Research Article

Entire Functions of Bounded L-Index: Its Zeros and Behavior of Partial Logarithmic Derivatives

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1. Introduction

In this paper, we find multidimensional sufficient conditions of boundedness of L-index in joint variables, which describe distribution of zeros and behavior of partial logarithmic derivatives. Recently, we published a paper [1] where some similar restrictions are established. Another approach was used by a slice function $F(z^0 + tb)$, where $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$, $b$ is a given direction in $\mathbb{C}^n \setminus \{0\}$, $F : \mathbb{C}^n \to \mathbb{C}$ is an entire function. It is a background for concept of function of bounded L-index in direction (see definition and properties in [2, 3]). We proved that if an entire function in $\mathbb{C}^n$ function $F$ is of bounded $l_j$-index in every direction $I_j = (0, \ldots, 0, 1, \ldots, 0, \ldots, 0)$, then $F$ is of bounded L-index in $j$th place joint variables for $L = (l_1, \ldots, l_n)$, $l_j : \mathbb{C}^n \to \mathbb{R}$ (Theorem 6, [1]). It helped us to find restrictions by directional logarithmic derivatives and distribution zeros in every direction $I_j$, $j \in \{1, \ldots, n\}$. We assumed that the logarithmic derivative in direction $I_j$ is bounded by a function $l_j$ outside some exceptional set, which contains all zeros of entire function $F$ (see definition of $G^*(F)$ below). Prof. Chyzhykov paid attention in conversation with authors that this exceptional set is too small because it does not contain neighborhoods of some zeros of the function in $\mathbb{C}^n$. Thus, it leads to the following question: is there sufficient conditions of boundedness of L-index in joint variables with larger exceptional sets? We give a positive answer to this question (Theorem 10). Moreover, we obtain sufficient conditions of boundedness of L-index in joint variables by estimating the maximum modulus of an entire function on the skeleton in polydisc by minimum modulus (Theorem 7). Theorems 9 and 10 present restrictions by a measure of zero set of an entire function $F$, under which $F$ has bounded L-index in joint variables. Nevertheless, we do not know whether the obtained conditions in Theorems 7–10 are necessary too in $\mathbb{C}^n$, $(n \geq 2)$. Note that these propositions are new even for entire functions of bounded index in joint variables, i. e. $L = (1, \ldots, 1)$ (see definition and properties in [4–8]).

It is known [9] that for every entire function $f$ with bounded multiplicities of zeros there exists a positive continuity on $[0; +\infty)$ function $l(r) (r = |z|)$ such that $f$ is of bounded L-index. This result can be easily generalized for entire functions in $\mathbb{C}^n$. Thus, the concept of bounded L-index in joint variables allows the study of growth properties of any entire functions with bounded multiplicities of zero points.

It should be noted that the concepts of bounded L-index in a direction and bounded L-index in joint variables have few
advantages in the comparison with traditional approaches to study properties of entire solutions of differential equations. In particular, if an entire solution has bounded index [10], then it immediately yields its growth estimates, a uniform distribution of its zeros, a certain regular behavior of the solution, and so forth. A full bibliography about application in theory of ordinary and partial differential equations is in [3, 11, 12].

The paper is devoted to two old problems in theory of entire and meromorphic functions. The first problem is the establishment of sharp estimates for the logarithmic derivative of the functions in the unit disc outside some exceptional set. Chyzhykov et al. [13–16] considered various formulations of the problem. The obtained estimates were used to study properties of holomorphic solutions of differential equations. Instead, the authors assume that partial logarithmic properties of holomorphic solutions of differential equations of the form

\[ F(z) = f(z) \prod_{j=1}^{n} \frac{1}{z - z_j}, \]

where \( f \) is an entire function, and \( z_1, \ldots, z_n \) are fixed points in \( \mathbb{C}^n \). They are also included in English monographs [3, 11].

Below we use results from Ukrainian papers [23, 24], but they are also included in English monographs [3, 11].

2. Main Definitions and Notations

We need some standard notations. Let \( \mathbb{R}_+ = [0, +\infty) \). Denote \( 0 = (0, \ldots, 0) \in \mathbb{R}_n^m, \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}_n^m, 2 = (2, \ldots, 2) \in \mathbb{R}_n^m, \mathbf{1}_j = (0, \ldots, 0, 1, \ldots, 0) = \mathbb{R}^n_j \).

For \( R = (r_1, \ldots, r_n) \in \mathbb{R}_n^m \) and \( K = (k_1, \ldots, k_n) \in \mathbb{Z}_n^m \) denote \( \|R\| = r_1 + \cdots + r_n \) and \( K! = k_1! \cdots k_n! \). For \((a_1, \ldots, a_m) \in \mathbb{C}^m, b = (b_1, \ldots, b_n) \in \mathbb{C}^n, z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), we will use formal notations without violating the existence of these expressions:

\[
\begin{align*}
|a| &= (|a_1|, |a_2|, \ldots, |a_m|), \\
A &\pm B = (a_1 \pm b_1, \ldots, a_m \pm b_m), \\
AB &= (a_1 b_1, \ldots, a_m b_m), \\
A &= \begin{pmatrix} a_1 & \cdots & a_m \\ b_1 & \cdots & b_m \end{pmatrix}, \\
A^R &= d_1^{a_1} d_2^{a_2} \cdots d_m^{a_m}, \\
dz &= dz_1 dz_2 \cdots dz_n.
\end{align*}
\]

If \( a, b \in \mathbb{R}^n \) the notation \( a < b \) means that \( a_j < b_j \) (\( j = 1, \ldots, n \)); similarly, the relation \( a \leq b \) is defined.

