Research Article

Coefficients Bounds for Certain Subclass of Biunivalent Functions Associated with Ruscheweyh q -Differential Operator

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We introduce in our present investigation a new subclass of analytic and biunivalent functions associated with Ruscheweyh q -differential operator in open unit disk E . We use the Faber polynomial expansions to find n th coefficients bounds of class of bisubordinate functions and also find initial coefficient estimates.

1. Introduction

Let $\mathcal{A}$ denote the class of all function $f(z)$ which is analytic in the open unit disk $E = \{z : |z| < 1\}$ and has the Taylor series expansion of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

By $\mathcal{S}$ we mean the class of all functions in $\mathcal{A}$ which are univalent in E . The Koebe one-quarter theorem [1] states that the image of E under every function f from $\mathcal{S}$ contains a disk of radius $1/4$. It is well known that every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} which is defined as

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad z \in E, \\ f(f^{-1}(w)) &= w, \\ |w| &< r_0(f), \\ r_0(f) &\geq \frac{1}{4}, \end{aligned} \quad (2)$$

where

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \end{aligned} \quad (3)$$

A function $f \in \mathcal{A}$ is said to be biunivalent in E if both f and f^{-1} are univalent in E .

If the functions f and g are analytic in E , then f is said to be subordinate to g , written as

$$f(z) \prec g(z), \quad z \in E \quad (4)$$

if there exists a Schwarz function $w(z)$, analytic in E , with $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z)), \quad z \in E. \quad (5)$$

Let Σ denote the class of analytic and biunivalent functions in E given by the Taylor-Maclaurin series expansion (1).

Here we give few examples of functions in the class Σ such that

$$\begin{aligned} u_1(z) &= \frac{z}{1-z}, \\ u_2(z) &= -\log(1-z), \\ u_3(z) &= \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \end{aligned} \quad (6)$$

$z \in E.$

However, the famous Koebe function $k(z) = z/(1-z)^2$ is not in Σ ; for more details we refer to [2]. For $f \in \Sigma$, the class of biunivalent analytic functions was first introduced and studied by Lewin where it was proved to show that $|a_2| < 1.51$. Brannan and Clunie [3] proved that $|a_2| \leq \sqrt{2}$. Netanyahu [4] showed that $\max|a_2| = 4/3$. Brannan and Taha [5] introduced certain subclass of the biunivalent functions class Σ . For a brief history and interesting examples of biunivalent functions we refer to [3, 5–11].

The Faber polynomials introduced by Faber [12]. Gong [13] and Schiffer [14] demonstrated the significance of the Faber polynomials in mathematical sciences, especially in geometric function theory. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic biunivalent functions given by (1) using Faber polynomial expansions. A very little is known about the bounds of Maclaurin's series coefficient $|a_n|$ for $n \geq 3$ by using a Faber polynomials. For more study about Faber polynomials we refer to [15–21].

Using the technique of convolution, Ruscheweyh [22] defined the operator R^λ on the class of analytic functions $\mathcal{A}$ as

$$R^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}, \quad (7)$$

$z \in E, \lambda \in \mathbb{R}, \lambda > -1.$

For $\lambda = m \in N_0 = NU\{0\}$, we obtain

$$R^m f(z) = \frac{z \left(z^{m-1} f(z) \right)^{(m)}}{m!}. \quad (8)$$

The expression $R^m f(z)$ is called an m th-order Ruscheweyh derivative of $f(z)$ and the symbol $*$ stands for Hadamard product (or convolution).

For $t \in \mathbb{R}$ and $q > 0, q \neq 1$, the number $[t]$ is defined in [23] as

$$[t] = \frac{1-q^t}{1-q}, \quad [0] = 0. \quad (9)$$

For any nonnegative integer n the q -number shift factorial is defined as

$$[n]! = [1][2][3] \cdots [n], \quad ([0]! = 1). \quad (10)$$

We have $\lim_{q \rightarrow 1} [n] = n$. Throughout in this paper we will assume q to be fixed number between 0 and 1.

The q -derivative operator or q -difference operator for $f \in \mathcal{A}$ is defined as

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in E. \quad (11)$$

It can easily be seen that for $n \in N = \{1, 2, 3, \dots\}$ and $z \in E$.

$$\begin{aligned} \partial_q z^n &= [n] z^{n-1}, \\ \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} &= \sum_{n=1}^{\infty} [n] a_n z^{n-1} \end{aligned} \quad (12)$$

The q -generalized Pochhammer symbol for $t \in \mathbb{R}$ and $n \in N$ is defined as

$$[t]_n = [t][t+1][t+2] \cdots [t+(n-1)], \quad (13)$$

and, for $t > 0$, let q -gamma function be defined as

$$\begin{aligned} \Gamma_q(t+1) &= [t] \Gamma_q(t), \\ \Gamma_q(1) &= 1. \end{aligned} \quad (14)$$

