Research Article

Riordan Matrix Representations of Euler’s Constant $\gamma$ and Euler’s Number $e$

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We show that the Euler-Mascheroni constant $\gamma$ and Euler’s number $e$ can both be represented as a product of a Riordan matrix and certain row and column vectors.

Dedicated to David Harold Blackwell (April 24, 1919–July 8, 2010)

1. Introduction

It was shown by Kenter [1] that the Euler-Mascheroni constant

$$\gamma = \lim_{n \to \infty} \left[ \left( \sum_{m=1}^{n} \frac{1}{m} \right) - \ln n \right] = 0.5772156649 \cdots \quad (1)$$

can be represented as a product of an infinite-dimensional row vector, the inverse of a lower triangular matrix, and an infinite-dimensional column vector:

$$\left( \begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{array} \right)^{-1} \cdot \left( \begin{array}{c} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \cdots \\ \frac{1}{n+1} \\ \cdots \\ \end{array} \right). \quad (2)$$

Kenter’s proof uses induction, definite integrals, convergence of power series, and Abel’s Theorem. In this paper, we recast this statement using the language of Riordan matrices. We exhibit another proof as well as a generalization. Our main result is the following.

**Theorem 1.** Consider sequences $\{a_0, a_1, \ldots, a_n, \ldots\}$, $\{b_0, b_1, \ldots, b_n, \ldots\}$, and $\{c_0, c_1, \ldots, c_n, \ldots\}$ of complex numbers such that $a_0, b_0, c_0 \neq 0$, as well as an integer exponent $d$. Assume that
(i) the power series \( a(x) = \sum_n a_n x^n \), \( b(x) = \sum_n b_n x^n \),
\( c(x) = \sum_n c_n x^n \), and \( b(x)^d \) are convergent in the
interval \(|x| < 1\);

(ii) the following complex residue exists:
\[
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right] = \frac{1}{2\pi i} \oint_{|z|=1} a(z) b(z^{-1})^d c(z^{-1}) \frac{1}{z} dz.
\]

Then, the matrix product
\[
\begin{pmatrix}
b_0 & a_1 & a_2 & \cdots & a_n & \cdots \\
b_1 & b_0 & & & & \cdots \\
b_2 & b_1 & b_0 & & & \cdots \\
\vdots & \vdots & \vdots & \ddots & & \vdots \\
b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots 
\end{pmatrix}^d
\]

\[
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_n \\
\vdots 
\end{pmatrix}
\]

is equal to the above residue.

The infinite-dimensional lower triangular matrix is an
element of a Riordan matrix. Specifically, it is that Riordan
matrix associated with the power series \( b(x)^d \). Kenter's result
follows by careful analysis of the power series:

\[
a(x) = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2} x + \frac{1}{3} x^2 + \cdots + \frac{1}{n+1} x^n + \cdots,
\]

\[
b(x)^{-1} = -\frac{x}{\log(1-x)} = 1 - \frac{1}{2} x - \frac{1}{12} x^2 - \frac{1}{24} x^3 - \cdots - L_n x^n - \cdots,
\]

\[
c(x) = \frac{a(x) - 1}{x} = \frac{1}{2} + \frac{1}{3} x + \frac{1}{4} x^2 + \cdots + \frac{1}{n+2} x^n + \cdots.
\]

The coefficients \( L_n \) are sometimes called the "logarithmic
numbers" or the "Gregory coefficients"; these are basically the
Bernoulli numbers of the second kind up to a choice of sign.
(Kenter employs the coefficients \( c_k = -L_k \).) The idea of this
paper is that we have the matrix product

\[
\begin{pmatrix}
1 & & & & & \\
1 & 1 & & & & \vdots \\
\vdots & \vdots & \ddots & & & \\
1 & 1 & \cdots & 1 & & \\
1 & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 & \\
\vdots & \vdots & \cdots & \cdots & \ddots & \vdots 
\end{pmatrix}^{-1}
\]

\[
\begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{4} \\
\vdots \\
\frac{1}{n+1} \\
\vdots 
\end{pmatrix}
\]

which is equivalent to the recursive identity \( \sum_{m=0}^{n-1} L_m/(n - m) = 0 \),
which is valid whenever \( n = 2, 3, 4, \ldots \). The matrix
product, and hence the recursive identity, can be derived
from properties of Riordan matrices. Kenter's result follows from
the identity \( \sum_{m=1}^{\infty} L_m/m = \gamma \), in which turn follows
from an identity involving a definite integral.

As another consequence of our main result, we can also
show that Euler's number

\[
e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = 2.7182818284 \cdots
\]

can be represented as a product of an infinite-dimensional
row vector, a lower triangular matrix, and an infinite-
dimensional column vector.

**Corollary 2.** For any integers \( p, q, \) and \( d \) with \( pq > 1 \), the
number

\[
\frac{pq}{pq - 1} \sqrt{e^d} = \lim_{n \to \infty} \left[ \frac{pq}{pq - 1} \left( 1 + \frac{1}{pn} \right)^{dn} \right]
\]

is equal to the matrix product
2. Introduction to Riordan Matrices

We wish to list several key results in the theory of Riordan matrices. To do so, we recast this theory using techniques from representation theory very much in the spirit of Bacher [2]. Our ultimate goal in this section is to explain how Riordan matrices are connected to a permutation representation \( \pi : G \rightarrow GL(V) \) of a certain group \( G \), namely, the Riordan group, acting on an infinite-dimensional vector space \( V \), namely, the collection of those formal power series \( h(x) \) in \( \mathbb{C}[x] \), where \( h(0) = 0 \).

2.1. Group Actions. Before developing the representation theoretic view, we give the definition of a Riordan matrix and few related useful properties. Let \( G \) be a field; it is customary to set \( k = \mathbb{C} \) as the set of complex numbers, but, in practice, \( k = \mathbb{Q} \) is the set of rational numbers. Set \( k[[x]] \) as the collection of formal power series in an indeterminate \( x \); we will view this as a \( k \)-vector space with countable basis \( \{1, x, x^2, \ldots, x^n, \ldots\} \). For most of this article, we will not be concerned with regions of convergence for these series.

There are three binary operations \( k[[x]] \times k[[x]] \rightarrow k[[x]] \) which will be of importance to us, namely, multiplication \( \bullet \), composition \( \circ \), and addition \( + \). Explicitly, if we write

\[
(f \cdot g)(x) = \sum_{n=0}^{\infty} f_n g_n x^n, \quad (f \circ g)(x) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} g_m f_{n-m} \right) x^n, \quad (f + g)(x) = \sum_{n=0}^{\infty} (f_n + g_n) x^n,
\]

then we have the formal power series

\[
(f \cdot g)(x) = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} f_m g_{n-m} \right] x^n,
\]

In the process of proving these generalizations, we present a representation theoretic view of Riordan matrices. That is, we consider the matrices as representations \( \pi : G \rightarrow GL(V) \) of a certain group \( G \), namely, the Riordan group, acting on an infinite-dimensional vector space \( V \), namely, the collection of those formal power series \( h(x) \) in \( \mathbb{C}[x] \), where \( h(0) = 0 \).

Proposition 3. Define the subsets

\[
H = \{ f(x) \in k[[x]] \mid f(0) \neq 0 \}, \\
K = \{ g(x) \in k[[x]] \mid g(0) = 0 \text{ yet } g'(0) \neq 0 \}, \\
V = \{ h(x) \in k[[x]] \mid h(0) = 0 \}.
\]

(i) \( H \) is a group under multiplication \( \bullet \), \( K \) is a group under composition \( \circ \), and \( V \) is a group under addition \( + \). In particular, \( V \) is a \( k \)-vector space with countable basis \( \{x, x^2, \ldots, x^n, \ldots\} \).

(ii) The map \( \varphi : K \rightarrow \text{Aut}(H) \) which sends \( g(x) \) to the automorphism \( \varphi_g : f(x) \mapsto (f \circ g)(x) \) is a group homomorphism, where \( \overline{g}(x) \) is the compositional inverse of \( g(x) \). In particular, \( G = H \rtimes K \) is a group under the binary operation \( \ast : G \times G \rightarrow G \) defined by

\[
(f_1, g_1) \ast (f_2, g_2) = (f_1 \cdot g_1 f_2 \circ g_2, g_1 \circ g_2).
\]

(iii) The map \( \ast : G \times V \rightarrow V \) defined by \( (f, g) \ast h = f \ast (h \circ \overline{g}) \) is a group action of \( G \) on \( V \).

