Research Article

Some Congruence Properties of a Restricted Bipartition Function $c_N(n)$

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Let $c_N(n)$ denote the number of bipartitions (λ, μ) of a positive integer n subject to the restriction that each part of μ is divisible by N . In this paper, we prove some congruence properties of the function $c_N(n)$ for $N = 7, 11$, and 5ℓ , for any integer $\ell \geq 1$, by employing Ramanujan's theta-function identities.

1. Introduction

A bipartition of a positive integer n is an ordered pair of partitions (λ, μ) such that the sum of all of the parts equals n . If $c_N(n)$ counts the number of bipartitions (λ, μ) of n subject to the restriction that each part of μ is divisible by N , then the generating function of $c_N(n)$ [1] is given by

$$\sum_{n=0}^{\infty} c_N(n) q^n = \frac{1}{(q; q)_{\infty} (q^N; q^N)_{\infty}}, \quad (1)$$

where

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n). \quad (2)$$

The partition function $c_N(n)$ is first studied by Chan [2] for the particular case $N = 2$ by considering the function $c_2(n)$ defined by

$$\sum_{n=0}^{\infty} c_2(n) q^n = \frac{1}{(q; q)_{\infty} (q^2; q^2)_{\infty}}. \quad (3)$$

Chan [2] proved that, for $n \geq 0$,

$$c_2(3n+2) \equiv 0 \pmod{3}. \quad (4)$$

Kim [3] gave a combinatorial interpretation (4). In a subsequent paper, Chan [4] showed that, for $k \geq 1$ and $n \geq 0$,

$$c_2(3^k n + s_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \quad (5)$$

where s_k is the reciprocal modulo 3^k of 8 and $\delta(k) = 1$ if k is even and 0 otherwise. Inspired by the work of Ramanujan on the standard partition function $p(n)$, Chan [4] asked whether there are any other congruence properties of the following form: $c_2(\ell n + k) \equiv 0 \pmod{\ell}$, where ℓ is prime and $0 \leq k \leq \ell$. Sinick [1] answered Chan's question in negative by considering restricted bipartition function $c_N(n)$ defined in (1). Liu and Wang [5] established several infinite families of congruence properties for $c_5(n)$ modulo 3. For example, they proved that

$$c_5\left(3^{2\alpha+1}n + \frac{7 \cdot 3^{2\alpha} + 1}{4}\right) \equiv 0 \pmod{3}, \quad (6)$$

$$\alpha \geq 1, n \geq 0.$$

Baruah and Ojah [6] also proved some congruence properties for some particular cases of $c_N(n)$ by considering the generalised partition function $p_{[c^l, d^m]}(n)$ defined by

$$\sum_{n=0}^{\infty} p_{[c^l, d^m]}(n) q^n = \frac{1}{(q^c; q^c)_{\infty}^l (q^d; q^d)_{\infty}^m} \quad (7)$$

and using Ramanujan's modular equations. Clearly, $c_N(n) = p_{[1^1 N^1]}(n)$. For example, Baruah and Ojah [6] proved that

$$\begin{aligned} p_{[1^1 3^1]}(4n+j) &\equiv 0 \pmod{2}, \quad \text{for } j = 2, 3, \\ p_{[1^1 7^1]}(8n+7) &\equiv 0 \pmod{2}. \end{aligned} \quad (8)$$

Ahmed et al. [7] investigated the function $C_N(n)$ for $N = 3$ and 4 and proved some congruence properties modulo 5 . They also gave alternate proof of some congruence properties due to Chan [2].

In this paper, we investigate the restricted bipartition function $c_N(n)$ for $n = 7, 11$, and 5ℓ , for any integer $\ell \geq 1$, and prove some congruence properties modulo $2, 3$, and 5 by using Ramanujan's theta-function identities. In Section 3, we prove congruence properties modulo 2 for $c_7(n)$. For example, we prove, for $\alpha \geq 0$,

$$c_7 \left(2^{2\alpha+1}n + \frac{5 \cdot 2^{2\alpha} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (9)$$

In Section 4, we deal with the function $c_{11}(n)$ and establish the notion that if p is an odd prime, $1 \leq j \leq p-1$, and $\alpha \geq 0$, then

$$c_{11} \left(4p^{2\alpha+1}(pn+j) + \frac{p^{2\alpha+2} + 1}{2} \right) \equiv 0 \pmod{2}. \quad (10)$$

In Section 5, we show that, for any integer $\ell \geq 1$, $c_{5\ell}(5n+4) \equiv 0 \pmod{5}$. We also prove congruence properties modulo 3 for $c_{15}(n)$. Section 2 is devoted to listing some preliminary results.

