta_i \).

Moreover, we have \( \gamma_i \circ \alpha_i = \alpha_{i+1} \).

Proposition 2. For any map \( iX : A \to X \), we have
\[ g_j(g_i(iX)) = g_{j+i+1}(iX). \] (5)

Proof. This is just an application of the "associativity of the join" (see [7, Theorem 4.8], for instance):
\[ g_j(g_i(iX)) = g_i(iX) \cdot X \cdot \cdots \cdot X g_j(iX) \quad (j + 1 \text{ times}) \]
\[ = (iX \cdot X \cdot \cdots \cdot X g_j(iX)) \cdots (iX \cdot X \cdot \cdots \cdot X g_j(iX)) \]
\[ = iX \cdot X \cdot \cdots \cdot X g_j(iX) \quad ((j + 1) (i + 1) \text{ times}) \]
\[ = g_{(j+1)(i+1)-1}(iX). \]
\[ \square \]

Definition 3. Let \( iX : A \to X \) be any map.

(1) The sectional category of \( iX \) is the least integer \( n \) such that the map \( g_n : G_n(iX) \to X \) has a homotopy section; that is, there exists a map \( \sigma : X \to G_n(iX) \) such that \( g_n \circ \sigma = \text{id}_X \).

(2) The relative category of \( iX \) is the least integer \( n \) such that the map \( g_n : G_n(iX) \to X \) has a homotopy section \( \sigma \) and \( \sigma \circ iX = \alpha_n \).

We denote the sectional category by \( \text{secat}(iX) \) and the relative category by \( \text{relcat}(iX) \). If \( A = \ast \), \( \text{secat}(iX) = \text{relcat}(iX) \) and it is denoted simply by \( \text{cat}(X) \); this is the "normalized" version of the Lusternik-Schnirelman category.

A lot about the integers \( \text{cat} \) and \( \text{secat} \) is collected in [8].

The integer \( \text{relcat} \) is introduced in [4] and further studied in [9, 10].

Proposition 4. For any map \( iX : A \to X \), we have
\[ \text{secat}_{g_k(iX)} = \left[ \text{secat}_{iX} \right] k + 1. \] (7)

Proof. By definition, \( \text{secat}_{g_k(iX)} \) is the least integer \( n \) such that \( g_k(g_k(iX)) \), that is, \( g_{k+n+1}(iX) \), has a homotopy section. Thus, if \( \text{secat}_{iX} = m, n \) will be such that \( kn + k + n \geq m \) and \( k(n-1) + k + (n-1) < m \); that is, \( n \geq m/(k+1) - k/(k+1) \) and \( n < m/(k+1) + 1/(k+1) \), so \( n = [m/(k+1)] \). \( \square \)
3. Higher Relative Category

For any map $t_X : A \to X$ and two integers $0 \leq k < r$, consider the following homotopy commutative diagram:

$$
\begin{array}{c}
H_k \\
\downarrow \\
G_r \\
\downarrow
\end{array}
\xymatrix{
G_k \\
\downarrow \\
G_r \\
\downarrow \\
X
\end{array}
$$

where the outside square is a homotopy pullback and the inside square is a homotopy pushout.

Because of the associativity of the join, we also have $y_k^r = y_{r-1} \circ y_{r-2} \circ \cdots \circ y_{k+1} \circ y_k$. For coherence, let $y_k^k = 0$.

**Definition 5.** Let $t_{i_X} : A \to X$ be any map. The relative category of order $k$ of $i_X$ is the least integer $n \geq k$ such that the map $g_n : G_n(i_X) \to X$ has a homotopy section $\sigma$ and $\sigma \circ g_k = y_k^n$.

We denote this integer by $\text{relcat}_{i_X}$. In order to avoid the prefix "rel" when $A = \ast$, we write $\text{cat}_X = \text{relcat}_{i_X}$ in this case.

