Research Article

Application of Quasi-Subordination for Generalized Sakaguchi Type Functions

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The author constructs a new class $S_q^\lambda(G, s, t)$ of univalent functions applying the Ruscheweyh derivative. Moreover, the coefficient estimates including a Fekete-Szegő inequality of this class were determined.

1. Introduction and Definitions

Let Δ be the unit disk

$$\{z : z \in \mathbb{C}, |z| < 1\}, \quad (1)$$

and let A be the class of functions analytic in Δ , satisfying the conditions

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1. \end{aligned} \quad (2)$$

Then each $f \in A$ has the Taylor expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (3)$$

Moreover, by S we shall represent the class of all functions in A which are univalent in Δ . Let $h(z)$ be an analytic function in Δ and $|h(z)| \leq 1$, such that

$$h(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \quad (4)$$

where all coefficients are real. Also, let φ be an analytic and univalent function with positive real part in Δ with $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps the unit disc Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (5)$$

where all coefficients are real and $B_1 > 0$. Let P be the class of functions consisting of form (5).

If the functions f and g are analytic in Δ , then f is said to be subordinate to g , written as

$$f(z) \prec g(z), \quad (z \in \Delta) \quad (6)$$

if there exists a Schwarz function $w(z)$, analytic in Δ , with

$$\begin{aligned} w(0) &= 0, \\ |w(z)| &< 1 \quad (z \in \Delta) \end{aligned} \quad (7)$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta). \quad (8)$$

In the year 1970, Robertson [1] introduced the concept of quasi-subordination. For two analytic functions f and g , the function f is said to be quasi-subordinate to g in Δ and written as

$$f(z) \prec_q g(z) \quad (z \in \Delta) \quad (9)$$

if there exists an analytic function $|h(z)| \leq 1$ such that $f(z)/h(z)$ analytic in Δ and

$$\frac{f(z)}{h(z)} \prec g(z) \quad (z \in \Delta); \quad (10)$$

that is, there exists a Schwarz function $w(z)$ such that $f(z) = h(z)g(w(z))$. Observe that if $h(z) = 1$, then $f(z) = g(w(z))$

so that $f(z) \prec g(z)$ in Δ . Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said that it is majorized by g and written by $f(z) \ll g(z)$ in Δ . Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see, e.g., [1–3] for works related to quasi-subordination).

In [4], Sakaguchi introduced the class S_s^* of starlike functions with respect to symmetric points in Δ , consisting of functions $f \in A$ that satisfy the condition $\Re(zf'(z)/(f(z) - f(-z))) > 0$, $z \in \Delta$. Similarly, in [5], Wang et al. introduced the class C_s of convex functions with respect to symmetric points in Δ , consisting of functions $f \in A$ that satisfy the condition $\Re((zf'(z))/(f'(z) + f'(-z))) > 0$, $z \in \Delta$. For different parametric values, we get the classes studied in the literature by Frasin [6], Goyal et al. [7], and Owa et al. [8].

The Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for normalized univalent functions of the form given by (3) is well known for its rich history in Geometric Function Theory. Its origin was in the disproof by Fekete and Szegő [9] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [9]). The Fekete-Szegő functional has $|a_3 - \mu a_2^2|$ since it received great attention, particularly in connection with many subclasses of the class of normalized analytic and univalent functions (see, e.g., [10–15]).

The object of the present paper is to introduce a new class of univalent functions applying the Ruscheweyh derivative, where Ruscheweyh [16] observed that

$$D^n f(z) = \frac{z [z^{n-1} f(z)]^{(n)}}{n!} \quad (11)$$

for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, \dots\}$. This symbol $D^n f(z)$, $n \in \mathbb{N}_0$, is called by Al-Amiri [17] the n th order Ruscheweyh derivative of $f(z)$. We note that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, and

$$D^n f(z) = z + \sum_{k=2}^{\infty} \binom{n+k-1}{n} a_k z^k. \quad (12)$$

2. Preliminary Results

The study of special functions plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Special functions can be categorized into three, namely, Ramp function, threshold function, and sigmoid function. The popular type among all is the sigmoid function because of its gradient descent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The sigmoid function of the form

$$\kappa(z) = \frac{1}{1 + e^{-z}} \quad (13)$$

is useful because it is differentiable. The Sigmoid function has very important properties, including the following (see [18]):

- (i) It outputs real numbers between 0 and 1.

- (ii) It maps a very large input domain to a small range of outputs.
- (iii) It never loses information because it is a one-to-one function.
- (iv) It increases monotonically.

Lemma 1 (see [19]). *Let the Schwarz function $w(z)$ be given by*

$$w(z) = w_1 z + w_1 z^2 + \dots, \quad (z \in \Delta); \quad (14)$$

then

$$|w_1| \leq 1, \quad (15)$$

$$|w_2 - 9w_1^2| \leq 1 + (|9| - 1)|w_1|^2 \leq \max\{1, |9|\},$$

where $9 \in \mathbb{C}$.