For $f \in \mathcal{A}$ Ruscheweyh q -differential operator was defined by Aldweby and Darus [24] (see also [23]), as

$$\begin{aligned} R_q^\lambda f(z) &= f(z) * F_{q,\lambda+1}(z), \quad z \in E, \lambda > -1. \\ F_{q,\lambda+1}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]! \Gamma_q(1+\lambda)} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} z^n \\ &= z + \sum_{n=2}^{\infty} \Psi_{n-1} z^n, \end{aligned} \quad (15)$$

$$\Psi_{n-1} = \frac{\Gamma_q(n+\lambda)}{[n-1]! \Gamma_q(1+\lambda)} = \frac{[\lambda+1]_{n-1}}{[n-1]!}, \quad (16)$$

$$z \partial_q (R_q^\lambda f(z)) = \left(1 + \frac{[\lambda]}{q^\lambda} \right) R_q^{\lambda+1} f(z) - \frac{[\lambda]}{q^\lambda} R_q^\lambda f(z). \quad (17)$$

If $q \rightarrow 1$, equality (17) implies

$$z (R^\lambda f(z))' = (1+\lambda) R^{\lambda+1} f(z) - \lambda R^\lambda f(z). \quad (18)$$

which is the well known recurrent formula for Ruscheweyh differential operator.

in the present paper we introduce new subclass of the function class Σ , involving Ruscheweyh q -differential operator $R_q^\lambda f(z)$. By using Faber polynomial coefficient techniques we determine estimates for the general coefficient bounds $|a_n|$ for $n \geq 3$ and also estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass of function class Σ . Several related classes are also considered, and connections to earlier known results are also defined.

Definition 1. A function $f \in \Sigma$, $0 \leq \gamma$, $-1 < \lambda$, $1 \leq \beta$, and $0 \leq \alpha < 1$; we introduce a new class of biunivalent functions $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$ as $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$ if and only if

$$\begin{aligned} & \frac{(1 - \beta) R_q^\lambda f(z) + \beta z \partial_q R_q^\lambda f(z) + \gamma z^2 \partial_q^2 R_q^\lambda f(z)}{z} \\ & < \frac{1 + Az}{1 + Bz}, \quad z \in E, \\ & \frac{(1 - \beta) R_q^\lambda g(w) + \beta w \partial_q R_q^\lambda g(w) + \gamma w^2 \partial_q^2 R_q^\lambda g(w)}{w} \\ & < \frac{1 + Aw}{1 + Bw}, \quad w \in E, \end{aligned} \quad (19)$$

where

$$\begin{aligned} A &= \{1 - \alpha(1 + q)\}, \\ B &= -q \end{aligned} \quad (20)$$

and $g(w) = f^{-1}(w)$ is defined by (3).

On specializing the parameters α , β , and γ , one can state the various new subclasses as illustrated in the following definition.

Definition 2. For $\beta = 1$, $-1 < \lambda$, $0 \leq \gamma$, and $0 \leq \alpha < 1$. A function, $f \in \Sigma$, is in the class $\mathcal{N}_{\Sigma}^{q(\alpha, \gamma, \lambda)}$ if the following conditions are satisfied:

$$\begin{aligned} \partial_q R_q^\lambda f(z) + \gamma z \partial_q^2 R_q^\lambda f(z) &< \frac{1 + Az}{1 + Bz}, \quad z \in E, \\ \partial_q R_q^\lambda g(w) + \gamma w \partial_q^2 R_q^\lambda g(w) &< \frac{1 + Aw}{1 + Bw}, \quad w \in E, \end{aligned} \quad (21)$$

where A, B are given by (20), and $g(w) = f^{-1}(w)$ is defined by (3).

Definition 3. For $\beta = 1$, $\gamma = 0$, $-1 < \lambda$, and $0 \leq \alpha < 1$. A function, $f \in \Sigma$, is in the class $\mathcal{N}_{\Sigma}^{q(\alpha, \lambda)}$ if the following conditions are satisfied:

$$\begin{aligned} \partial_q R_q^\lambda f(z) &< \frac{1 + Az}{1 + Bz}, \quad z \in E, \\ \partial_q R_q^\lambda g(w) &< \frac{1 + Aw}{1 + Bw}, \quad w \in E, \end{aligned} \quad (22)$$

where A, B are given by (20) and $g(w) = f^{-1}(w)$ is defined by (3).

Definition 4. For $\gamma = 0$, $-1 < \lambda$ and $0 \leq \alpha < 1$. A function, $f \in \Sigma$, is in the class $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \lambda)}$ if the following conditions are satisfied:

$$\begin{aligned} (1 - \beta) \frac{R_q^\lambda f(z)}{z} + \beta \partial_q R_q^\lambda f(z) &< \frac{1 + Az}{1 + Bz}, \quad z \in E, \\ (1 - \beta) \frac{R_q^\lambda g(w)}{w} + \beta \partial_q R_q^\lambda g(w) &< \frac{1 + Aw}{1 + Bw}, \quad w \in E, \end{aligned} \quad (23)$$

where A, B are given by (20) and $g(w) = f^{-1}(w)$ is defined by (3).