**Remark 6.** Notice that $\text{relcat}_{i_X} = \text{cat}_X$ and that, clearly, $k \leq \text{relcat}_{i_X} \leq \text{relcat}_{i_{k+1}X}$ for any $k$. Also notice that $\text{relcat}_{i_X} = k$ if and only if $g_k(i_X)$ is a homotopy equivalence. In particular, $\text{cat}_X = k$ for any $k$.

Following the same reasoning as in Proposition 4, we have the following.

**Proposition 7.** For any map $i_X : A \to X$, we have

$$\text{relcat}_k(i_X) = \left\lfloor \frac{\text{relcat}_{i_X}}{k+1} \right\rfloor. \tag{8}$$

**Proposition 8.** For any map $i_X : A \to X$, any $k$, we have

$$\text{secat} i_X \leq \text{relcat} i_X \leq \text{secat} i_X + k + 1. \tag{9}$$

**Proof.** Only the second inequality needs a proof. Let $n = \text{secat} i_X$ and let $\sigma$ be a homotopy section of $g_n$. Consider the following homotopy commutative diagram:

$$
\begin{array}{c}
G_k \\
\downarrow \\
G_n \\
\downarrow
\end{array}
\xymatrix{
G_k \\
\downarrow \\
G_n \\
\downarrow \\
X
\end{array}
$$

where $\sigma' = (\sigma \circ g_k, \text{id}_{G_n})$ is the whisker map induced by the right homotopy pullback. We have $g' \circ \sigma' = \text{id}_{G_k}$ and the left square is a homotopy pullback by the Prism lemma (see [7, Lemma 1.3], for instance). The map $\sigma' = y_n^{k+1} \circ \sigma$

is a homotopy section of $g_{n+k+1}$ and, moreover, $\sigma' \circ g_k = y_k^{n+k+1} \circ g' \circ \sigma' = y_k^{n+k+1}$. So $\text{relcat}_{i_X} \leq n + k + 1$. \hfill \Box

**Theorem 9.** For any map $i_X : A \to X$, any $k$, we have

$$k \leq \text{relcat} i_X \leq \text{relcat}_{i_{k+1}X} \leq \text{relcat} i_X + 1. \tag{10}$$

**Proof.** The first two inequalities are our Remark 6; only the third needs a proof. Let $n = \text{relcat} i_X$ and let $\sigma$ be a homotopy section of $g_n$ such that $\sigma \circ g_k = y_k^n$. Consider the following homotopy commutative diagram:

$$
\begin{array}{c}
F_k \\
\downarrow \\
G_n \\
\downarrow
\end{array}
\xymatrix{
F_k \\
\downarrow \\
G_n \\
\downarrow \\
X
\end{array}
$$

The map $\sigma' = y_n \circ \sigma$ is a homotopy section of $g_{n+1}$ and $\sigma' \circ g_k = y_k^{n+1}$, so $\text{relcat}_{i_{k+1}X} \leq n + 1$.

So $\text{relcat}_{i_X}$ increases at most by one when $k$ increases by one.

**Corollary 10.** For any map $i_X : A \to X$, any $k$, we have

$$\text{relcat} i_X \leq \text{relcat} i_X \leq \text{relcat} i_X + k. \tag{11}$$

**Remark 11.** As a consequence of Theorem 9 and Corollary 10, if $n = \text{relcat} i_X$, there are at most $n$ integers $k$ for which $\text{relcat}_{i_{k+1}X} = \text{relcat} i_X$.

**Example 12.** If $i_X$ is a homotopy equivalence, then $g_k$ is a homotopy equivalence for all $k$. So $\text{relcat} i_X = k$ for all $k$.