The polydisc \( \{ z \in \mathbb{C}^n : |z - z_j^0| < r_j, j = 1, \ldots, n \} \) is denoted by \( \mathbb{D}^n(z^0,R) \), its skeleton \( \{ z \in \mathbb{C}^n : |z - z_j^0| = r_j, j = 1, \ldots, n \} \) is denoted by \( \mathbb{T}^n(z^0,R) \), and the closed polydisc \( \{ z \in \mathbb{C}^n : |z - z_j^0| \leq r_j, j = 1, \ldots, n \} \) is denoted by \( \mathbb{D}^n(z^0,R) \).

For \( K = (k_1, \ldots, k_n) \in \mathbb{Z}_n^m \) and partial derivatives of entire function \( F(z) = F(z_1, \ldots, z_n) \), we will use the notation

\[
F^{(K)}(z) = \frac{\partial^{|K|} F}{\partial z^K} = \frac{\partial^{k_1+\cdots+k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}.
\]

Let \( L(z) = (l_1(z_1), \ldots, l_n(z_n)) \), where \( l_j(z) \) are positive continuous functions of \( z \in \mathbb{C}^n, j = 1, \ldots, n \). An entire function, \( F(z) \in \mathbb{C}^n \), is called a function of bounded \( L \)-index in joint variables if it exists a number \( m \in \mathbb{Z}_+ \) such that for all \( z \in \mathbb{C}^n \) and \( J = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}_n^m \)

\[
\bigg\| \frac{F^{(J)}}{J!L(J)}(z) \bigg\| \leq \max \bigg\{ \bigg\| \frac{F^{(K)}}{K!L(K)}(z) \bigg\| : K \in \mathbb{Z}_n^m, \|K\| \leq m \bigg\}.
\]

If \( l_j = l_j(|z_j|) \) then we obtain a concept of entire functions of bounded \( L \)-index in a sense of definition given in [24]. If \( l_j(z_j) = 1, j = 1, 2, \ldots, n \), then the entire function is called a function of bounded index in joint variables [4–8, 25].

The least integer \( m \) for which inequality (3) holds is called \( L \)-index in joint variables of the function \( F \) and is denoted by \( N(F,L) \).

For \( R \in \mathbb{R}_n^m, j = 1, \ldots, n \) and \( L(z) = (l_1(z), \ldots, l_n(z)) \) we define

\[
\lambda_{1,j}(R) = \inf_{z \in \mathbb{C}^n} \inf_{z^0} \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[ z^0, \frac{R}{L(z^0)} \right] \right\},
\]

\[
\lambda_{2,j}(R) = \sup_{z \in \mathbb{C}^n} \sup_{z^0} \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^n \left[ z^0, \frac{R}{L(z^0)} \right] \right\}.
\]

Let \( \lambda_1(R) = (\lambda_{1,1}(R), \ldots, \lambda_{1,n}(R)) \),

\[
\Lambda_1(R) = (\lambda_{1,1}(R), \ldots, \lambda_{1,n}(R)).
\]

By \( Q^n \) we denote a class of functions \( L(z) \) which for every \( R \in \mathbb{R}_n^m \) and \( j = 1, \ldots, n \) satisfy the condition

\[
0 < \lambda_{1,j}(R) \leq \lambda_{2,j}(R) < +\infty.
\]

If \( n = 1 \) then \( Q \equiv Q^1 \).

Let \( \tilde{L}(z) = (\tilde{l}_1(z), \ldots, \tilde{l}_n(z)) \). A notation \( \tilde{L} \equiv \tilde{L} \) means that there exist \( \tilde{\Theta}_1 = (\tilde{\Theta}_{1,1}, \ldots, \tilde{\Theta}_{1,n}) \in \mathbb{R}^n, \tilde{\Theta}_2 = (\tilde{\Theta}_{2,1}, \ldots, \tilde{\Theta}_{2,n}) \in \mathbb{R}_n^m \) such that \( \forall z \in \mathbb{C}^n \tilde{\Theta}_{1,j}(z) \leq \tilde{l}_j(z) \leq \tilde{\Theta}_{2,j}(z) \).

3. Auxiliary Propositions

We need the following theorems.

**Theorem 1** ([11, p. 158, Th. 4.2], see also [23]). Let \( L \in Q^n \) and \( L \equiv \tilde{L} \). An entire function \( F : \mathbb{C}^n \to \mathbb{C} \) has bounded \( L \)-index in joint variables if and only if \( F \) has bounded \( \tilde{L} \)-index in joint variables.

**Theorem 2** (see [1]). Let \( L \in Q^n \). An entire function \( F \) is of bounded \( L \)-index in joint variables if and only if, for any \( R' \),
Lemma 5. If $L : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function such that $(\forall R \in \mathbb{R}_+) \Lambda_2(R) < \infty$ then $(\forall R \in \mathbb{R}_+) \Lambda_1(R) \geq 1/\Lambda_2(R)\Lambda(R) > 0$.