2. Preliminary Results

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (11)$$

Three important special cases of $f(a, b)$ are

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad (12)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}, \quad (13)$$

$$\begin{aligned} f(-q) &:= f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= (q; q)_{\infty}. \end{aligned} \quad (14)$$

Ramanujan also defined the function $\chi(q)$ as

$$\chi(q) = (-q; q^2)_{\infty}. \quad (15)$$

Lemma 1. For any prime p and positive integer m , one has

$$(q^{pm}; q^{pm})_{\infty} \equiv (q^m; q^m)_{\infty}^p \pmod{p}. \quad (16)$$

Proof. It follows easily from the binomial theorem. $\square$

Lemma 2 (see [8, page 315]). One has

$$\begin{aligned} \psi(q) \psi(q^7) &= \phi(q^{28}) \psi(q^8) + q \psi(q^{14}) \psi(q^2) \\ &\quad + q^6 \psi(q^{56}) \phi(q^4). \end{aligned} \quad (17)$$

Lemma 3. One has

$$\psi(q) \psi(q^7) \equiv (q; q)_{\infty}^3 (q^7; q^7)_{\infty}^3 \pmod{2}. \quad (18)$$

Proof. From (13), we have

$$\psi(q) \psi(q^7) = \frac{(q^2; q^2)_{\infty}^2 (q^{14}; q^{14})_{\infty}^2}{(q; q)_{\infty} (q^7; q^7)_{\infty}}. \quad (19)$$

Simplifying (19) using Lemma 1 with $p = 2$, we arrive at the desired result. $\square$

Lemma 4 (see [9, page 286, Equation (60)]). One has

$$\phi(-q) = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}}, \quad (20)$$

$$\psi(-q) = \frac{(q; q)_{\infty} (q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}}, \quad (21)$$

$$f(q) = \frac{(q^2; q^2)_{\infty}^3}{(q; q)_{\infty} (q^4; q^4)_{\infty}}, \quad (22)$$

$$\chi(q) = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty} (q^4; q^4)_{\infty}}. \quad (23)$$

Lemma 5 (see [10, page 372]). One has

$$\begin{aligned} \psi(q) \psi(q^{11}) &= \phi(q^{66}) \psi(q^{12}) \\ &\quad + q f(q^{44}, q^{88}) f(q^2, q^{10}) \\ &\quad + q^{22} f(q^{22}, q^{110}) f(q^8, q^4) \\ &\quad + q^{15} \psi(q^{132}) \phi(q^6). \end{aligned} \quad (24)$$

Lemma 6 (see [8, page 350, Equation (13)]). One has

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}, \quad (25)$$

where

$$\chi(-q) = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (26)$$

Lemma 7. One has

$$f(q^{11}, q^{22}) \equiv (q^{11}; q^{11})_{\infty} \pmod{2}. \quad (27)$$

Proof. Employing (20) in Lemma 6 and performing simplification using Lemma 1 with $p = 2$, we obtain

$$f(q; q^2) \equiv (q; q)_\infty \pmod{2}. \quad (28)$$

Replacing q by q^{11} in (28), we arrive at the desired result. $\square$

Lemma 8 (see [8, page 51, Example (v)]). *One has*

$$f(q, q^5) = \psi(-q^3) \chi(q). \quad (29)$$

Lemma 9. *One has*

$$f(q, q^5) \equiv \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} \pmod{2}. \quad (30)$$

Proof. Employing (21) and (23) in Lemma 8, we obtain

$$f(q, q^5) = \frac{(q^3; q^3)_\infty (q^{12}; q^{12})_\infty (q^2; q^2)_\infty^2}{(q^6; q^6)_\infty (q; q)_\infty (q^4; q^4)_\infty}. \quad (31)$$

Simplifying (31) using Lemma 1 with $p = 2$, we complete the proof. $\square$

Lemma 10 (see [11, page 5, Equation (15)]). *One has*

$$\frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} + q \frac{(q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty}. \quad (32)$$