We use \( \overline{g}(x) \) to denote the compositional inverse \( g^{-1}(x) \) so that we will not confuse this with the multiplicative inverse \( g(x)^{-1} \). Later, we will show that \( G \) is isomorphic to the Riordan group \( R \). Moreover, we will show that \( H \), a normal subgroup of \( G \), is isomorphic to the Appell subgroup of \( R \). The motivation of this result is to use the action of \( G \) on \( V \) to write down a permutation representation \( \pi : G \rightarrow GL(V) \) and then use the canonical basis \( \{x, x^2, \ldots, x^n, \ldots\} \) of \( V \) to list infinite-dimensional matrices.
Proof. We show (i) to fix some notation to be used in the sequel. Since \((f \circ g)(0) = f(0)g(0) \neq 0\) for any \(f, g) \in H, we see that \(\bullet : H \times H \rightarrow H\) is an associative binary operation. The identity is the constant power series \(e(x) = 1\), and the inverse of \(f(x)\) is its reciprocal, seen to be a power series in \(x\) by definition:

\[
\frac{1}{f(x)} = 1 - \sum_{n=1}^{\infty} \left( -\frac{f_n}{f_0} \right) x^n.
\]

Since \((f \circ g)(0) = f(0)g(0) = 0\) and \((f \circ g)'(0) = f'(0)g'(0) \neq 0\) for any \(f, g) \in K, we see that \(\circ : K \times K \rightarrow K\) is an associative binary operation. The identity is the constant power series \(1 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n\) having the implicitly defined coefficients

\[
\begin{align*}
\bar{g}_0 &= 0, \\
\bar{g}_1 &= \frac{1}{g_1},
\end{align*}
\]

\[
\sum_{m=0}^{n} \bar{g}_m \left[ \sum_{n_1 + \cdots + n_m = m} g_{n_1} \cdots g_{n_m} \right] = 0 \quad \text{for } n = 2, 3, \ldots
\]

Since \((f + g)(0) = f(0) + g(0) = 0\) for any \(f, g) \in V, we see that \(+: V \times V \rightarrow V\) is an associative binary operation. The identity is the constant power series \(0\), and the inverse of \(g(x)\) is its compositional inverse \(\bar{g}(x) = \sum_{n=0}^{\infty} \frac{g_n}{n!} x^n\) having the coefficient relation

\[
\begin{align*}
\bar{g}_0 &= 0, \\
\bar{g}_1 &= \frac{1}{g_1},
\end{align*}
\]

\[
\sum_{m=0}^{n} \frac{\bar{g}_m}{g_1} \left[ \sum_{n_1 + \cdots + n_m = m} g_{n_1} \cdots g_{n_m} \right] = 0 \quad \text{for } n = 2, 3, \ldots
\]

The semidirect product \(G = H \ltimes K\) is indeed a group homomorphism. The semidirect product \(G = H \ltimes K\) consists of pairs \((f(x), g(x))\) with \(f(x) \in H\) and \(g(x) \in K\), where the binary operation \(* : G \times G \rightarrow G\) is defined by

\[
(f_1(x), g_1(x)) \ast (f_2(x), g_2(x)) = (f_1(x) f_2(\bar{g}(x)), g_1(g_2(x))).
\]

Finally, we show (iii). The map \(* : G \times V \rightarrow V\) is defined as the formal identity

\[
(f(x), g(x)) \ast h(x) = f(x) h(\bar{g}(x)).
\]