Proof. Let $(\forall R \in \mathbb{R}_+) \Lambda_2(R) < \infty$ i.e. $\forall j \in \{1, \ldots, n\} \lambda_2, j(R) := \lambda(R)$. Hence, we have $l_j(z) \leq l_j(R)$ for $z \in D^n(z_0, R/L(z_0))$. This means that $|z_j - z_0| \leq r_j/l_j(z_0) \leq r_j\lambda_2(R)/l_j(z)$. Using definition of $\Lambda_1(R)$, we deduce

$$
\inf \left\{ \frac{l_j(z)}{l_j(z_0)} : z \in D^n(z_0, R/L(z_0)) \right\} = \frac{1}{\sup \left\{ l_j(z_0) l_j(z) : z \in D^n(z_0, R/L(z_0)) \right\}} 
\geq \frac{1}{\Lambda_2(R)}.
$$

Thus, $\Lambda_1(R) \geq 1/\Lambda_2(R \Lambda_2(R))$.

Remark 6. By Lemma 5 the left inequality in (5) is excessive because the condition $\Lambda_2(R) < +\infty$ implies $\Lambda_1(R) > 0$. But in our considerations we will use so $\Lambda_1(R)$ as $\Lambda_2(R)$. It is convenient.

4. Estimate Maximum Modulus on a Skeleton in Polydisc

Let $Z_R$ be a zero set of entire function $F$. We denote

$$
G_R(F) = \bigcup_{z \in Z_R} \left\{ z \in \mathbb{C}^n : |z_j - z_0| < \frac{r_j}{l_j(z_0)} \right\} \forall j \in \{1, 2, \ldots, n\} = \bigcup_{z \in Z_R} D^n(z_0, R/L(z_0)).
$$

Theorem 7. Let $L : \mathbb{C}^n \to \mathbb{R}_+$ be an entire function in $\mathbb{C}_+$. If $\exists R > 0 \exists p_2 \geq 1 \exists \Theta \in \mathbb{R}_+, 0 < \Theta < R, \exists R' > 0, (R' = 0$ for $Z_R = \emptyset$) such that $\forall z_0 \in \mathbb{C}^n \exists R_0 = R_0(z_0) \in \mathbb{R}_+, \Theta \leq R' \leq R$, for which

$$
\text{meas} \left\{ \mathbb{T}^n \left( z_0, \frac{R_0}{L(z_0)} \right) \cap G_R(F) \right\} < \left( \frac{2\pi}{3} \right)^n \prod_{j=1}^n \lambda_2, j(2(R + 1)) l_j(z_0).
$$

$$
\max \left\{ |F(z)| : z \in \mathbb{T}^n \left( z_0, \frac{R_0}{L(z_0)} \right) \right\} \leq p_2 \min \left\{ |F(z)| : z \in \mathbb{T}^n \left( z_0, \frac{R_0}{L(z_0)} \right) \right\}
$$

then the function $F$ has bounded $L$-index in joint variables (meas is the Lebesgue measure on the skeleton in the polydisc).

Proof. By Theorem 4, we will show that $\exists p_1 > 0 \forall z_0 \in \mathbb{C}^n$

$$
\max \left\{ |F| : z \in \mathbb{T}^n \left( z_0, \frac{R + 1}{L(z_0)} \right) \right\} \leq p_1 \max \left\{ |F| : z \in \mathbb{T}^n \left( z_0, \frac{R}{L(z_0)} \right) \right\}.
$$
Denote $l'_j = \max\{l_j(z) : z \in \mathbb{D}^n[z^0, 2(R + 1)/L(z^0)]\}$, $\rho_{j,0} = r_j/l_j(z^0)$, $\rho_{j,k} = \rho_{j,0} + (k \cdot \theta_j)/l'_j$, $k \in \mathbb{N}$, $j \in \{1, \ldots, n\}$. The following estimate holds

$$\frac{\theta_j}{l'_j} \leq \frac{r_j}{l_j(z^0)} \leq \frac{2r_j + 2}{l'_j(z^0)} \leq \frac{r_j + 1}{l_j(z^0)}. \quad (14)$$

Hence, there exists $S^* = (s_1^*, \ldots, s_n^*) \in \mathbb{N}$ independent of $z^0$ such that

$$\rho_{j,m_j - 1} < \frac{r_j + 1}{l'_j(z^0)} < \rho_{j,m_j} \leq \frac{2r_j + 2}{l'_j(z^0)} \quad (15)$$

for some $m_j = m_j(z^0) \leq s_j^*$ because $L \in Q^n$. Indeed,

$$\left(\frac{(2r_j + 2)/l_j(z^0) - \rho_{j,0}}{\theta_j/l'_j}\right) = \left(2r_j + 2/r_j - 1\right) = \frac{r_j + 2}{\theta_j} \max\left\{\frac{l_j(z)}{l'_j(z^0)} : z \in \mathbb{D}^n[z^0, 2(R + 1)/L(z^0)]\right\} \leq \frac{r_j + 2}{\theta_j} \lambda_{2,j}(2(R + 1)). \quad (16)$$

Thus, $s_j^* = \max\{(l_j + 2)/\theta_j, \lambda_{2,j}(2(R + 1))\}$, where $[x]$ is the integer part of $x \in \mathbb{R}$.