Lemma 11 (see [12, Theorem 2.1]). *For any odd prime p ,*

$$\begin{aligned} & \psi(q) \\ &= \sum_{k=0}^{(p-3)/2} q^{(k^2+k)/2} f\left(q^{(p^2+(2k+1)p)/2}, q^{(p^2-(2k+1)p)/2}\right) \\ &+ q^{(p^2-1)/8} \psi(q^{p^2}), \end{aligned} \quad (33)$$

where, for $0 \leq k \leq (p-3)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}. \quad (34)$$

Lemma 12 (see [12, Theorem 2.2]). *For any prime $p \geq 5$, one has*

$$\begin{aligned} f(-q) &= \sum_{\substack{k=(-p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k \\ &\cdot q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) \\ &+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f(-q^{p^2}), \end{aligned} \quad (35)$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases} \quad (36)$$

Lemma 13 (see [13]). *One has*

$$\begin{aligned} \frac{1}{(q; q)_\infty} &= \frac{(q^{25}; q^{25})_\infty^6}{(q^5; q^5)_\infty^6} (F^4(q^5) + qF^3(q^5) \\ &+ 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) \\ &+ 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)), \end{aligned} \quad (37)$$

where $F(q) := q^{-1/5}R(q)$ and $R(q)$ is Rogers-Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}}, \quad |q| < 1. \quad (38)$$

Lemma 14 (see [8, page 345, Entry 1(iv)]). *One has*

$$(q; q)_\infty^3 = (q^9; q^9)_\infty^3 (4q^3W^2(q^3) - 3q + W^{-1}(q^3)), \quad (39)$$

where $W(q) = q^{-1/3}G(q)$ and $G(q)$ is Ramanujan's cubic continued fraction defined by

$$G(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \dots}}}, \quad |q| < 1. \quad (40)$$

3. Congruence Identities for $c_7(n)$

Theorem 15. *One has*

$$\sum_{n=0}^{\infty} c_7(2n+1) q^n \equiv (q; q)_\infty (q^7; q^7)_\infty \pmod{2}. \quad (41)$$

Proof. For $N = 7$ in (1), we have

$$\sum_{n=0}^{\infty} c_7(n) q^n = \frac{1}{(q; q)_\infty (q^7; q^7)_\infty}. \quad (42)$$

Employing (19) in (42), we obtain

$$\sum_{n=0}^{\infty} c_7(n) q^n = \frac{\psi(q) \psi(q^7)}{(q^2; q^2)_\infty^2 (q^{14}; q^{14})_\infty^2}. \quad (43)$$

Employing Lemma 2 in (43), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7(n) q^n &= \frac{1}{(q^2; q^2)_\infty^2 (q^{14}; q^{14})_\infty^2} [\phi(q^{28}) \psi(q^8) \\ &+ q\psi(q^{14}) \psi(q^2) + q^6\psi(q^{56}) \phi(q^4)]. \end{aligned} \quad (44)$$

Extracting the terms involving q^{2n+1} , dividing by q , and replacing q^2 by q in (44), we get

$$\sum_{n=0}^{\infty} c_7(2n+1) q^n = \frac{1}{(q; q)_\infty^2 (q^7; q^7)_\infty^2} [\psi(q^7) \psi(q)]. \quad (45)$$

Employing Lemma 3 in (45), we complete the proof. $\square$

Theorem 16. One has

$$(i) \sum_{n=0}^{\infty} c_7 (4n+3) q^n \equiv (q^2; q^2)_{\infty} (q^{14}; q^{14})_{\infty} \pmod{2}, \quad (46)$$

$$(ii) c_7 (8n+7) \equiv 0 \pmod{2}.$$

Proof. From Theorem 15, we obtain

$$\sum_{n=0}^{\infty} c_7 (2n+1) q^n \equiv \frac{(q^7; q^7)_{\infty}^3 (q; q)_{\infty}^3}{(q^7; q^7)_{\infty}^2 (q; q)_{\infty}^2} \pmod{2}. \quad (47)$$

Employing Lemma 3 in (47), we obtain

$$\sum_{n=0}^{\infty} c_7 (2n+1) q^n \equiv \frac{\psi(q) \psi(q^7)}{(q^2; q^2)_{\infty} (q^{14}; q^{14})_{\infty}} \pmod{2}. \quad (48)$$

Employing Lemma 2 in (48), extracting the terms involving q^{2n+1} , dividing by q , and replacing q^2 by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 (4n+3) q^n \\ \equiv \frac{1}{(q; q)_{\infty} (q^7; q^7)_{\infty}} \psi(q) \psi(q^7) \pmod{2}. \end{aligned} \quad (49)$$

Employing Lemma 3 in (49) and performing simplification using Lemma 1 with $p = 2$, we arrive at (i).

All the terms on the right hand side of (i) are of the form q^{2n} . Extracting the terms involving q^{2n+1} on both sides of (i), we complete the proof of (ii). $\square$

Theorem 17. For all $n \geq 0$, one has

- (i) $c_7(14n+7) \equiv 0 \pmod{2}$,
- (ii) $c_7(14n+9) \equiv 0 \pmod{2}$,
- (iii) $c_7(14n+13) \equiv 0 \pmod{2}$.

Proof. Employing (14) in Theorem 15, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 (2n+1) q^n \\ \equiv (q^7; q^7)_{\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} \pmod{2}. \end{aligned} \quad (50)$$

Extracting those terms on each side of (50) whose power of q is of the forms $7n+3$, $7n+4$, and $7n+6$ and employing the fact that there exists no integer n such that $n(3n+1)/2$ is congruent to 3, 4, and 6 modulo 7, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 (14n+7) q^{7n+3} &\equiv \sum_{n=0}^{\infty} c_7 (14n+9) q^{7n+4} \\ &\equiv \sum_{n=0}^{\infty} c_7 (14n+13) q^{7n+6} \\ &\equiv 0 \pmod{2}. \end{aligned} \quad (51)$$

Now, (i), (ii), and (iii) are obvious from (51). $\square$

Theorem 18. For $\alpha \geq 1$, one has

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1} n + \frac{2^{2\alpha+1} + 1}{3} \right) q^n \\ \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}. \end{aligned} \quad (52)$$

Proof. We proceed by induction on α . Extracting the terms involving q^{2n} and replacing q^2 by q in Theorem 16(i), we obtain

$$\sum_{n=0}^{\infty} c_7 (8n+3) q^n \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}, \quad (53)$$

which corresponds to the case $\alpha = 1$. Assume that the result is true for $\alpha = k \geq 1$, so that

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2k+1} n + \frac{2^{2k+1} + 1}{3} \right) q^n \\ \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}. \end{aligned} \quad (54)$$

Employing Lemma 3 in (54), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2k+1} n + \frac{2^{2k+1} + 1}{3} \right) q^n \\ \equiv \frac{\psi(q) \psi(q^7)}{(q; q)_{\infty}^2 (q^7; q^7)_{\infty}^2} \pmod{2}. \end{aligned} \quad (55)$$

Employing Lemma 2 in (55) and extracting the terms involving q^{2n+1} , dividing by q , and replacing q^2 by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2k+1} (2n+1) + \frac{2^{2k+1} + 1}{3} \right) q^n \\ \equiv \frac{\psi(q) \psi(q^7)}{(q; q)_{\infty} (q^7; q^7)_{\infty}} \pmod{2}. \end{aligned} \quad (56)$$

Simplifying (56) using Lemmas 3 and 1 with $p = 2$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2(k+1)} n + \frac{2^{2(k+1)+1} + 1}{3} \right) q^n \\ \equiv (q^2; q^2)_{\infty} (q^{14}; q^{14})_{\infty} \pmod{2}. \end{aligned} \quad (57)$$

Extracting the terms involving q^{2n} and replacing q^2 by q in (57), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2(k+1)+1} n + \frac{2^{2(k+1)+1} + 1}{3} \right) q^n \\ \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}, \end{aligned} \quad (58)$$

which is the $\alpha = k+1$ case. Hence, the proof is complete. $\square$

Theorem 19. For $\alpha \geq 0$, one has

$$c_7 \left(2^{2\alpha+1} n + \frac{5 \cdot 2^{2\alpha} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (59)$$

Proof. All the terms in the right hand side of (57) are of the form q^{2n} , so, extracting the coefficients of q^{2n+1} on both sides of (57) and replacing k by α , we obtain

$$c_7 \left(2^{2(\alpha+1)+1} n + \frac{5 \cdot 2^{2(\alpha+1)} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (60)$$

Replacing $\alpha + 1$ by α in (60) completes the proof. $\square$

Theorem 20. *If any prime $p \geq 5$, $(-7/p) = -1$, and $\alpha \geq 0$, then*

$$c_7 \left(2^{2\alpha+1} p^2 n + \frac{2^{2\alpha+1} p(3j+p) + 1}{3} \right) \equiv 0 \pmod{2}, \quad (61)$$

where $1 \leq j \leq p-1$.

Proof. Employing Lemma 12 in (52), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1} n + \frac{2^{2\alpha+1} + 1}{3} \right) q^n &\equiv \left(\sum_{\substack{k=(-p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k \right. \\ &\cdot q^{(3k^2+k)/2} f \left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2} \right) \\ &\left. + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f \left(-q^{p^2} \right) \right) \\ &\cdot \left(\sum_{\substack{k=(-p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^m \right. \\ &\cdot q^{7 \cdot (3m^2+m)/2} f \left(-q^{7 \cdot (3p^2+(6m+1)p)/2}, \right. \\ &\left. -q^{7 \cdot (3p^2-(6m+1)p)/2} \right) + (-1)^{(\pm p-1)/6} \\ &\left. \cdot q^{7 \cdot (p^2-1)/24} f \left(-q^{7p^2} \right) \right) \pmod{2}. \end{aligned} \quad (62)$$

We consider the congruence

$$\frac{3k^2 + k}{2} + 7 \cdot \frac{3m^2 + m}{2} \equiv \frac{8p^2 - 8}{24} \pmod{p}, \quad (63)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$. The congruence (63) is equivalent to

$$(6k+1)^2 + 7(6m+1)^2 \equiv 0 \pmod{p} \quad (64)$$

and, for $(-7/p) = -1$, the congruence (64) has unique solution $k = m = (\pm p - 1)/6$. Extracting terms containing

$q^{pn+(p^2-1)/3}$ from both sides of (62) and replacing q^p by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1} pn + \frac{2^{2\alpha+1} p^2 + 1}{3} \right) q^n \\ \equiv (q^p; q^p)_{\infty} (q^{7p}; q^{7p}) \pmod{2}. \end{aligned} \quad (65)$$

Extracting the coefficients of q^{pn+j} , for $1 \leq j \leq p-1$, on both sides of (65) and performing simplification, we arrive at the desired result. $\square$

4. Congruence Identities for $c_{11}(n)$

Theorem 21. *One has*

$$\sum_{n=0}^{\infty} c_{11}(4n+1) q^n \equiv \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}} = \psi(q) \pmod{2}. \quad (66)$$

Proof. Setting $N = 11$ in (1), we obtain

$$\sum_{n=0}^{\infty} c_{11}(n) q^n = \frac{1}{(q; q)_{\infty} (q^{11}; q^{11})_{\infty}}. \quad (67)$$

Employing (13) in (67), we obtain

$$\sum_{n=0}^{\infty} c_{11}(n) q^n = \frac{\psi(q) \psi(q^{11})}{(q^2; q^2)_{\infty}^2 (q^{22}; q^{22})_{\infty}^2}. \quad (68)$$

Employing Lemma 5 in (68), extracting the terms involving q^{2n+1} , dividing by q , and replacing q^2 by q , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_{11}(2n+1) q^n \\ = \frac{1}{(q; q)_{\infty}^2 (q^{11}; q^{11})_{\infty}^2} \left[f(q^{22}, q^{44}) f(q, q^5) \right. \\ \left. + q^7 \psi(q^{66}) \phi(q^3) \right]. \end{aligned} \quad (69)$$

Employing Lemmas 9 and 10 in (69), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} c_{11}(2n+1) q^n \\ \equiv \frac{1}{(q^2; q^2)_{\infty} (q^{22}; q^{22})_{\infty}} \left[f(q^{22}, q^{44}) \right. \\ \cdot \left(\frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} + q \frac{(q^{12}; q^{12})_{\infty}^3}{(q^4; q^4)_{\infty}} \right. \\ \left. + q^7 \psi(q^{66}) \phi(q^3) \right] \pmod{2}. \end{aligned} \quad (70)$$

Extracting the terms involving q^{2n} and replacing q^2 by q on both sides of (70) and performing simplification using Lemma 1 with $p = 2$, we obtain

$$\sum_{n=0}^{\infty} c_{11} (4n+1) q^n \equiv \frac{1}{(q; q)_{\infty} (q^{11}; q^{11})_{\infty}} f(q^{11}; q^{22}) \cdot (q^2; q^2)_{\infty}^2 \pmod{2}. \quad (71)$$

Employing Lemma 7 in (71) and using (13), we complete the proof. $\square$

Theorem 22. For any odd prime p and any integer $\alpha \geq 0$, one has

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2\alpha} n + \frac{p^{2\alpha} + 1}{2} \right) q^n \equiv \psi(q) \pmod{2}. \quad (72)$$

Proof. We proceed by induction on α . The case $\alpha = 0$ corresponds to the congruence theorem (Theorem 21). Suppose that the theorem holds for $\alpha = k \geq 0$, so that

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2k} n + \frac{p^{2k} + 1}{2} \right) q^n \equiv \psi(q) \pmod{2}. \quad (73)$$

Employing Lemma 11 in (73), extracting the terms involving $q^{pn+(p^2-1)/8}$ on both sides of (73), dividing by $q^{(p^2-1)/8}$, and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2k+1} n + \frac{p^{2(k+1)} + 1}{2} \right) q^n \equiv \psi(q^p) \pmod{2}. \quad (74)$$

Extracting the terms containing q^{pn} from both sides of (74) and replacing q^p by q , we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} c_{11} \left(4p^{2(k+1)} n + \frac{p^{2(k+1)} + 1}{2} \right) q^n \\ \equiv \psi(q) \pmod{2}, \end{aligned} \quad (75)$$

which shows that the theorem is true for $\alpha = k + 1$. Hence, the proof is complete. $\square$

Theorem 23. For any odd prime p and integers $\alpha \geq 0$ and $1 \leq j \leq p - 1$, one has

$$c_{11} \left(4p^{2\alpha+1} (pn+j) + \frac{p^{2\alpha+2} + 1}{2} \right) \equiv 0 \pmod{2}. \quad (76)$$

Proof. Extracting the coefficients of q^{pn+j} for $1 \leq j \leq p - 1$ on both sides of (74) and replacing k by α , we arrive at the desired result. $\square$

5. Congruence Identities for $c_{5l}(n)$

Theorem 24. For any positive integer ℓ , one has

$$c_{5l}(5n+4) \equiv 0 \pmod{5}. \quad (77)$$

Proof. Setting $N = 5l$ in (1), we obtain

$$\sum_{n=0}^{\infty} c_{5l}(n) q^n = \frac{1}{(q; q)_{\infty} (q^{5l}; q^{5l})_{\infty}}. \quad (78)$$

Using Lemma 13 in (78) and extracting the terms involving q^{5n+4} , dividing by q^4 , and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} c_{5l}(5n+4) q^n = 5 \frac{(q^5; q^5)_{\infty}^6}{(q^l; q^l)_{\infty} (q; q)_{\infty}^6}. \quad (79)$$

The desired result follows easily from (79). $\square$

Theorem 25. For all $n \geq 0$, one has

- (i) $c_{15}(5n+4) \equiv 0 \pmod{5}$,
- (ii) $c_{15}(15n+9) \equiv 0 \pmod{3}$,
- (iii) $c_{15}(15n+14) \equiv 0 \pmod{3}$.

Proof. Setting $N = 15$ in (1), we obtain

$$\sum_{n=0}^{\infty} c_{15}(n) q^n = \frac{1}{(q; q)_{\infty} (q^{15}; q^{15})_{\infty}}. \quad (80)$$

Employing Lemma 13 in (80), extracting terms involving q^{5n+4} , dividing by q^4 , and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} c_{15}(5n+4) q^n = 5 \frac{(q^5; q^5)_{\infty}^6}{(q^3; q^3)_{\infty} (q; q)_{\infty}^6}. \quad (81)$$

Now, (i) follows from (81).

Simplifying (81) by using Lemma 1 with $p = 3$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_{15}(5n+4) q^n &\equiv 2 \frac{(q^{15}; q^{15})_{\infty}^2 (q; q)_{\infty}^3}{(q; q)_{\infty}^3 (q^3; q^3)_{\infty}^6 (q; q)_{\infty}^3} \\ &= 2 \frac{(q^{15}; q^{15})_{\infty}^2 (q; q)_{\infty}^3}{(q^3; q^3)_{\infty}^4} \pmod{3}. \end{aligned} \quad (82)$$

Employing Lemma 14 in (82) and performing simplification, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_{15}(5n+4) q^n &\equiv 2 \\ &\cdot \frac{(q^{15}; q^{15})_{\infty}^2 (q^9; q^9)_{\infty}^3}{(q^3; q^3)_{\infty}^4} [q^3 W^2(q^3) \\ &+ W^{-1}(q^3)] \pmod{3}. \end{aligned} \quad (83)$$

Extracting terms involving q^{3n+1} and q^{3n+2} on both sides of (83), we arrive at (ii) and (iii), respectively. $\square$

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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