Since \([(f, g) \ast h](0) = f(0)h(\bar{g}(0)) = f(0)h(0) = 0\), we see that the map \(* : G \times V \rightarrow V\) is well defined. As the identity element of \(G\) is \((e(x), id(x)) = (1, x)\), we see that \((e(x), id(x)) \ast h(x) = h(x)\) so that it acts trivially on \(V\). Given two elements \((f_1, g_1), (f_2, g_2) \in G\) and \(h(x) \in V\), we have the identity

\[
\begin{align*}
(f_1(x), g_1(x)) \ast [(f_2(x), g_2(x)) \ast h(x)] &= (f_1(x), g_1(x)) \ast [f_2(x) h(\bar{g}(x))] \\
&= f_1(x) f_2(\bar{g}(x)) h(\bar{g}(x)) \\
&= f_1(x) f_2(\bar{g}(x)) h(\bar{g}(x)) \\
&= (f_1(x), g_1(x)) \ast (f_2(x), g_2(x)) \ast h(x).
\end{align*}
\]

Similarly, given two elements \(h_1(x), h_2(x) \in V\) and \((f, g) \in G\), we have the identity

\[
\begin{align*}
(f(x), g(x)) \ast [h_1(x) + h_2(x)] &= f(x) [h_1(\bar{g}(x)) + h_2(\bar{g}(x))] \\
&= (f(x), g(x)) \ast h_1(x) + (f(x), g(x)) \ast h_2(x).
\end{align*}
\]

Hence, \(* : G \times V \rightarrow V\) is indeed a group action.

\[
\text{2.2. Riordan Matrices.} \text{ Recall that the set}
\]

\[
V = \{h(x) \in k[x] \mid h(0) = 0\}
\]

is a \(k\)-vspan \(\{x, x^2, \ldots, x^m, \ldots\}\). Since the semidirect product \(G = H \ltimes K\) acts on \(V\), we have a “permutation” representation \(\pi : G \rightarrow GL(V)\). Explicitly, this representation is defined on the basis elements of \(V\) via the formal identity

\[
(f(x), g(x)) \ast x^m = f(x) [\bar{g}(x)]^m = \sum_{n=1}^{\infty} l_{nm} x^n
\]

(Recall that \(\bar{g}(x)\) is the compositional inverse of \(g(x)\)). The matrix with respect to the basis \(\{x, x^2, \ldots, x^m, \ldots\}\) is given by the lower triangular matrix

\[
\pi(f(x), g(x)) = \begin{pmatrix}
1_{1,1} & l_{2,1} & l_{2,2} \\
0 & 1_{3,1} & l_{3,2} & l_{3,3} \\
\vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & l_{n,1} & l_{n,2} & \cdots & l_{n,m}
\end{pmatrix}.
\]

Recall that \(g(0) = 0\) yet \(f(0), g'(0) \neq 0\). The following result explains the main multiplicative property of these matrices.
Theorem 4. Continue notation as above.

(i) \( \pi : G \to GL(V) \) is a group homomorphism. That is,

\[
\pi(f_1(x), g_1(x)) \pi(f_2(x), g_2(x)) = \pi(f_1(x) f_2(\overline{g_1}(x)), g_1(g_2(x))).
\]

(ii) For a generating function \( t(x) = t_0 + t_1 x + t_2 x^2 + \cdots \) with \( t_0 \neq 0 \),

\[
\pi(f(x), g(x)) \pi(t(x), id(x)) = \sum_{p=1}^{m} l_{n,p} t_{p-m}.
\]

Such matrices \( \pi(f, g) \) are called the Riordan matrices associated with the pair \((f, g)\). The collection \( R \) of Riordan matrices is a group which is isomorphic to \( G = H \rtimes \phi K \); this is the Riordan group. The collection of matrices \( \pi(f, id) \) is a group which is isomorphic to \( H \); this normal subgroup is the Appell subgroup of \( R \).

Proof. We show (i). In the proof of Proposition 3, we found that for each \( h(x) \in V \) we have the following formal identity involving power series as elements of \( k[x] \):

\[
(f_1(x), g_1(x)) * [(f_2(x), g_2(x)) * h(x)] = [(f_1(x), g_1(x)) * (f_2(x), g_2(x))] * h(x)
\]

In particular, this holds for the basis elements \( h(x) = x^n \), so the result follows.

Now, we show (ii). For a generating function \( t(x) = t_0 + t_1 x + t_2 x^2 + \cdots \), we have the product

\[
(t(x), id(x)) * x^m = t(x) x^m = \sum_{n=1}^{\infty} t_{n-m} x^n;
\]

so matrices in the Appell subgroup are in the form

\[
\pi(t(x), id(x)) = \begin{pmatrix} t_0 & t_1 & t_0 & t_1 & t_0 & \cdots \ \cdots & \cdots & \cdots & \cdots \ t_{n-1} & t_{n-2} & \cdots & t_0 \ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\]

This gives the matrix product

\[
\pi(f(x), g(x)) \pi(t(x), x) = (l_{np})_{n,p \geq 1} (t_{p-m})_{p,m \geq 1}
\]

so the result follows.
Since we may use Theorem 4 to conclude that \(\pi(f, \text{id})^{-1} = \pi(1/f, \text{id})\), we find the identity
\[
\begin{pmatrix}
1 & 1 & 1 & \cdots \\
1/2 & 1 & 1 & \cdots \\
1/3 & 1/2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
1/n & 1/(n-1) & 1/(n-2) & \cdots & 1
\end{pmatrix}^{-1}
\]
(36)
These matrices are elements of the Appell subgroup of \(\mathbb{R}\).

2.4. Relation with Standard Notation. Standard references for Riordan matrices are Shapiro et al. [3] and Sprugnoli [4, 5]. The notation \(\pi(f, g)\) employed above is not the typical one, so we explain the connection. Consider sequences \(\{4, 5\}\). Then the notation for Riordan matrices are Shapiro et al. [3] and Sprugnoli [2.4]. Relation with Standard Notation. The matrices are elements of the Appell subgroup of \(\mathbb{R}\).

(i) The product of Riordan matrices is again a Riordan matrix. Explicitly, their product satisfies the relation
\[
\begin{bmatrix}
G_1(x) & F_1(x) \\
G_2(x) & F_2(x)
\end{bmatrix}
= \begin{bmatrix}
G_1(x)G_2(F_1(x)) & F_2(F_1(x))
\end{bmatrix}
\]
(39)

(ii) For a generating function \(T(x) = T_0 + T_1x + T_2x^2 + \cdots\) with \(T_0 \neq 0\), one has the product
\[
\begin{bmatrix}
G(x) & F(x)
\end{bmatrix}
[T(x), x] = \left( \sum_{p=1}^{m} \sum_{l=0}^{p} t_{p-m} \right)_{nm \geq 1}
\]
(40)

Proof. Statement (i) is shown in [3, Eq. 5] and [6, Proof of Thm. 2.1], but we give an alternate proof. Upon denoting \(f_i(x) = G_i(x)\) and \(g_i(x) = \overline{T_i}(x)\) for \(i = 1\) and \(2\), we find the relation
\[
\begin{bmatrix}
G_1(x) & F_1(x) \\
G_2(x) & F_2(x)
\end{bmatrix}
\]
= \begin{bmatrix}
\pi(f_1(x), g_1(x)) \pi(f_2(x), g_2(x)) \\
\pi(f_1(x), g_2(x)) \pi(g_1(x), g_2(x))
\end{bmatrix}
(41)
which follows directly from Theorem 4. Statement (ii) is also shown in [6], but it follows directly from Theorem 4 as well.

3. Proof of Kenter’s Result and Generalizations

3.1. Main Result. We now prove Theorem 1.

Proof of Theorem 1. With the three power series \(a(x) = \sum_{n=0}^{\infty} a_n x^n\), \(b(x) = \sum_{n=0}^{\infty} b_n x^n\), and \(c(x) = \sum_{n=0}^{\infty} c_n x^n\) convergent in the interval \(|x| < 1\), consider the power series
\[
f(x) = b(x)^d c(x) = \sum_{n=0}^{\infty} f_n x^n \quad \text{where} \quad |x| < 1.
\]
(42)
As elements of the Appell subgroup of \(\mathbb{R}\), we invoke Theorem 4 to see that we have the matrix product \(\pi(f(x), x) = \pi(b(x), x)^d \pi(c(x), x)\). In particular, the first column is given by
\[
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}
\]
Corollary 5 (fundamental theorem of the Riordan group [3, 5, 6]). Continue notation as above.
The corollary follows now from Theorem 1.

Kenter's result is also an application of Theorem 1.

3.2. Applications. We explain how to use Theorem 1 in order to express Euler's number $e = 2.7182818284 \cdots$ in terms of Riordan matrices.