Let $M_0 = (m_1, \ldots, m_n)$ and $r_k^{**}$ be such a point in $C^n$ that

$$|F(r_k^{**})| = \max\{|F(z) : z \in T^n(z^0, \mathcal{R}_K)|, \quad (17)$$

where $K = (k_1, \ldots, k_n), \mathcal{R}_K = (\rho_{j,k}, \ldots, \rho_{n,k})$ and $r_j^{**}$ is the intersection point in $C$ of the segment $[z^0, r_j^{**}]$ with $|z_j - z_j^0| = \rho_j$ for $j \geq 1$. We construct a sequence of polydisc $D^n(z^0, \mathcal{R}_K)$ with $K \leq M_0, \mathcal{R}_K = R/L(z^0) = (\rho_1, \ldots, \rho_n)$ and $L(z^0) = (\theta_1/l'_1, \ldots, \theta_n/l'_n)$ (see Figures 1 and 2).

Denote $\alpha^{(j)}_K = (r_{1,k}^{**}, \ldots, r_{j-1,k}^{**}, r_{j,k}^{**}, r_{j+1,k}^{**}, \ldots, r_{n,k}^{**})$. Hence, for every $r_j > \theta_j$ and $K \leq S^* : |z_j - z_j^0| = \theta_j/l'_j \leq r_j/l_j(\alpha^{(j)}_K)$. Thus, for some $R_0 = R(\alpha^{(j)}_K) \in \mathbb{R}^n, \Theta \leq R_0 \leq R$, we deduce

$$|F(r_k^{**})| \leq \max\{|F(z) : z \in T^n(\alpha^{(j)}_K, R_0/L(\alpha^{(j)}_K))|, \quad$$

$$\leq \min\{|F(z) : z \in T^n(\alpha^{(j)}_K, R_0/L(\alpha^{(j)}_K)) \cap G_{R'}(F) \cap D^n(z^0, \mathcal{R}_K) \}, \quad (18)$$

To deduce (18), we implicitly used that

$$\left(\min\{|F(z) : z \in T^n(\alpha^{(j)}_K, R_0/L(\alpha^{(j)}_K)) \cap D^n(z^0, \mathcal{R}_K) \}, \quad (19)$$

Condition (11) provides (19). Indeed, we will find a lower estimate of measure of the set $T^n(\alpha^{(j)}_K, R_0/L(\alpha^{(j)}_K)) \cap D^n(z^0, \mathcal{R}_K)$ and will show that the measure is not lesser than a left part of inequality (11).
The set $\mathcal{T}^n(\alpha^{(j)}_K, R^0/L(\alpha^{(j)}_K)) \cap \mathbb{D}^n[\zeta^0, R_{K-1}]$ is a Cartesian product of the following arcs on circles: for every $m \in \{1, \ldots, n\}$, $m \neq j$ (see Figure 3)

$$\left\{ z_m \in \mathbb{C} : \left| z_m - z_{m, K}^* \right| = \frac{r_m^0}{l_m(\alpha^{(j)}_K)} \right\} \cap \left\{ z_m \in \mathbb{C} : \left| z_m - z_m^0 \right| \leq \rho_{m, k_m} \right\}$$

and for $m = j$ (see Figure 4)

$$\left\{ z_j \in \mathbb{C} : \left| z_j - z_{j, K}^* \right| = \frac{r_j^0}{l_j(\alpha^{(j)}_K)} \right\} \cap \left\{ z_j \in \mathbb{C} : \left| z_j - z_j^0 \right| \leq \rho_{j, k_{j-1}} \right\}.$$  

It is easy to prove that the length of arc equals

$$\frac{2r_m^0}{l_m(\alpha^{(j)}_K)} \cdot \arccos \frac{r_m^0}{2l_m(\alpha^{(j)}_K) \rho_{m, k_m}} \quad \text{for } m \neq j. \quad (22)$$

$$\frac{2r_j^0}{l_j(\alpha^{(j)}_K)} \cdot \arccos \frac{r_j^0}{2l_j(\alpha^{(j)}_K) \rho_{j, k_{j-1}}} \quad \text{for } m = j. \quad (23)$$

But for $m \neq j$ $r_m^0/l_m(\alpha^{(j)}_K) \leq \rho_{m, k_m}$ and $r_j^0/l_j(\alpha^{(j)}_K) \leq \rho_{j, k_{j-1}}$ the argument in arccosine from (23) and (22) does not exceed $1/2$. This means that the length of arc is not lesser than

$$\frac{2r_m^0}{l_m(\alpha^{(j)}_K)} \cdot \arccos \frac{1}{2} \geq \frac{2\theta_m \pi}{3l_m(z_j^0) \lambda_{2, m}(2(R + 1))}$$

for every $m \in \{1, 2, \ldots, n\}$, because $L \in Q^0$. Accordingly, the measure of the set

$$\mathcal{T}^n(\alpha^{(j)}_K, R^0/L(\alpha^{(j)}_K)) \cap \mathbb{D}^n[\zeta^0, R_{K-1}]$$

on the skeleton of polydisc is always not less than $\prod_{m=1}^n (2\theta_m \pi/3l_m(z_j^0) \lambda_{2, m}(2(R + 1)))$. Assuming a strict inequality in (11), we deduce that (19) is valid.

Applying (18) $m_j$ th times in every variable $z_j$, we obtain

$$\max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R^1, \frac{R}{L(z_j^0)} \right) \right\}$$

$$\leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_{M_j - 1} \right) \right\} \leq \cdots \leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_{M_j - m_{j-1}} \right) \right\} \leq \cdots \leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_0 \right) \right\}$$

$$\leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_{M_j - m_{j-1}} \right) \right\} \leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_0 \right) \right\}$$

$$\leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_{M_j - m_{j-1}} \right) \right\} \leq \max \left\{ |F(z)| : z \in \mathbb{T}^n \left( \zeta^0, R_0 \right) \right\}$$

By Theorem 2 the function $F$ has bounded $L$-index in joint variables.

Let us denote $c(z', r) = \{ z \in \mathbb{C} : |z - z'| = r/l(z') \}$. For $n = 1$ Theorem 7 implies the following corollary.

**Corollary 8.** Let $l \in Q$, $f$ be an entire function. If $\exists r > 0, \exists r^0 \geq 0, \exists p_2 \geq 1$ $\forall \theta \in (0, r)$, such that $\forall \zeta^0 \in \mathbb{C}$ $\exists p_0 = r^0(\zeta^0) \in [\theta; r]$, and $\text{meas} (\{ \zeta^0, \rho^{0} \} \cap G_{r^0}(F)) < 2\pi \theta / 3(\zeta^0) \lambda_{2}(2r + 2)$ and

$$\max \left\{ |f(z)| : z \in c(\zeta^0, r^0) \right\} \leq p_2 \min \left\{ |f(z)| : z \in c(\zeta^0, r^0) \setminus G_{r^0}(f) \right\}$$

then the function $f$ has bounded 1-index (here $\text{meas}$ means the Lebesgue measure on the circle).
In a some sense, this corollary is new even for an entire function of one variable because the circle theorem 9.

5. Behavior of Partial Logarithmic Derivatives

Denote $\mathcal{J} = \{(j_1, \ldots, j_n) : j_i \in \{0, 1\}, i \in \{1, \ldots, n\}\} \setminus \emptyset$.

Theorem 9. Let $L \in \mathbb{Q}^n$. If an entire function $F$ satisfies the following conditions

(1) for every $R > 0$ there exists $p_1 = p_1(R) > 0$ such that for all $z \in \mathbb{C}^n \setminus G_R(F)$ and for all $J \in \mathcal{J}$

$$|\ln F(z)| \leq p_1 L^J(z),$$

where $\ln F(z)$ is the principal value of logarithm.

(2) for every $R > 0$ and $R' \geq 0$ exists $p_2 = p_2(R, R') \geq 1$ that for all $z^0 \in \mathbb{C}^n$ such that $T^n(z^0, R/L(z^0)) \cap G_R(F) = \bigcup_i C_i \neq \emptyset$, where the sets $C_i$ are connected disjoint sets, and either (a) $\max \{\min_{eC_i} |F(z)| : |F(z)| \leq p_2 \min_{eC_i} |F(z)|\}$, or (b) $\max \{\min_{eC_i} |F(z)| : |F(z)| \leq p_2 \max_{eC_i} |F(z)|\}$, or (c) $|F(z^*)| = \max \{\min_{eC_i} |F(z)|, |F(z^*)| = \min \{\max_{eC_i} |F(z)|,\}$ and $z^*, z^{**}$ belong to the same set $C_i$.

(3) for every $R > 0$ there exists $n^*(R) > 0$ such that for all $z \in \mathbb{C}^n$

$$\text{meas}\left(\left[Z_R \cap \bigcap_{j=1}^{n^*(R)} \frac{R}{L(z)} \right] \right) \leq n^*(R).$$

then $F$ has bounded $L$-index in joint variables (here $\text{meas}$ is $(2n - 2)$-dimensional of the Lebesgue measure).

Proof. Let $z^0 \in \mathbb{C}^n$ be arbitrarily chosen point. In view of Theorem 7 we need to prove that

$$\text{meas}\left(\left[T^n(z^0, R/L(z^0)) \cap G_R(F) \right] \right) \leq \frac{2\pi n}{3} \prod_{j=1}^{n} \frac{\theta_j}{\lambda_{2,j}(2(R + 1))},$$

for some $R^0 = R^0(z^0)$.

Let $R > 0$ be arbitrary radius. We choose $\Theta, R' \in \mathbb{R}^n$, such that $\theta_j < 2r_j/(2 + 2\lambda_{2,j}(2(R + 1)))$,

$$r_j' < \min\left\{\theta_j, \frac{2\lambda_{1,j}^2(R + RA_{2,j}(R)/\lambda_{1,j}(R)) \theta_j(r_j - \theta_j)}{3(n^*(R + RA_{2,j}(R)/\lambda_{1,j}(R)))^{1/n} \lambda_{2,j}(2(R + 1))}\right\}.$$

Let $dS = ds_1 \cdots ds_n, S = (s_1, \ldots, s_n), \omega_2$ be a volume measure in $\mathbb{R}^{2n}$. Clearly, (see [28, p. 75-76])

$$\int_{\mathbb{R}^{(\mathbb{C}^R)}} u(z) d\omega_2 = \int_0^\infty \cdot \left(\int_0^{2\pi} \int_0^{2\pi} u(z^0 + Se^{i\theta}) d\theta_1 \cdots d\theta_n\right) ds_1 \cdots ds_n$$