It is well known that

$$f(z) < \frac{1 + (1 - 2\alpha)z}{1 - z} \quad \text{iff } \operatorname{Re}(f(z)) > \alpha. \quad (24)$$

Special Cases

- (i) For $\lambda = 0$ and $q \rightarrow 1$, the class $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)} = \mathcal{N}_{\Sigma}^{(\alpha, \beta, \gamma)}$; see [21].
- (ii) For $\beta = 1$, $\lambda = 0$, and $q \rightarrow 1$, the class $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)} = \mathcal{B}_{\Sigma}^{(\alpha, \gamma)}$; see [25].
- (iii) For $\beta = 1$, $\lambda = 0$, $\gamma = 0$, and $q \rightarrow 1$, the class $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)} = \mathcal{H}_{\Sigma}^{\alpha}$; see [2].
- (iv) For $\lambda = 0$, $\gamma = 0$, and $q \rightarrow 1$, the class $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)} = \mathcal{N}_{\Sigma}^{(\alpha, \beta)}$; see [26].

Lemma 5 (see [27]). *Let the Schwarz function $w(z)$ be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots, \quad (z \in E); \quad (25)$$

then

$$\begin{aligned} |w_1| &\leq 1, \\ |w_2| &\leq 1 - |w_1|^2, \\ |w_2 - tw_1^2| &\leq 1 + (|t| - 1)|w_1|^2. \end{aligned} \quad (26)$$

2. Main Results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [16] given by

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n, \quad (27)$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-5)!} a_2^{n-1} \\ &+ \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \quad (28)$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, \dots, |a_n|$; see [28]. In particular, the first three terms of K_{n-1}^p are in general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^p (for details we refer [16, 29]), which is as follows:

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!} E_{n-1}^3 + \dots + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1}, \quad (29)$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, \dots)$ and, by [28],

$$E_{n-1}^m(a_2, \dots, a_n) = \sum_{n=2}^{\infty} \frac{m! (a_2)^{\mu_1} \dots (a_n)^{\mu_{n-1}}}{\mu_1! \dots \mu_{n-1}!}, \quad (30)$$

for $m \leq n$,

while $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, \dots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \dots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \dots + (n-1)\mu_{n-1} &= n-1. \end{aligned} \quad (31)$$

Evidently, $E_{n-1}^{n-1}(a_2, \dots, a_n) = a_2^{n-1}$ [15], or, equivalently,

$$E_n^m(a_1, a_2, \dots, a_n) = \sum_{n=1}^{\infty} \frac{m! (a_1)^{\mu_1} \dots (a_n)^{\mu_n}}{\mu_1! \dots \mu_n!}, \quad (32)$$

for $m \leq n$,

while $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, \dots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \dots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \dots + n\mu_n &= n. \end{aligned} \quad (33)$$

It is clear that $E_n^n(a_1, \dots, a_n) = E_1^n$; the first and last polynomials are $E_n^n = a_1^n$, $E_n^1 = a_n$.

Theorem 6. For $0 \leq \gamma$, $1 \leq \beta$, $-1 < \lambda$, and $0 \leq \alpha < 1$. If $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$, if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{(1-\alpha)(1+q)[n-1]!\Gamma_q(1+\lambda)}{\{(1-\beta) + \beta[n] + \gamma[n][n-1]\}\Gamma_q(n+\lambda)}, \quad (34)$$

for $n \geq 3$.

Proof. For the function $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$ of the form (1), we have

$$\begin{aligned} & \frac{(1-\beta)R_q^\lambda f(z) + \beta(z\partial_q R_q^\lambda f(z) + \gamma z^2 \partial_q^2 R_q^\lambda f(z))}{z} \\ &= 1 + \sum_{n=2}^{\infty} \{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \\ & \quad \times \Psi_{n-1} a_n z^{n-1}, \end{aligned} \quad (35)$$

and, for its inverse map $g = f^{-1}$, we have

$$\begin{aligned} & \frac{(1-\beta)R_q^\lambda g(w) + \beta(w\partial_q R_q^\lambda g(w) + \gamma w^2 \partial_q^2 R_q^\lambda g(w))}{w} \\ &= 1 + \sum_{n=2}^{\infty} \{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \\ & \quad \times \Psi_{n-1} b_n w^{n-1}, \end{aligned} \quad (36)$$

where Ψ_{n-1} is given by (16) and $b_n = (1/n)K_{n-1}^{-n}(a_2, a_3, \dots, a_n)$.