Lemma 2 (see [18]). *Let κ be a sigmoid function and*

$$G(z) = 2\kappa(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m, \quad (16)$$

and then $G(z) \in P$, $|z| < 1$, where $G(z)$ is a modified sigmoid function.

Lemma 3 (see [18]). *Let*

$$G_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m, \quad (17)$$

and then $|G_{n,m}(z)| < 2$.

3. Main Result and Its Consequences

Definition 4. A function $f \in A$ is said to be in the class $S_q^\lambda(G, s, t)$, if the following quasi-subordination holds:

$$\left[(D^n f(z))' \left(\frac{(s-t)z}{D^n f(sz) - D^n f(tz)} \right)^\lambda - 1 \right] \prec_q (G(z) - 1), \quad z \in \Delta, \quad (18)$$

where $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and $\lambda \geq 0$.

From the definition, it follows that $f \in S_q^\lambda(G, s, t)$ if and only if there exists an analytic function $h(z)$ with $|h(z)| \leq 1$, such that

$$\frac{(D^n f(z))' \left(\frac{(s-t)z}{D^n f(sz) - D^n f(tz)} \right)^\lambda - 1}{h(z)} \prec (G(z) - 1). \quad (19)$$

If, in the subordination condition (19), $h(z) = 1$, then the class $S_q^\lambda(G, s, t)$ is denoted by $S^\lambda(G, s, t)$ and the functions therein satisfy the condition that

$$(D^n f(z))' \left(\frac{(s-t)z}{D^n f(sz) - D^n f(tz)} \right)^\lambda \prec G(z), \quad (20)$$

$z \in \Delta$.

Theorem 5. Let f of the form (3) be in the class $S_q^\lambda(G, s, t)$. Then

$$|a_2| \leq \frac{1}{2(n+1)|2-\lambda(s+t)|} \quad (21)$$

and for some $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3-\lambda(s^2+st+t^2)|} \cdot \max \left\{ 1, \left| \frac{1}{4} \left(\frac{\eta(n+2)(3-\lambda(s^2+st+t^2))}{(n+1)(2-\lambda(s+t))^2} - \frac{\lambda(1+(2-(s+t))/(2-\lambda(s+t)))(s+t)}{2-\lambda(s+t)} \right) \right| \right\} \quad (22)$$

and the result is sharp.

From (27), it follows that

Proof. Let $f \in S_q^\lambda(G, s, t)$. In view of Definition 4, we can write

$$(D^n f(z))' \left(\frac{(s-t)z}{D^n f(sz) - D^n f(tz)} \right)^\lambda - 1 < h(z)(G(w(z)) - 1), \quad (23)$$

where the function $G(z)$ is a modified sigmoid function given by

$$G(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \dots \quad (24)$$

Combining (4), (14), and (24), we obtain

$$h(z)(G(w(z)) - 1) = \frac{1}{2}h_0w_1z + \frac{1}{2}(h_0w_2 + h_1w_1)z^2 + \dots \quad (25)$$

In the light of (23) and (25), we get

$$(n+1)[2-\lambda(s+t)]a_2 = \frac{1}{2}h_0w_1, \quad (26)$$

$$\begin{aligned} & \frac{(n+1)(n+2)}{2}[3-\lambda(s^2+st+t^2)]a_3 \\ & - \lambda(n+1)^2(s+t) \left(2 - \frac{(\lambda+1)(s+t)}{2} \right) a_2^2 \\ & = \frac{1}{2}h_0w_2 + \frac{1}{2}h_1w_1. \end{aligned} \quad (27)$$

Now, (26) gives

$$a_2 = \frac{h_0w_1}{2(n+1)[2-\lambda(s+t)]}. \quad (28)$$

$$\begin{aligned} a_3 &= \frac{1}{(n+1)(n+2)[3-\lambda(s^2+st+t^2)]} \left[h_1w_1 \right. \\ &+ h_0 \left(w_2 \right. \\ &+ \left. \frac{h_0\lambda(1+(2-(s+t))/(2-\lambda(s+t)))(s+t)}{4[2-\lambda(s+t)]} \right) \\ &\left. \cdot w_1^2 \right]. \end{aligned} \quad (29)$$

For some $\eta \in \mathbb{C}$, we obtain from (28) and (29)

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{1}{(n+1)(n+2)[3-\lambda(s^2+st+t^2)]} \\ &\times \left[h_0w_2 + h_1w_1 \right. \\ &- \frac{1}{4} \left(\frac{\eta(n+2)(3-\lambda(s^2+st+t^2))}{(n+1)(2-\lambda(s+t))^2} \right. \\ &- \left. \frac{\lambda(1+(2-(s+t))/(2-\lambda(s+t)))(s+t)}{2-\lambda(s+t)} \right) \\ &\left. \cdot h_0^2w_1^2 \right]. \end{aligned} \quad (30)$$

Since $h(z)$ given by (4) is analytic and bounded in Δ , therefore, on using [20] (p 172), we have for some y ($|y| \leq 1$)

$$\begin{aligned} |h_0| &\leq 1, \\ h_1 &= (1 - h_0^2)y. \end{aligned} \quad (31)$$

On putting the value of h_1 from (31) into (30), we get

$$a_3 - \eta a_2^2 = \frac{1}{(n+1)(n+2)[3 - \lambda(s^2 + st + t^2)]} \times \left[h_o w_2 + y w_1 - \left[\frac{1}{4} \left(\frac{\eta(n+2)(3 - \lambda(s^2 + st + t^2))}{(n+1)(2 - \lambda(s+t))^2} - \frac{\lambda(1 + (2 - (s+t))/(2 - \lambda(s+t)))(s+t)}{2 - \lambda(s+t)} \right) w_1^2 + y w_1 \right] h_o^2 \right]. \quad (32)$$

If $h_o = 0$ in (32), we obtain

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3 - \lambda(s^2 + st + t^2)|}. \quad (33)$$

If $h_o \neq 0$ in (32), let

$$T(h_o) = h_o w_2 + y w_1 - \left[\frac{1}{4} \left(\frac{\eta(n+2)(3 - \lambda(s^2 + st + t^2))}{(n+1)(2 - \lambda(s+t))^2} - \frac{\lambda(1 + (2 - (s+t))/(2 - \lambda(s+t)))(s+t)}{2 - \lambda(s+t)} \right) w_1^2 + y w_1 \right] h_o^2$$

$$- \frac{\lambda(1 + (2 - (s+t))/(2 - \lambda(s+t)))(s+t)}{2 - \lambda(s+t)} \Big) w_1^2 + y w_1 \Big] h_o^2 \quad (34)$$

which is a polynomial in h_o and hence analytic in $|h_o| \leq 1$, and maximum $|T(h_o)|$ is attained at $h_o = e^{i\theta}$, $(0 \leq \theta < 2\pi)$. We find that

$$\max_{0 \leq \theta < 2\pi} |T(h_o)| = K(1),$$

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3 - \lambda(s^2 + st + t^2)|} \times \left| w_2 - \frac{1}{4} \left(\frac{\eta(n+2)(3 - \lambda(s^2 + st + t^2))}{(n+1)(2 - \lambda(s+t))^2} - \frac{\lambda(1 + (2 - (s+t))/(2 - \lambda(s+t)))(s+t)}{2 - \lambda(s+t)} \right) w_1^2 \right| \quad (35)$$

which on using Lemma 1 shows that

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3 - \lambda(s^2 + st + t^2)|} \cdot \max \left\{ 1, \left| \frac{1}{4} \left(\frac{\eta(n+2)(3 - \lambda(s^2 + st + t^2))}{(n+1)(2 - \lambda(s+t))^2} - \frac{\lambda(1 + (2 - (s+t))/(2 - \lambda(s+t)))(s+t)}{2 - \lambda(s+t)} \right) w_1^2 \right| \right\}. \quad (36)$$

□ **Corollary 6.** Let f of form (3) be in the class $S_q^\lambda(G, 1, t)$. Then

$$|a_2| \leq \frac{1}{2(n+1)|2 - \lambda(1+t)|} \quad (37)$$

For the case when $s = 1$, one has the following.

and for some $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3 - \lambda(1+t+t^2)|} \cdot \max \left\{ 1, \left| \frac{1}{4} \left(\frac{\eta(n+2)(3 - \lambda(1+t+t^2))}{(n+1)(2 - \lambda(1+t))^2} - \frac{\lambda(1 + (2 - (1+t))/(2 - \lambda(1+t)))(1+t)}{2 - \lambda(1+t)} \right) \right| \right\} \quad (38)$$

and the result is sharp.

Putting $t = -1$ in Corollary 6, we obtain the following corollary.

Corollary 7. Let f of form (3) be in the class $S_q^\lambda(G, 1, -1)$. Then

$$|a_2| \leq \frac{1}{4(n+1)} \quad (39)$$

and for some $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{(n+1)(n+2)|3-\lambda|} \cdot \max \left\{ 1, \left| \left(\frac{\eta(n+2)(3-\lambda)}{16(n+1)} \right) \right| \right\} \quad (40)$$

and the result is sharp.

Setting $\lambda = 0$ in Corollary 7, we have the following.

Corollary 8. Let f of form (3) be in the class $S_q(G, 1, -1)$. Then

$$|a_2| \leq \frac{1}{4(n+1)} \quad (41)$$

and for some $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{3(n+1)(n+2)} \max \left\{ 1, \left| \left(\frac{3\eta(n+2)}{16(n+1)} \right) \right| \right\} \quad (42)$$

and the result is sharp.

Setting $\lambda = 1$ in Corollary 7, we have the following.

Corollary 9. