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Research Article

Fixed Point Theorems for Geraghty Type Contractive Mappings and Coupled Fixed Point Results in 0-Complete Ordered Partial Metric Spaces

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We establish new fixed point theorems in 0-complete ordered partial metric spaces. Also, we give remark on coupled generalized Banach contraction. Some examples illustrate the usability of our results. The theorems presented in this paper are generalizations and improvements of the several well known results in the literature.

1. Introduction and Preliminaries

Henceforward, the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will indicate the set of real numbers, the set of nonnegative real numbers, and the set of positive integer numbers, respectively.

Definition 1 (see [1]). A partial metric on a nonempty set X is a function $P : X^2 \to \mathbb{R}^+$ such that, for all $x, y, z \in X$, $(P1)$ $x = y \Leftrightarrow P(x,x) = P(x,y) = P(y,y)$, $(P2)$ $P(x,x) \le P(x,y)$, $(P3)$ $P(x,y) = P(y,x)$, and $(P4)$ $P(x,y) \le P(x,z) + P(z,y) - P(z,z)$. The pair (X,P) is called a partial metric space.

If P is a partial metric on X, then the function $P^s : X^2 \to \mathbb{R}^+$ given by $P^s(x,y) = -P(y,y) - P(x,x) + 2P(x,y)$ is a metric on X. Each partial metric P on X introduces a T_0 topology τ_P on X which has as a base the family of open balls $B_P(x,\varepsilon) = \{p \in X : P(x,p) < P(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Let (X,P) be a partial metric space, and let $\{x_n\}$ be any sequence in X and $x \in X$. Then (i) a sequence $\{x_n\}$ is convergent to x with respect to τ_P, if $P(x_n,x) \to P(x,x)$ as $n \to \infty$, (ii) a sequence $\{x_n\}$ is a Cauchy sequence in (X,P) if $\lim_{n,m\to\infty} P(x_n,x_m)$ exists and is finite; (iii) (X,P) is called complete if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $P(x_n,x_m) \to P(x,x)$ as $n,m \to \infty$.

Romaguera [2] introduced the notion of 0-Cauchy sequence, 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness. After that many authors extended the results of [2] and studied fixed point theorems in 0-complete partial metric space (see [2–10]).

Definition 2 (see [2]). Let (X,P) be a partial metric space. A sequence $\{x_n\}$ in X is called a 0-Cauchy sequence if $P(x_n,x_m) \to 0$ as $n,m \to \infty$. The space (X,P) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_P to a point $x \in X$ such that $P(x,x) = 0$.

Remark 3 (see [11, 12]). (1) Let (X,P) be a partial metric space. If $P(x_n,\rho) \to P(\rho,\rho) = 0$ as $n \to \infty$, then $P(x_n,y) \to P(\rho,y)$ as $n \to \infty$ for all $y \in X$.

(2) If $f : X \to X$ is a continuous at ρ, then for each sequence $\{x_n\}$ in X, we have $P(x_n,\rho) \to P(\rho,\rho) = 0$ as $n \to \infty \Rightarrow P(fx_n,f\rho) \to P(f\rho,f\rho) = 0$ as $n \to \infty$ (see [5]).

Let Ŧ be the class of functions $\theta : [0,\infty) \to [0,1)$ with $\theta(t_n) \to 1$ implying $t_n \to 0$. Amini-Harandi and Emami [13] presented the following results.

Theorem 4 (see [13]). *Let $(X,\preccurlyeq)$ be an ordered set endowed with a metric d and let $f : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(i) *(X,d) is complete.*

(ii)

(1) *f is continuous or*

(2) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*

(iii) *f is nondecreasing.*

(iv) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$.*

(v) *There exists $\theta \in \mathbb{F}$ such that for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(fx, fy) \leq \theta(d(x, y))\, d(x, y). \tag{1}$$

Then f has a fixed point. Moreover, if for all $(x, y) \in X^2$ there exists a $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

In this paper, we establish new fixed point theorems in 0-complete ordered partial metric spaces (briefly 0-*COPMS*). Also, we give remark on coupled generalized Banach contraction. Some examples illustrate the usability of our results. The theorems presented in this paper are generalizations and improvements of the several well known results in the literature.

2. Main Results

Theorem 5. *Let (X, P) be a 0-COPMS. Let $f : X \to X$ be a nondecreasing mapping such that*

$$P(fx, fy) \leq \theta(P(x, y))\, P(x, y) \tag{2}$$

for all $x, y \in X$ with $x \succcurlyeq y$ (or $y \succcurlyeq x$) and $\theta \in \mathbb{F}$. Also suppose that there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$ (or $x_0 \succcurlyeq fx_0$). One assumes

(1) *f is continuous or*

(2) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preccurlyeq x$ (or $x_n \succcurlyeq x$) for all $n \in \mathbb{N}$.*

Then f has a fixed point ρ.

Proof. By assumption there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$. Define $x_1 \in X$ as $x_1 = fx_0$. Then we have $x_1 \succcurlyeq x_0$. In a similar manner, we get $x_2 \in X$ as $x_2 = fx_1$. In that case, $x_2 = fx_1 \succcurlyeq fx_0 = x_1$. Continuing this procedure we have $\{x_n\}$ in X such that

$$fx_n = x_{n+1}, \quad n \in \mathbb{N}. \tag{3}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then the proof is completed. Suppose farther that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Consider, as f is nondecreasing, we obtain that

$$\begin{aligned} x_0 \preccurlyeq fx_0 = x_1 \preccurlyeq fx_1 = x_2 \preccurlyeq \cdots \preccurlyeq x_n = fx_{n-1} \preccurlyeq x_{n+1} \\ = fx_n \preccurlyeq \cdots. \end{aligned} \tag{4}$$

From (2), (3), and (4), for all $n \geq 1$, we get that

$$\begin{aligned} P(x_{n+1}, x_{n+2}) &= P(fx_n, fx_{n+1}) \\ &\leq \theta(P(x_n, x_{n+1}))\, P(x_n, x_{n+1}) \\ &\leq P(x_n, x_{n+1}). \end{aligned} \tag{5}$$

Then $\{P(x_n, x_{n+1})\}$ is a monotone decreasing. Hence $P(x_n, x_{n+1}) \to c \geq 0$ as $n \to \infty$. Assume $c > 0$. Then by (2) we have

$$\frac{P(x_{n+1}, x_{n+2})}{P(x_n, x_{n+1})} \leq \theta(P(x_n, x_{n+1})). \tag{6}$$

Equation (6) yields $\theta(P(x_n, x_{n+1})) \to 1$ as $n \to \infty$. By virtue of $\theta \in \mathbb{F}$, this implies that

$$P(x_n, x_{n+1}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{7}$$

Now we claim that $\{x_n\}$ is a 0-Cauchy sequence. Conversely, suppose that

$$\lim \sup_{n,m\to\infty} P(x_n, x_m) > 0. \tag{8}$$

By $(P4)$ and (2), we have, for $n > m$,

$$\begin{aligned} P(x_n, x_m) &\leq P(x_n, x_{n+1}) + P(x_{n+1}, x_{m+1}) \\ &\quad + P(x_{m+1}, x_m) - P(x_{m+1}, x_{m+1}) \\ &\quad - P(x_{n+1}, x_{n+1}) \\ &\leq P(x_n, x_{n+1}) + P(x_{m+1}, x_m) \\ &\quad + \theta(P(x_n, x_m))\, P(x_n, x_m) \\ &\leq \frac{P(x_n, x_{n+1}) + P(x_{m+1}, x_m)}{1 - \theta(P(x_n, x_m))}. \end{aligned} \tag{9}$$

Owing to (7) and (8), we get that

$$\lim_{n,m\to\infty} \frac{1}{1 - \theta(P(x_n, x_m))} = \infty, \tag{10}$$

from which we have $\lim_{n,m\to\infty}\theta(P(x_n, x_m)) \geq 1$ which implies $\lim_{n,m\to\infty}\theta(P(x_n, x_m)) = 1$. Since $\theta \in \mathbb{F}$, we obtain $\lim_{n,m\to\infty}P(x_n, x_m) = 0$. It is a contradiction. Thus $\{x_n\}$ is a 0-Cauchy sequence. As (X, P) is 0-complete, it follows that there exists $\rho \in X$ such that $x_n \to \rho$ in (X, P) and $P(\rho, \rho) = 0$. Furthermore,

$$\lim_{n\to\infty} P(x_n, \rho) = P(\rho, \rho) = 0. \tag{11}$$

We will show that $f\rho = \rho$. Consider two cases.

Case 1. If f is continuous, then

$$f\rho = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = \rho; \tag{12}$$

hence $f\rho = \rho$.

Case 2. If (2) holds, then,

$$\begin{aligned} P(\rho, f\rho) &\leq P(\rho, x_{n+1}) + P(x_{n+1}, f\rho) - P(x_{n+1}, x_{n+1}) \\ &\leq P(\rho, x_{n+1}) + P(x_{n+1}, f\rho) \\ &\leq P(\rho, x_{n+1}) + \theta(P(x_n, \rho))\, P(x_n, \rho) \\ &\leq P(\rho, x_{n+1}) + P(x_n, \rho). \end{aligned} \tag{13}$$

In view of $P(x_n, \rho) \to 0$ as $n \to \infty$, then we have $f\rho = \rho$. □

The following is example which illustrate Theorem 5 and that the generalizations are proper.

Example 6. Let $X = [0, +\infty) \cap \mathbb{Q}$, and let $P : X \times X \to \mathbb{R}^+$ be defined by $P(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, P) is a 0-*COPMS*. Yet it is not complete partial metric space. We endow X with the partial order

$$\begin{aligned} &x \preccurlyeq y \Longleftrightarrow \\ &x = y \\ &\text{or } x, y \in [0, 1] \quad \text{with } x \le y. \end{aligned} \tag{14}$$

Let $\theta(t) = (t + 1)^{-1}$ for all $t \ge 0$. Then it is clear that $\theta \in \mathbb{F}$. Define $f : X \to X$ as

$$fx = \begin{cases} \dfrac{x^3}{1 + x^3} & x \in [0, 1], \\ \dfrac{1}{x^2} & x > 1. \end{cases} \tag{15}$$

Assume that $y \preccurlyeq x$. Then we have two cases.

Case 1. If $x \in [0, 1]$ $(y \in [0, 1])$, then

$$\begin{aligned} P(fx, fy) &= \max\left\{\frac{x^3}{1 + x^3}, \frac{y^3}{1 + y^3}\right\} = \frac{x^3}{1 + x^3}, \\ P(x, y) &= \max\{x, y\} = x. \end{aligned} \tag{16}$$

Therefore, we have

$$\begin{aligned} \theta(P(x, y))\, P(x, y) - P(fx, fy) &= \frac{x}{1 + x} - \frac{x^3}{1 + x^3} \\ &= \frac{x - x^2}{1 + x^3} \ge 0. \end{aligned} \tag{17}$$

Hence, for $x \in [0, 1]$, $P(fx, fy) \le \theta(P(x, y))P(x, y)$.

Case 2. If $x > 1$ $(y = x)$, then

$$\begin{aligned} P(fx, fy) &= \max\left\{\frac{1}{x^2}, \frac{1}{y^2}\right\} = \frac{1}{x^2}, \\ P(x, y) &= \max\{x, y\} = x. \end{aligned} \tag{18}$$

Hence, we get

$$\begin{aligned} \theta(P(x, y))\, P(x, y) - P(fx, fy) &= \frac{x}{1 + x} - \frac{1}{x^2} \\ &= \frac{x^3 - x - 1}{x^3 + x^2} = 1 - k \ge 0, \end{aligned} \tag{19}$$

where

$$0 < k = \frac{x^2 + x + 1}{x^3 + x^2} < 1. \tag{20}$$

Thus, for $x > 1$, $P(fx, fy) \le \theta(P(x, y))P(x, y)$.

Moreover, by Cases 1 and 2, it is clear that both assumptions (1) and (2) of Theorem 5 are satisfied, and for $x_0 = 0$, we have $x_0 \preccurlyeq fx_0$. Hence, all assumptions of Theorem 5 are satisfied, and f has a fixed point $\rho = 0$.

On the contrary, consider Example 6 in the standard metric $d(x, y)$. If $x = 1/3$ and $y = 1$, then

$$\begin{aligned} d\left(f\frac{1}{3}, f1\right) &= \left|\frac{(1/3)^3}{1 + (1/3)^3} - \frac{1}{2}\right| = 0.46, \\ d\left(\frac{1}{3}, 1\right) &= \left|\frac{1}{3} - 1\right| = 0.66 \end{aligned} \tag{21}$$

and so

$$d\left(f\frac{1}{3}, f1\right) = 0.46 > 0,4 = \theta\left(d\left(\frac{1}{3}, 1\right)\right) d\left(\frac{1}{3}, 1\right). \tag{22}$$

Thus, $d(fx, fy) \le \theta(d(x, y))d(x, y)$ is not satisfied.

3. Remark on Coupled Generalized Banach Contraction

The following result generalizes and extends Theorem 2.1. in [14]. When making the proof of the theorem, Radenović's technique [15] is used.

Theorem 7. *Let (X, p) be a 0-COPMS and let $f : X^2 \to X$ be a mapping. Suppose that, for all $x, y, a, b \in X$ and $\theta \in \mathbb{F}$, the following condition*

$$\begin{aligned} p(F(x, y), F(a, b)) &\le \theta\left((p(x, a) + p(y, b)) \times 2^{-1}\right) \\ &\quad \cdot \left((p(x, a) + p(y, b)) \times 2^{-1}\right) \end{aligned} \tag{23}$$

holds. Then F has a fixed point.

Proof. Consider the metrics $P : X^2 \times X^2 \to [0, \infty)$ defined by

$$\begin{aligned} P(A, B) &= p(x, a) + p(y, b), \\ &\qquad \forall A = (x, y), B = (a, b) \in X^2. \end{aligned} \tag{24}$$

If (X^2, p) is complete, then (X^2, P) is complete (resp. 0-complete), too. Now, define the operator $T : X^2 \to X^2$ by

$$T(A) = (F(x, y), F(y, x)), \quad \forall A = (x, y) \in X^2. \tag{25}$$

Let $P' : X^2 \times X^2 \to [0, \infty)$ be a metric on X^2 defined by $P'(A, B) = P(A, B)/2$ for all $A = (x, y), B = (a, b) \in X^2$.

From (23), for all $(x, y), (a, b) \in X^2$ with $x \preccurlyeq a$ and $y \succcurlyeq b$, we get

$$\begin{aligned} p(F(x, y), F(a, b)) &\le \theta\left((p(x, a) + p(y, b)) \times 2^{-1}\right) \\ &\quad \cdot \left((p(x, a) + p(y, b)) \times 2^{-1}\right), \\ p(F(b, a), F(y, x)) &\le \theta\left((p(x, a) + p(y, b)) \times 2^{-1}\right) \\ &\quad \cdot \left((p(x, a) + p(y, b)) \times 2^{-1}\right). \end{aligned} \tag{26}$$

This implies that for all $(x, y), (a, b) \in X^2$ with $x \preccurlyeq a$ and $y \succcurlyeq b$,

$$\begin{aligned}&\{p(F(x,y), F(a,b)) + p(F(b,a), F(y,x))\} \times 2^{-1}\\&\quad \le \theta\left(\left(p(x,a) + p(y,b)\right) \times 2^{-1}\right)\\&\quad \cdot \left(\left(p(x,a) + p(y,b)\right) \times 2^{-1}\right);\end{aligned} \tag{27}$$

that is,

$$P'(T(A), T(B)) \le \theta\left(P'(A,B)\right) P'(A,B) \tag{28}$$

which is in fact condition (2), for all $A = (x, y)$, $B = (a, b) \in X^2$ with $A = (x, y) \succcurlyeq (a, b) = B$. Hence, all conditions of Theorem 5 are satisfied. In this case, applying Theorem 5, we have that T has a fixed point. From the definition of T, we have $x = F(x, y)$ and $y = F(y, x)$; that is to say, (x, y) is a coupled fixed point of F. □

Remark 8. If $\theta(t) = k$, $0 \le k < 1$ in the inequality (23), then we obtain results of Bhaskar and Lakshmikantham [16] in 0-*COPMS*.

The following example illustrates the case when Theorem 7 is applicable, while Theorem 2.1. in [14] is not.

Example 9. Let $X = [0, 1]$, and let $p : X^2 \to \mathbb{R}^+$ be defined by $p(x, y) = \max\{x, y\} + |x - y|$ for all $x, y \in X$. Then (X, P) is a 0-*COPMS*. Yet it is not complete partial metric space. We consider the following order relation on X:

$$\begin{aligned}&x, y \in X, \quad x \preccurlyeq y \Longleftrightarrow\\&\quad x = y\\&\text{or } (x, y) \in \{(0,0), (0,1), (1,1)\}.\end{aligned} \tag{29}$$

Let $\theta(t) = (t + 1)^{-1}$ for all $t \ge 0$. Then it is clear that $\theta \in \text{Ŧ}$. Define $F : X^2 \to X$ as $F(x, y) = (y - x)/5$ for all $x, y \in X$.

We have the following cases.

Case 1. For $(x, y) = (a, b) = (0, 0)$ or $(x, y) = (a, b) = (1, 1)$ or $(x, y) = (0, 0)$ and $(a, b) = (1, 1)$, we have $p(F(x, y), F(a, b)) = 0$. Thus, (23) holds.

Case 2. For $(x, y) = (0, 0)$ and $(a, b) = (0, 1)$, we have

$$\begin{aligned}p(F(0,0), F(0,1)) &= \frac{2}{5} < \frac{1}{2}\\&= \theta\left(\left(p(0,0) + p(0,1)\right) \times 2^{-1}\right)\\&\quad \cdot \left(\left(p(0,0) + p(0,1)\right) \times 2^{-1}\right).\end{aligned} \tag{30}$$

Thus, (23) holds.

Case 3. For $(x, y) = (0, 1)$ and $(a, b) = (1, 1)$, we have

$$\begin{aligned}p(F(0,1), F(1,1)) &= \frac{2}{5} < \frac{3}{5}\\&= \theta\left(\left(p(0,1) + p(1,1)\right) \times 2^{-1}\right)\\&\quad \cdot \left(\left(p(0,1) + p(1,1)\right) \times 2^{-1}\right).\end{aligned} \tag{31}$$

Thus, (23) holds.

Therefore, all the conditions of Theorem 7 are satisfied and $(0, 0)$ is a coupled fixed point of F.

Competing Interests

The author declares that she has no competing interests.

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