(32)

where $u$ is plurisubharmonic function. Hence,

$$\int_0^{2\pi} \text{meas}\left(T^n(z^0, S) \cap G_R(F)\right) dS$$

$$= \text{meas}\left(\mathbb{D}^n \left[z^0, \frac{R}{L(z^0)}\right] \cap G_R(F)\right).$$

(33)

Obviously, there can exist points $z' \in Z_F \cap \mathbb{D}^n[z^0, R/L(z^0)]$ such that

$$\mathbb{D}^n \left[z^0, \frac{R}{L(z^0)}\right] \cap \mathbb{D}^n \left[z', \frac{R'}{L(z')}\right] \neq \emptyset.$$  

(34)

Let $z_j'$ be the intersection point of the segment $[z_j^0, z_j']$ and the circle $|z_j - z_j^0| = r_j/l_j(z^0)$, $j \in \{1, \ldots, n\}$. Then $|z_j'' - z_j'| \leq r_j/l_j(z^0)$ and $z'' \in T^n(z_j^0, R/L(z^0))$. Using $L \in \mathbb{Q}^n$, we estimate maximum distance between $z_j$ and $z_j'$:

$$l_j(z') \geq l_j(z'') \geq \frac{l_j(z^0) \cdot l_j(z''(z^0))}{l_j(z^0)} \geq \frac{\lambda_{1,j}(R)}{\lambda_{2,j}(R')} l_j(z^0),$$

$$\left|z_j^0 - z_j'\right| \leq \left|z_j^0 - z_j''\right| + \left|z_j'' - z_j'\right| \leq \frac{r_j}{l_j(z^0)} + \frac{r_j}{l_j(z_j')} \leq \frac{r_j}{l_j(z^0)} + \frac{\lambda_{1,j}(R)}{\lambda_{2,j}(R')}.$$ 

(35)}
\[
\int_{\Theta(L(z^0))}^{R/|L(z)|} \text{meas} \left\{ T^n \left( z^0, S \right) \cap G_{R'}(F) \right\} \, dS \leq \int_{Z_{\Theta \cap D \cap [z_0, R'/|L(z^0)|]}} \chi_F(z) n^n \prod_{j=1}^{n} \left( \frac{r_j}{l_j(z^0)} \right)^2 \, dV_{2n-2} \\
\leq n^n \prod_{j=1}^{n} \left( \frac{r_j}{l_j(z^0)} \right)^2 \left( R'' \right) \int_{Z_{\Theta \cap D \cap [z_0, R'/|L(z^0)|]}} \chi_F(z) \, dV_{2n-2} \leq n^* \left( R'' \right) \prod_{j=1}^{n} \left( \frac{r_j}{l_j(z^0)} \right)^2 \left( R'' \right) \\
< n^* \left( R'' \right) n^n \prod_{j=1}^{n} \left( \frac{2\lambda_{1,j}^2 (R + RA_j(R)/\Lambda_j(R)) \theta_j (r_j - \theta_j)}{3} \right) \\
= \left( \frac{2\pi}{3} \right)^n \prod_{j=1}^{n} \frac{\theta_j^2 (r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1))^2 l_j(z^0)}.
\]

Besides, we have that
\[
\int_{\Theta(L(z^0))}^{R/|L(z)|} \text{meas} \left\{ T^n \left( z^0, S \right) \cap G_{R'}(F) \right\} \, dS \\
= \text{meas} \left\{ \prod^n \left[ z^0, \frac{\Theta}{L(z^0)} \right] \cap G_{R'}(F) \right\} \\
\leq n^n \prod_{j=1}^{n} \frac{\theta_j^2 (r_j - \theta_j)}{l_j(z^0)}.
\]

Hence, the following difference is positive
\[
\left( \frac{2\pi}{3} \right)^n \prod_{j=1}^{n} \frac{\theta_j (r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1))^2 l_j(z^0)} \\
- \int_{\Theta(L(z^0))}^{R/|L(z)|} \text{meas} \left\{ T^n \left( z^0, S \right) \cap G_{R'}(F) \right\} \, dS \\
\geq \left( \frac{2\pi}{3} \right)^n \prod_{j=1}^{n} \frac{\theta_j (r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1))^2 l_j(z^0)} \\
- n^n \prod_{j=1}^{n} \frac{\theta_j^2}{l_j(z^0)} \\
= n^n \prod_{j=1}^{n} \frac{\theta_j (2r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1)) l_j(z^0) (2 + 3\lambda_{2,j} (2 (R + 1)))} > 0
\]

because \( \theta_j < 2r_j / (2 + 3\lambda_{2,j} (2 (R + 1))) \). From (36) it follows that
\[
\int_{\Theta(L(z^0))}^{R/|L(z)|} \text{meas} \left\{ T^n \left( z^0, S \right) \cap G_{R'}(F) \right\} \, dS \\
< \left( \frac{2\pi}{3} \right)^n \prod_{j=1}^{n} \frac{\theta_j (r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1))^2 l_j(z^0)} \\
= \left( \frac{2\pi}{3} \right)^n \prod_{j=1}^{n} \frac{\theta_j (r_j - \theta_j)}{\lambda_{2,j}^2 (2 (R + 1))^2 l_j(z^0)}.
\]
and deduce
\[
\ln \left| \frac{F(z^*)}{F(z^{**})} \right| \leq \int_{z^{**}}^{z^*} |\ln F(z)| \, |dz^j|
\]
\[
\leq \int_{z^{**}}^{z^*} p_1 L^j (z) \, |dz^j|
\]
\[
\leq p_1 L^j (z) \Lambda_2^j (R).
\]