Since both function f and its inverse map $g = f^{-1}$ are in $\mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$, by the definition of subordination there exist two Schwarz functions $p(z) = \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = \sum_{n=1}^{\infty} d_n w^n$, where $z, w \in E$. We have

$$\begin{aligned} & \frac{(1-\beta)R_q^\lambda f(z) + \beta(z\partial_q R_q^\lambda f(z) + \gamma z^2 \partial_q^2 R_q^\lambda f(z))}{z} \\ &= \frac{1 + A(p(z))}{1 + B(p(z))}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} & \frac{1 + A(p(z))}{1 + B(p(z))} \\ &= 1 - \sum_{n=1}^{\infty} (A-B)K_n^{-1}(c_1, c_2, \dots, c_n, B)z^n, \end{aligned} \quad (38)$$

$$\begin{aligned} & \frac{(1-\beta)R_q^\lambda g(w) + \beta\{w\partial_q R_q^\lambda g(w) + \gamma w^2 \partial_q^2 R_q^\lambda g(w)\}}{w} \\ &= \frac{1 + A(q(w))}{1 + B(q(w))}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} & \frac{1 + A(q(w))}{1 + B(q(w))} \\ &= 1 - \sum_{n=1}^{\infty} (A-B)K_n^{-1}(d_1, d_2, \dots, d_n, B)w^n. \end{aligned} \quad (40)$$

In general [15, 16] for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of $K_n^p(k_1, k_2, \dots, k_n, B)$,

$$\begin{aligned} K_n^p(k_1, k_2, \dots, k_n, B) &= \frac{p!}{(p-n)!n!} k_1^n (B)^{n-1} \\ &+ \frac{p!}{(p-n+1)!(n-2)!} k_1^{n-2} k_2 (B)^{n-2} \\ &+ \frac{p!}{(p-n+2)!(n-3)!} \times k_1^{n-3} k_3 (B)^{n-3} \end{aligned}$$

$$\begin{aligned}
& + \frac{p!}{(p-n+3)!(n-4)!} \\
& \cdot k_1^{n-4} \left[k_4(B)^{n-4} + \frac{p-n+3}{2} k_3^2(B) \right] \\
& + \frac{p!}{(p-n+4)!(n-5)!} \\
& \cdot k_1^{n-5} \left[k_5(B)^{n-5} + (p-n+4) k_3 k_4(B) \right] \\
& + \sum_{j \geq 6} k_1^{n-1} X_j,
\end{aligned} \tag{41}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, \dots, k_n$.

For the coefficients of the Schwarz functions $p(z)$ and $q(w)$, $|c_n| \leq 1$ and $|d_n| \leq 1$ [9].

Comparing the corresponding coefficients of (35) and (37), we have

$$\begin{aligned}
& \{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Psi_{n-1} a_n \\
& = -(A-B) K_{n-1}^{-1}(c_1, c_2, \dots, c_{n-1}, B).
\end{aligned} \tag{42}$$

Similarly, corresponding to coefficients of (36) and (39), we have

$$\begin{aligned}
& \{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Psi_{n-1} b_n \\
& = -(A-B) K_{n-1}^{-1}(d_1, d_2, \dots, d_{n-1}, B).
\end{aligned} \tag{43}$$

Note that, for $a_m = 0$, $2 \leq m \leq n-1$, we have $b_n = -a_n$

$$\begin{aligned}
& \{(1-\beta) + \beta[n] + \beta\gamma[n][n-1]\} \Psi_{n-1} a_n \\
& = -(A-B) c_{n-1},
\end{aligned} \tag{44}$$

$$\begin{aligned}
& -\{(1-\beta) + \beta[n] + \beta\gamma[n][n-1]\} \Psi_{n-1} a_n \\
& = -(A-B) d_{n-1}.
\end{aligned} \tag{45}$$

Taking the absolute values of (44) and (45), we have

$$|a_n| \leq \frac{|A-B|}{\{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Psi_{n-1}} |c_{n-1}| \tag{46}$$

$$= \frac{|A-B|}{\{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Psi_{n-1}} |d_{n-1}|,$$

$$|a_n| \leq \frac{|A-B|}{\{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Psi_{n-1}}, \tag{47}$$

for $n \geq 3$.

By using $A = \{1 - \alpha(1+q)\}$, $B = -q$, from (20) and $\Psi_{n-1} = \Gamma_q(n+\lambda)/[n-1]!\Gamma_q(1+\lambda)$ from (16) in (47), we have

$$|a_n| \leq \frac{(1-\alpha)(1+q)[n-1]!\Gamma_q(1+\lambda)}{\{(1-\beta) + \beta[n] + \gamma[n][n-1]\} \Gamma_q(n+\lambda)}, \tag{48}$$

for $n \geq 3$,

which completes the proof of theorem. $\square$

By putting $\beta = 1$, in Theorem 6, we have the following corollary.

Corollary 7. For $0 \leq \gamma$, $-1 < \lambda$, and $0 \leq \alpha < 1$, if $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \gamma, \lambda)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{(1-\alpha)(1+q)[n-1]!\Gamma_q(1+\lambda)}{[n]\{1+\gamma[n-1]\}\Gamma_q(n+\lambda)}, \text{ for } n \geq 3. \tag{49}$$

For $\beta = 1$, $\gamma = 0$, in Theorem 6, we have the following corollary.

Corollary 8. For $0 \leq \alpha < 1$, $-1 < \lambda$, if $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \lambda)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{(1-\alpha)(1+q)[n-1]!\Gamma_q(1+\lambda)}{[n]\Gamma_q(n+\lambda)}, \text{ for } n \geq 3. \tag{50}$$

For $\lambda = 0$, $q \rightarrow 1$ in Theorem 6, we obtain the following corollary.

Corollary 9 (see [21]). For $1 \leq \beta$ and $0 \leq \gamma$, $0 \leq \alpha < 1$, if $f \in \mathcal{N}_{\Sigma}^{(\alpha, \beta, \gamma)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{2(1-\alpha)}{1+\beta(n-1)+\gamma n(n-1)}, \text{ for } n \geq 3. \tag{51}$$

For $\beta = 1$, $\lambda = 0$, and $q \rightarrow 1$ in Theorem 6, we obtain the following corollary.

Corollary 10 (see [25]). For $0 \leq \alpha < 1$ and $0 \leq \gamma$, if $f \in \mathcal{B}_{\Sigma}^{(\alpha, \gamma)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{2(1-\alpha)}{n\{1+\gamma(n-1)\}}, \text{ for } n \geq 3. \tag{52}$$

For $\beta = 1$, $\gamma = 0$, $\lambda = 0$, $q \rightarrow 1$ in Theorem 6, we obtain the following corollary.

Corollary 11. For $0 \leq \alpha < 1$, if $f \in \mathcal{H}_{\Sigma}^{(\alpha)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{2(1-\alpha)}{n}, \text{ for } n \geq 3. \tag{53}$$

For $\gamma = 0$, $\lambda = 0$, $q \rightarrow 1$ in Theorem 6, we obtain the following corollary.

Corollary 12 (see [30]). For $1 \leq \beta$ and $0 \leq \alpha < 1$, if $f \in \mathcal{N}_{\Sigma}^{(\alpha, \beta)}$ and if $a_m = 0$, $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\beta}, \text{ for } n \geq 3. \tag{54}$$

Theorem 13. For $0 \leq \alpha < 1$, $-1 < \lambda$, $1 \leq \beta$, and $0 \leq \gamma$, if $f \in \mathcal{N}_{\Sigma}^{q(\alpha, \beta, \gamma, \lambda)}$, then

$$\begin{aligned}
|a_2| &\leq \min \left\{ \frac{(1-\alpha)(1+q)[1]!\Gamma_q(1+\lambda)}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Gamma_q(2+\lambda)}, \right. \\
&\quad \left. \sqrt{\frac{2|q(1-\alpha)(1+q)|[2]!\Gamma_q(1+\lambda)}{[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Gamma_q(3+\lambda)}}, \right. \\
|a_3| &\leq \min \left\{ \frac{[2]}{([1]+[1])} \left[\frac{(1-\alpha)(1+q)[1]!\Gamma_q(1+\lambda)}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Gamma_q(2+\lambda)} \right]^2, \right. \\
&\quad \left. \frac{[2]}{([1]+[1])} \left[\frac{2q(1-\alpha)(1+q)[2]!\Gamma_q(1+\lambda)}{[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Gamma_q(3+\lambda)} \right], \right. \\
|a_3 - [2]a_2^2| &\leq \frac{q(1-\alpha)(1+q)[2]!\Gamma_q(1+\lambda)}{\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Gamma_q(3+\lambda)}.
\end{aligned} \tag{55}$$

Proof. Replacing n by 2 and 3 in (44) and (45), respectively, we have

$$\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Psi_1 a_2 = -(A-B)c_1, \tag{56}$$

$$\begin{aligned}
&\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2 a_3 \\
&= -(A-B)(Bc_1^2 - c_2),
\end{aligned} \tag{57}$$

$$-\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Psi_1 a_2 = -(A-B)d_1, \tag{58}$$

$$\begin{aligned}
&\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2 \{[2]a_2^2 - a_3\} \\
&= -(A-B)(Bd_1^2 - d_2).
\end{aligned} \tag{59}$$

From (56) and (58) we have

$$c_1 = -d_1, \tag{60}$$

$$\begin{aligned}
|a_2| &\leq \frac{|A-B|}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Psi_1} |c_1| \\
&= \frac{|A-B|}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Psi_1} |d_1|, \\
&\leq \frac{|A-B|}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Psi_1}.
\end{aligned} \tag{61}$$

Using (20) and (16) in (61), we have

$$|a_2| \leq \frac{(1-\alpha)(1+q)[1]!\Gamma_q(1+\lambda)}{\{(1-\beta)+\beta[2]+\gamma[2][1]\}\Gamma_q(2+\lambda)}. \tag{62}$$

Adding (57) and (59) we have

$$\begin{aligned}
&[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2 a_2^2 \\
&= -(A-B)\{B(c_1^2 + d_1^2) - (c_2 + d_2)\}.
\end{aligned} \tag{63}$$