**Example 13.** Let $A \neq \ast$ and consider the map $i_X : A \to \ast$. We have $\text{secat} i_X = 0$ because $i_X$ has a (unique) section. By Proposition 8, $\text{relcat} i_X = k + 1$. Indeed, for any $k$, the map $g_{k+1} : A \ast \cdots \ast A (k + 1 \text{ times}) \to A \ast \cdots \ast A (k + 2 \text{ times})$ is homotopic to the null map, so $\sigma \circ g_k = y_k^k$, where $\sigma : \ast \to G_k(i_X)$. But we cannot have $\text{relcat} i_X = k$ unless $g_k(i_X) : A \ast \cdots \ast A (k + 1 \text{ times}) \to \ast$ is a homotopy equivalence.

If we choose a space $A$ such that $A \neq \ast$ but $\Sigma A = \ast$ (the 2-skeleton of the Poincaré homology 3 spheres, for instance), then $A \ast A = \Sigma A \land A = \ast$ and $g_k$ is a homotopy equivalence for all $k > 0$, so $\text{relcat} i_X = 1$ and $\text{relcat} i_X = k$ for all $k > 0$. However, if we chose a simply connected CW-complex $A$ (in order that $A \ast \cdots \ast A (k+1 \text{ times}) \neq A \ast \cdots \ast A (k \text{ times})$), then $\text{relcat} i_X = k + 1$ for all $k$.

**Example 14.** Consider any CW-complex $X$ with $\text{cat} X = 1$ and the map $i_X : \ast \to X$. We have $\text{secat} i_X = \text{relcat} i_X = \text{cat} X = 1$. 


Let us compute $\text{cat}_1 X = \text{relcat}_1 X$. Notice that $G_1(X) = \Sigma \Omega X$. By Theorem 9, we know that $1 \leq \text{cat}_1 X \leq 2$. But we cannot have $\text{cat}_1 X = 1$ because $g_1$ is not a homotopy equivalence, so $\text{cat}_1 X = 2$. By the way, we can say that $\gamma_k^2 : \Sigma \Omega X \to G_2(X)$ factorizes up to homotopy through $g_1 : \Sigma \Omega X \to X$.

Example 15. More generally, if $\text{relcat}_1 X = 1$, we have $k \leq \text{relcat}_1 X \leq 1 + k$ for any $k$ by Corollary 10. Thus, $\text{relcat}_1 X = k + 1$ while $g_1(\text{relcat}_1 X)$ is not a homotopy equivalence (and if any $n$ exists such that $g_n(\text{relcat}_1 X)$ is a homotopy equivalence, then $\text{relcat}_k X = k$ for all $k \geq n$).

Suppose we are given any map $\iota_X : A \to X$ with $\text{secat}(\iota_X) \leq n$ and any homotopy section $\sigma : X \to G_n$ of $g_n : G_n \to X$. For any $k \leq n$, consider the following homotopy pullbacks:

\[
\begin{array}{c}
Q \xrightarrow{\pi} G_k \\
\downarrow \quad \downarrow \\
G_k \xrightarrow{\sigma} H^k_n \xrightarrow{\eta_k^n} G_k \\
\downarrow \quad \downarrow \\
X \xrightarrow{\sigma} G_n \xrightarrow{\gamma_k^n} G_k
\end{array}
\]

where $\theta_k^n = (\gamma_k^n, \text{id}_{G_k})$ is the whisker map induced by the homotopy pullback $H^k_n$. Notice that $\eta_k^n \circ \theta_k^n = \text{id}_{G_k}$. By the Prism lemma, we know that the homotopy pullback of $\sigma$ and $\eta_k^n$ is indeed $G_k$ and that $\eta_k^n \circ \sigma = \text{id}_{G_k}$. Also notice that $\pi = \pi'$ since $\pi = \eta_k^n \circ \theta_k^n \circ \pi = \eta_k^n \circ \sigma \circ \pi' = \pi'$.