Hence,
\[
\max \left\{ |F(z)| : z \in T^n \left( z^0, \frac{R_0}{L(z^0)} \right) \right\} = |F(z^*)|
\]
\[
\leq \exp \left\{ \pi^n p_1 R^j \Lambda_2^j (R) \right\} |F(z^{**})|
\]
\[
= \exp \left\{ \pi^n p_1 R^j \Lambda_2^j (R) \right\} \min_{z \in C_i} |F(z)|
\]
\[
\leq \exp \left\{ \pi^n p_1 R^j \Lambda_2^j (R) \right\} \min_{z \in C_i} |F(z)|
\]
\[
= \exp \left\{ \pi^n p_1 R^j \Lambda_2^j (R) \right\} \min_{z \in C_i} |F(z)|
\]
\[
\cdot \min \left\{ |F(z)| : z \in T^n \left( z^0, \frac{R_0}{L(z^0)} \right) \right\} \setminus G_R(F).
\]

By Theorem 7 the function $F$ has bounded $L$-index in joint variables.

Let us denote $\Delta$ as Laplace operator. We will consider $\Delta \ln |F|$ as generalized function. Using some known results from potential theory, we can rewrite Theorem 9 as follows.

**Theorem 10.** Let $L \in Q^n$. If an entire function $F$ satisfies the following conditions

1. For every $R > 0$ there exists $p_1 = p_1(R) > 0$ such that for all $z \in C^n \setminus G_R(F)$ and for every $j \in \{1, \ldots, n\}$
   \[
   \left| \frac{\partial \ln F(z)}{\partial z_j} \right| \leq p_1 L_j (z),
   \]
   where $\ln F(z)$ is the principal value of logarithm.

2. For every $R > 0$ and $R' \geq 0$ exists $p_2 = p_2(R, R') \geq 1$ that for all $z^0 \in C^n$ such that $T^n(z^0, R/L(z^0)) \setminus G_R(F) = \bigcup_i C_i = \emptyset$, where the sets $C_i$ are connected disjoint sets, and either (a) $\max_{x \in C_i} |F(z)| \leq p_2 \min_{x \in C_i} |F(z)|$, or (b) $\max_{x \in C_i} |F(z)| \leq p_2 \min_{x \in C_i} |F(z)|$, or (c) $|F(z^*)^j| = \max_{x \in C_i} |F(z)|$, $|F(z^{**})^j| = \min_{x \in C_i} |F(z)|$, and $z^*, z^{**}$ belong to the same set $C_i$.

3. For every $R > 0$ there exists $n^*(R) > 0$ such that for all $z \in C^n$
   \[
   \int_{D^n(z^0, R/L(z^0))} \Delta \ln |F| \, dV_{2n} \leq n^*(R)
   \]
then $F$ has bounded $L$-index in joint variables.

**Proof.** Ronkin [28, p. 230] deduced the following formula for entire function:
\[
\int_{D^n(0, R)} \Delta \ln |F| \, dV_{2n} = 2\pi \int_{|z| < R} \gamma_F(z) \, dV_{2n-2},
\]
where $\gamma_F(z)$ is a multiplicity of zero point of the function $F$ at point $z$, $R^* \in R^n$ is arbitrary radius. Let $\chi_F(z)$ be a characteristic function of zero set of $F$. Then $\chi_F(z) \leq \gamma_F(z)$. Hence,
\[
\int_{D^n(0, R)} \Delta \ln |F| \, dV_{2n} \leq \frac{n^*(R)}{2\pi},
\]
that is, inequality (29) holds.

Now we want to prove that (45) implies (28). For every $J \in \mathcal{F} \setminus \bigcup_{k=1}^n I_k$ and $z^0 \in C^n \setminus G_R(F)$, Cauchy’s integral formula can be written in the following form
\[
\left( \frac{\ln |F(z)|}{L(z)} \right)^{(n)}
\]
\[
= \frac{1}{(2\pi)^n} \int_{T^n(z^0, R/L(z^0))} \left( \ln |F(z)| \right)^{(1_m)} (z - z^0)^{-1-m+1} dz,
\]
where $m$ is such that $j_m = 1$. If $z^0 \in C^n \setminus G_R(F)$ and $z' \in Z_F \subset G_R(F)$, then for every $j \in \{1, \ldots, n\}$
\[
|z^0_j - z'_j| \geq \frac{r_j}{l_j(z^0)} \geq \frac{r_j A_{1,j}(R)}{l_j(z^0)} \geq \frac{r_j A_{1,j}(R)}{2L(z^0)}.
\]
Let us consider the set $A = \bigcup_{x \in C^n \setminus G_R(F)} T^n(z^0, R A_1(R)/2L(z^0))$. We want to find the greatest radius $R^* \in R^n$ such that $G_R(F) \cap A = \emptyset$.

\[
\frac{R}{L(z^0)} - \frac{RA_1(R)}{2L(z^0)} \geq \frac{R}{2A_1(R)L(z^0)}.
\]
Thus, for $R^* = R/3 \ A \subset \mathbb{C}^n \setminus G_R(F)$. Using (45), we obtain that for every $z^0 \in \mathbb{C}^n \setminus G_R(F)$
\[
\left(\ln F(z^0)\right)^{(1/n)} \leq \frac{1}{(2\pi)^n} \cdot \int_{\Gamma(z^0, R(\mathbb{R})/2\mathbb{L}(z^0))} \left|\ln F(z)\right|^{1/n} |dz| \leq \frac{1}{(2\pi)^n}
\]
\[
\cdot \int_{\Gamma(z^0, R(\mathbb{R})/2\mathbb{L}(z^0))} \left(\frac{2L(z^0)}{R\Lambda_1(R)}\right)^{j-1}p_1(\frac{1}{3}R)\lambda_{2,m}(0.5\Lambda_1(R))|dz|
\]
\[
= \frac{r_m\lambda_{1,m}(R)}{(2\pi)^n} \left(\frac{2L(z^0)}{R\Lambda_1(R)}\right)^{j}p_1(\frac{1}{3}R)\lambda_{2,m}(0.5\Lambda_1(R))
\]
\[
\cdot \left(\frac{2L(z^0)}{R\Lambda_1(R)}\right)^{j} \leq C(R)L_j(z^0),
\]

where $C(R) = 0.5p_1((1/3)R)\max_{x \in J_1}(\lambda_{2,m}(0.5\Lambda_1(R))r_m \lambda_{1,m}(R)(2/\Lambda_1(R))^j)$. Thus, we proved that inequality (28) is valid.

For $n = 1$ Theorem 9 implies the following corollary.

**Corollary 11.** Let $l \in \mathbb{Q}$, $f$ be an entire in $\mathbb{C}$ function, $n(r, z^0, f)$ a number of zeros of the $f$ in the disc $|z - z_0| \leq r / L(z^0)$. If the function $f$ satisfies the following conditions:

(1) for every $r > 0$ there exists $p_1 = p_1(r) > 0$ such that for all $z \in \mathbb{C} \setminus G_r(f)$
\[
\left|\frac{f'(z)}{f(z)}\right| \leq p_1(l(z)),
\]

(2) for every $r > 0$ and $r' \geq 0$ exists $p_2 = p_2(r, r') \geq 1$ that for all $z^0 \in \mathbb{C}$ such that $|z - z^0| = r / L(z^0)$ \ $G_r(f) = \bigcup_i C_i \neq \emptyset$, where the sets $C_i$ are connected disjoint sets, and either (a) $\max_{x \in C_i} f(z) \leq p_2, \min_{x \in C_i} f(z), \text{ or (b) } \max_{x \in C_i} f(z) \leq p_2, \min_{x \in C_i} f(z), \text{ or (c) } |f(z^*)| = \max_{x \in C_i} f(z), \text{ or (d) } |f(z^*)| = \min_{x \in C_i} f(z)^*$ and $z^*, z^*$ belong to the same set $C_i$

(3) for every $r > 0$ there exists $n^*(r) > 0$ such that for all $z^0 \in \mathbb{C} \ n(r, z^0, f) \leq n^*(r)$,

then $f$ has bounded $l$-index.

It is known (see [12, 27, 29]) that in one-dimensional case conditions (1) and (3) of Corollary 11 are necessary and sufficient for boundedness of $l$-index or index. Thus, condition (2) is excessive in the case. But for $\mathbb{C}^n (n \geq 2)$, it is required because $\mathbb{D}^n[z^0, R/L(z^0)] \setminus G_R(F)$ is a multiply connected domain, when $\mathbb{D}^n[z^0, R/L(z^0)]$ contains zeros of the function $F$.

We need some notations from [1]. Let $b \in \mathbb{C}^n \setminus \{0\}$ be a given direction. For a given $z^0 \in \mathbb{C}^n$ we denote $g_{z^0}(t) = F(z^0 + t b)$. If one has $g_{z^0}(t) \neq 0$ for all $t \in \mathbb{C}$, then $G_{t}(F, z^0) = \emptyset$; if $g_{z^0}(t) = 0$, then $G_{t}(F, z^0) = \{z^0 + t b : t \in \mathbb{C}\}$. And if $g_{z^0}(t) \neq 0$ and $d_k^0$ are zeros of the function $g_{z^0}(t)$, then $G_{t}(F, z^0) = \bigcup_k \{z^0 + t b : |t - d_k^0| \leq r / L(z^0 + d_k^0 b)\}, r > 0$. Let
\[
C_{t}^b(F) = \bigcup_{z^0 \in \mathbb{C}^n} C_{t}^b(F, z^0),
\]

**Remark 12.** In [1, Theorem 8], sufficient conditions of boundedness of $l$-index in joint variables were obtained, which are similar to Theorem 10. Particularly, we assumed the validity of inequality (45) for all $z \in \mathbb{C}^n \setminus G_{b, j}(F)$, $j \in \{1, 2, \ldots, n\}$. However, $G_{b, j}(F) \subset G_{b, j}(F)$, where $R = (r_1, \ldots, r_n)$. Thus, condition (1) in Theorem 10 is weaker than the corresponding assumption in Theorem 8 from [1].

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