Taking absolute values of both sides of (63) and applying the estimates $|c_2| \leq 1 - |c_1|^2$, $|d_2| \leq 1 - |d_1|^2$ of Lemma 5 and $|c_1| \leq 1$, $|d_1| \leq 1$, we have

$$\begin{aligned}
|a_2|^2 &\leq \frac{2|A-B||B|}{[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2}, \\
|a_2| &\leq \sqrt{\frac{2|A-B||B|}{[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2}},
\end{aligned} \tag{64}$$

using (20) and (16) in (64), we have

$$|a_2| \leq \sqrt{\frac{2q(1-\alpha)(1+q)[2]!\Gamma_q(1+\lambda)}{[2]\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Gamma_q(3+\lambda)}}. \tag{65}$$

Now, in order to find $|a_3|$, we subtract (59) from (57) and we have

$$\begin{aligned}
a_3 &= \frac{(A-B)\{B(d_1^2 - c_1^2) - (c_2 - d_2)\}}{([1]+[1])\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2} \\
&\quad + \frac{[2]}{([1]+[1])} a_2^2.
\end{aligned} \tag{66}$$

Using (60) in (66), we have

$$\begin{aligned}
a_3 &= \frac{-(A-B)(c_2 - d_2)}{([1]+[1])\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2} \\
&\quad + \frac{[2]}{([1]+[1])} a_2^2.
\end{aligned} \tag{67}$$

Taking the modulus of (67), we have

$$\begin{aligned}
|a_3| &\leq \frac{|A-B|(|c_2| + |d_2|)}{([1]+[1])\{(1-\beta)+\beta[3]+\gamma[3][2]\}\Psi_2} \\
&\quad + \frac{[2]}{([1]+[1])} |a_2|^2.
\end{aligned} \tag{68}$$

By using the estimates $|c_2| \leq 1 - |c_1|^2$, $|d_2| \leq 1 - |d_1|^2$, of Lemma 5, and $|c_1| \leq 1$, $|d_1| \leq 1$, in (68), we have

$$|a_3| \leq \frac{[2]}{([1] + [1])} |a_2|^2, \quad (69)$$

and using (56) in (69) we have

$$|a_3| \leq \frac{[2]}{([1] + [1])} \left[\frac{(1 - \alpha)(1 + q)}{\{(1 - \beta) + \beta[2] + \gamma[2][1]\} \Psi_1} \right]^2. \quad (70)$$

Using (20) and (16) in (70), we have

$$|a_3| \leq \frac{[2]}{([1] + [1])} \left[\frac{(1 - \alpha)(1 + q)[1]! \Gamma_q(1 + \lambda)}{\{(1 - \beta) + \beta[2] + \gamma[2][1]\} \Gamma_q(2 + \lambda)} \right]^2. \quad (71)$$

Again using (64) in (69) we have

$$|a_3| \leq \frac{[2]}{([1] + [1])} \left[\frac{2q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[2]\{(1 - \beta) + \beta[3] + \gamma[3][2]\} \Gamma_q(3 + \lambda)} \right]. \quad (72)$$

From (59) we have

$$|a_3 - [2]a_2^2| \leq \frac{|(A - B)B||d_1|^2 + |A - B||d_2|}{\{(1 - \beta) + \beta[3] + \gamma[3][2]\} \Psi_2}. \quad (73)$$

Using $|d_2| \leq 1 - |d_1|^2$, $|d_1|$ of Lemma 5 and (20), (16), on (73), we have

$$|a_3 - [2]a_2^2| \leq \frac{q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{\{(1 - \beta) + \beta[3] + \gamma[3][2]\} \Gamma_q(3 + \lambda)}. \quad (74)$$

□

For $\beta = 1$, in Theorem 13, we obtain the following corollary.

Corollary 14. For $-1 < \lambda$, $0 \leq \gamma$, and $0 \leq \alpha < 1$, if $f \in \mathcal{N}_\Sigma^{q(\alpha, \gamma, \lambda)}$, then

$$|a_2| \leq \min \left\{ \begin{aligned} &\frac{(1 - \alpha)(1 + q)[1]! \Gamma_q(1 + \lambda)}{[2]\{1 + \gamma\} \Gamma_q(2 + \lambda)}, \\ &\sqrt{\frac{2q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[2][3]\{1 + \gamma[2]\} \Gamma_q(3 + \lambda)}}, \end{aligned} \right.$$

$$|a_3| \leq \min \left\{ \begin{aligned} &\frac{[2]}{([1] + [1])} \left[\frac{(1 - \alpha)(1 + q)[1]! \Gamma_q(1 + \lambda)}{[2]\{1 + \gamma[1]\} \Gamma_q(2 + \lambda)} \right]^2, \\ &\frac{[2]}{([1] + [1])} \left[\frac{2q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[2][3]\{1 + \gamma[2]\} \Gamma_q(3 + \lambda)} \right], \end{aligned} \right. \quad (75)$$

$$|a_3 - [2]a_2^2| \leq \frac{q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[3]\{1 + \gamma[2]\} \Gamma_q(3 + \lambda)}.$$

For $\beta = 1$ and $\gamma = 0$, in Theorem 13, we obtain the following corollary.