Proposition 16. For any map $\iota_X : A \to X$ with $\text{secat}(\iota_X) \leq n$ and any homotopy section $\sigma : X \to G_n$ of $g_n : G_n \to X$, with the same definitions and notations as above, the following conditions are equivalent:

(i) $\sigma \circ g_k = \gamma_k^n$.
(ii) $\pi$ has a homotopy section.
(iii) $\pi$ is a homotopy epimorphism.
(iv) $\theta_k^n = \sigma$.

Proof. We have the following sequence of implications:

(i) $\Rightarrow$ (ii): since $\sigma \circ g_k = \gamma_k^n = \eta_k^n \circ \theta_k^n \circ \text{id}_{G_k}$, we have a whisker map $(g_k, \text{id}_{G_k}) : G_k \to Q$ induced by the homotopy pullback $Q$ which is a homotopy section of $\pi$.

(ii) $\Rightarrow$ (iii): it is obvious.

(iii) $\Rightarrow$ (iv): we have $\theta_k^n \circ \pi = \sigma \circ \pi' = \sigma \circ \pi$ since $\pi = \pi'$. Thus, $\theta_k^n = \sigma$ since $\pi$ is a homotopy epimorphism.

(iv) $\Rightarrow$ (i): we have $\sigma \circ g_k = \eta_k^n \circ \sigma = \eta_k^n \circ \theta_k^n = \gamma_k^n$.

\[
\square
\]

Theorem 17. Let $\iota_X : A \to X$ be a $(q - 1)$-connected map. If for some $k \leq \text{secat}_1 \iota_X$, $G_k$ has the homotopy type of a CW-complex with dimension strictly less than $(\text{secat}_1 \iota_X + 1)q - 1$, then $\text{relcat}_1 \iota_X = \text{secat}_1 \iota_X$ for all $j \leq k$.

This is an immediate consequence of the following.

Proposition 18. Let $\iota_X : A \to X$ be a $(q - 1)$-connected map with $\text{secat}_1 \iota_X \leq n$. If for some $k \leq n$, $G_k$ has the homotopy type of a CW-complex with dimension strictly less than $(n + 1)q - 1$, then $\sigma \circ g_k = \gamma_k^n$ for any homotopy section $\sigma$ of $g_n$, so $\text{relcat}_1 \iota_X \leq n$.

Proof. Recall that, for any $i \geq 0$, $g_i$ is the $(i + 1)$-fold join of $\iota_X$. Thus, by [11, Theorem 47], we obtain that $g_i : G_i \to X$ is $(i + 1)q - 1$-connected. As $g_i$ and $\eta_i^n$ have the same homotopy fiber, which is $(i + 1)q - 2$-connected, we see that $\eta_i^n : H^i_n \to G_k$ is $(i + 1)q - 1$-connected, too. By [12, Theorem IV.7.16], this means that, for every CW-complex $K$ with $\dim K < (i + 1)q - 1$, $H^i_n$ induces a one-to-one correspondence $[K, H^i_n] \to [K, G_k]$. Apply this to $K = G_n$ and $i = n$ since $\theta_k^n$ and $\sigma$ are both homotopy sections of $\eta_k^n$, we obtain $\theta_k^n = \sigma$, and Proposition 16 gives the desired result.

Example 19. Let $X$ be the Eilenberg-Mac Lane space $K(Q, 1)$. It is known that $\text{cat}(X) = 2$ and that $G_1(X) = \Sigma \Omega X$ has the homotopy type of a wedge of circles (see [8, Example 1.9 and Remark 1.62], for instance). By Theorem 9, we know that $2 \leq \text{cat}_1 X \leq 3$. Because $\dim G_1(X) = 1 < \text{cat}(X + 1) - 1 = 2$, we have $\sigma \circ g_k = \gamma_k^n$ for any homotopy section $\sigma$ of $g_k(X)$ and, thus, $\text{cat}_1 X = 2$. Moreover, $g_k$ is never a homotopy equivalence, so $\text{cat}_k X > k$ for any $k$; thus, $\text{cat}_k X = k + 1$ for $k \geq 1$.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References