**Proof of Corollary 2.** The coefficients of the matrices in (9) correspond to the three power series

$$
a(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!},
$$

$$
b(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},
$$

$$
c(x) = \frac{1}{1-\frac{x}{q}} = \sum_{n=0}^{\infty} \frac{x^n}{q^n},
$$

where $|x| < 1$.

For a complex number $z$ with $|z| < 1$, we have the identity

$$
a(z) b(z^{-1})^d c(z^{-1})
= \left[ \frac{1}{1-z/p} \right]^{d-1} \left[ \frac{1}{1-z^{-1}/q} \right]^{d-1}
= \sum_{n=-\infty}^{\infty} \sum_{n_1, n_2, n_3 = 0}^{\infty} \frac{d^{n_3}}{p^{n_1} q^{n_2}} z^n.
$$

The residue corresponds to the coefficient of the $z^{-1}$ term, so we consider the terms where $n = -1$:

$$
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^d c(z^{-1})}{z} \right] = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \frac{d^{n_3}}{p^{n_1} q^{n_2}}
$$

The corollary follows now from Theorem 1.
Corollary 6 (see [1]). The Euler-Mascheroni constant
\[ \gamma = \lim_{n \to \infty} \left( \sum_{m=1}^{n} \frac{1}{m} - \ln n \right) = 0.5772156649 \cdots \] (50)
is equal to the matrix product
\[ \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \frac{1}{n} & \cdots \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 & \cdots & \frac{1}{n-1} & \frac{1}{n-2} & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & \cdots & \frac{1}{n-2} & \frac{1}{n-3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \\ \frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 & \cdots & 1 \end{pmatrix}^{-1} \] (51)

Proof. The coefficients of the matrices above correspond to the three power series
\[ a(x) = -\log \left( \frac{1-x}{x} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \]
\[ b(x) = -\log \left( \frac{1-x}{x} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \] (52)
\[ c(x) = \frac{a(x) - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n+2} \]
where \( |x| < 1 \).

We will choose the exponent \( d = -1 \). We will express the reciprocal as the power series
\[ \frac{x}{\log(1-x)} = -1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^3 + \frac{19}{720}x^4 \]
\[ + \frac{3}{160}x^5 + \cdots = \sum_{n=0}^{\infty} L_n x^n \] (53)
which is also convergent in the interval \( |x| < 1 \). (Recall that the coefficients \( L_n \) are sometimes called the "logarithmic numbers" or the "Gregory coefficients.") For a complex number \( z \) with \( |z| < 1 \), we have the identity
\[ \frac{a(z)b\left(\frac{z^{-1}}{d}\right)c\left(\frac{z^{-1}}{d}\right)}{z} = -\frac{\log(1-z)}{z} + \left[ -\frac{\log(1-z)}{z} \right] \]
\[ \cdot \frac{z^{-1}}{\log(1-z^{-1})} \]
\[ = \sum_{n=0}^{\infty} \frac{z^n}{n+1} + \sum_{n=-\infty}^{\infty} \left[ \sum_{m=-n}^{\infty} \frac{L_m}{n+m+1} \right] z^n. \]

The residue corresponds to the coefficient of the \( z^{-1} \) term, so we consider the terms where \( n = -1 \):
\[ \text{Res}_{z=0} \left[ \frac{a(z)b\left(\frac{z^{-1}}{d}\right)c\left(\frac{z^{-1}}{d}\right)}{z} \right] = \sum_{m=1}^{\infty} \frac{L_m}{m} \] (54)
\[ = \int_{0}^{1} \left[ \sum_{m=1}^{\infty} L_m x^{m-1} \right] dx \]
\[ = \int_{0}^{1} \left[ \frac{1}{x} + \frac{1}{\log(1-x)} \right] dx = \gamma. \] (55)
The corollary follows now from Theorem 1.

We conclude by stating that Theorem 1 can also be used to show Riordan matrix representations for \( \ln 2 \) and \( \pi^2/6 \). Finding matrix representations of other constants, like \( \sqrt{2} \), \( \pi \), and the Golden Ratio \( \phi \), is of interest.

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Competing Interests
The authors declare that they have no competing interests.

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