Corollary 15. For $-1 < \lambda$, $0 \leq \alpha < 1$, if $f \in \mathcal{N}_\Sigma^{q(\alpha, \gamma, \lambda)}$, then

$$|a_2| \leq \min \left\{ \begin{aligned} &\frac{(1 - \alpha)(1 + q)[1]! \Gamma_q(1 + \lambda)}{[2] \Gamma_q(2 + \lambda)}, \\ &\sqrt{\frac{2q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[2][3] \Gamma_q(3 + \lambda)}}, \end{aligned} \right.$$

$$|a_3| \leq \min \left\{ \begin{aligned} &\frac{[2]}{([1] + [1])} \left[\frac{(1 - \alpha)(1 + q)[1]! \Gamma_q(1 + \lambda)}{[2] \Gamma_q(2 + \lambda)} \right]^2, \\ &\frac{[2]}{([1] + [1])} \left[\frac{2q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[2][3] \Gamma_q(3 + \lambda)} \right], \end{aligned} \right. \quad (76)$$

$$|a_3 - [2]a_2^2| \leq \frac{q(1 - \alpha)(1 + q)[2]! \Gamma_q(1 + \lambda)}{[3] \Gamma_q(3 + \lambda)}.$$

For $\lambda = 0$, $q \rightarrow 1$, in Theorem 13, we obtain the following corollary.

Corollary 16 (see [21]). For $0 \leq \alpha < 1$, $1 \leq \beta$, and $0 \leq \gamma$, if $f \in \mathcal{N}_\Sigma^{(\alpha, \beta, \gamma)}$, then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \alpha)}{1 + 2\beta + 6\gamma}}, & 0 \leq \alpha < 1 - \frac{(1 + \beta + 2\gamma)^2}{2(1 + 2\beta + 6\gamma)}, \\ \frac{2(1 - \alpha)}{1 + \beta + 2\gamma}, & 1 - \frac{(1 + \beta + 2\gamma)^2}{2(1 + 2\beta + 6\gamma)} \leq \alpha < 1, \end{cases} \quad (77)$$

$$|a_3| \leq \frac{2(1 - \alpha)}{1 + 2\beta + 6\gamma},$$

$$|a_3 - 2a_2^2| \leq \frac{2(1 - \alpha)}{1 + 2\beta + 6\gamma}.$$

For $\lambda = 0$, $\gamma = 0$, $q \rightarrow 1$, in Theorem 13, we obtain the following corollary.

Corollary 17 (see [30]). For $0 \leq \alpha < 1$ and $1 \leq \beta$, if $f \in \mathcal{N}_\Sigma^{(\alpha, \beta)}$, then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \alpha)}{1 + 2\beta}}, & 0 \leq \alpha < 1 - \frac{(1 + \beta)^2}{2(1 + 2\beta)}, \\ \frac{2(1 - \alpha)}{1 + \beta}, & 1 - \frac{(1 + \beta)^2}{2(1 + 2\beta)} \leq \alpha < 1, \end{cases} \quad (78)$$

$$|a_3| \leq \frac{2(1 - \alpha)}{1 + 2\beta},$$

$$|a_3 - 2a_2^2| \leq \frac{2(1 - \alpha)}{1 + 2\beta}.$$

For $\lambda = 0$, $\beta = 1$, $\gamma = 0$, $q \rightarrow 1$, in Theorem 13, we obtain the following [21].

Corollary 18 (see [21]). For $0 \leq \alpha < 1$, if $f \in \mathcal{H}_{\Sigma}^{(\alpha)}$, then

$$\begin{aligned} |a_2| &\leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \leq \alpha < \frac{1}{3}, \\ (1-\alpha), & \frac{1}{3} \leq \alpha < 1. \end{cases} \\ |a_3| &\leq \frac{2(1-\alpha)}{3}, \\ |a_3 - 2a_2^2| &\leq \frac{2(1-\alpha)}{3}. \end{aligned} \quad (79)$$

For $\lambda = 0$, $q \rightarrow 1$, $\beta = 1$, in Theorem 13, we obtain the following corollary.

Corollary 19 (see [25]). For $0 \leq \alpha < 1$, $1 \leq \beta$, and $0 \leq \gamma$, if $f \in \mathcal{B}_{\Sigma}^{(\alpha, \gamma)}$, then

$$\begin{aligned} |a_2| &\leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3(1+2\gamma)}}, & 0 \leq \alpha < \frac{1+2\gamma-2\gamma^2}{3(1+2\gamma)}, \\ \frac{2(1-\alpha)}{1+2\gamma}, & \frac{1+2\gamma-2\gamma^2}{3(1+2\gamma)} \leq \alpha < 1. \end{cases} \\ |a_3| &\leq \frac{2(1-\alpha)}{3(1+2\gamma)}. \end{aligned} \quad (80)$$

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] P. L. Duren, *Univalent Functions*, Grundlehren der Math. Wiss., Springer, New York, NY, USA, 1983.
- [2] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 23, no. 10, pp. 1188–1192, 2010.
- [3] D. A. Brannan and J. Clunie, "Aspects of contemporary complex analysis," in *Proceedings of the NATO Advanced study Institute Held at University of Durham*, Academic Press, New York, NY, USA, 1979.
- [4] E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $-\pi < \arg z < \pi$," *Archive for Rational Mechanics and Analysis*, vol. 32, no. 2, pp. 100–112, 1969.
- [5] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent functions," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 31, no. 2, pp. 70–77, 1986.
- [6] S. Altinkaya and S. Yalcin, "Initial coefficient bounds for a general class of biunivalent functions," *International Journal of Analysis*, vol. 2014, Article ID 867871, 4 pages, 2014.
- [7] B. S. Keerthi and B. Raja, "Coefficient inequality for certain new subclasses analytic bi-univalent functions," *Theory. Math. Appl.*, vol. 3, no. 1, pp. 1–10, 2013.
- [8] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, pp. 63–68, 1967.
- [9] X.-F. Li and A.-P. Wang, "Two new subclasses of bi-univalent functions," *International Mathematical Forum*, vol. 7, pp. 1495–1504, 2012.
- [10] N. Magesh and J. Yamini, "Coefficient bounds for certain subclasses of bi-univalent functions," *International Mathematical Forum. Journal for Theory and Applications*, vol. 8, no. 25–28, pp. 1337–1344, 2013.
- [11] S. Porwal and M. Darus, "On a new subclass of bi-univalent functions," *Journal of the Egyptian Mathematical Society*, vol. 21, no. 3, pp. 190–193, 2013.
- [12] G. Faber, "Über polynomische Entwicklungen," *Mathematische Annalen*, vol. 57, no. 3, pp. 1569–1573, 1903.
- [13] S. Gong, "The Bieberbach conjecture, translated from the 1989 Chinese original and revised by the author," in *AMS/IP Studies in Advanced Mathematics*, vol. 12, American Mathematical Society, Providence, RI, USA, 1999.
- [14] M. Schiffer, "Sur un problème d'extrémum de la représentation conforme," *Bull. Soc. Math. France*, vol. 66, pp. 48–55, 1938.
- [15] H. Airault, "Remarks on Faber polynomials," *International Mathematical Forum*, vol. 3, no. 9, pp. 449–456, 2008.
- [16] H. Airault and A. Bouali, "Differential calculus on the Faber polynomials," *Bulletin des Sciences Mathématiques*, vol. 130, no. 3, pp. 179–222, 2006.
- [17] S. Bulut, "Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions," *Comptes Rendus Mathématique*, vol. 352, no. 6, pp. 479–484, 2014.
- [18] S. G. Hamidi and J. M. Jahangiri, "Faber polynomial coefficient estimates for analytic bi-close-to-convex functions," *Comptes Rendus Mathématique*, vol. 352, no. 1, pp. 17–20, 2014.
- [19] S. G. Hamidi, S. Abd Halim, and J. M. Jahangiri, "Faber polynomial coefficient estimates for meromorphic Bi-starlike functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2013, Article ID 498159, 2013.
- [20] S. G. Hamidi and J. M. Jahangiri, "Faber polynomial coefficients of bi-subordinate functions," *Comptes Rendus Mathématique*, vol. 354, no. 4, pp. 365–370, 2016.
- [21] S. Bulut, "Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions," *Filomat*, vol. 30, no. 6, pp. 1567–1575, 2016.
- [22] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.
- [23] S. Kanas and D. Răducanu, "Some class of analytic functions related to conic domains," *Mathematica Slovaca*, vol. 64, no. 5, pp. 1183–1196, 2014.
- [24] H. Aldweby and M. Darus, "Some subordination results on q -analogue of Ruscheweyh differential operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 958563, 6 pages, 2014.
- [25] H. M. Srivastava, S. S. Eker, and R. M. Ali, "Coefficient bounds for a certain class of analytic and bi-univalent functions," *Filomat*, vol. 29, no. 8, pp. 1839–1845, 2015.

- [26] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 24, no. 9, pp. 1569–1573, 2011.
- [27] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 8–12, 1969.
- [28] H. Airault, "Symmetric sums associated to the factorizations of Grunsky coefficients," in *Proceedings of the Groups and Symmetries*, Montreal, Canada, April 2007.
- [29] H. Airault and J. Ren, "An algebra of differential operators and generating functions on the set of univalent functions," *Bulletin des Sciences Mathématiques*, vol. 126, no. 5, pp. 343–367, 2002.
- [30] J. M. Jahangiri and S. G. Hamidi, "Coefficient estimates for certain classes of bi-univalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2013, Article ID 190560, 2013.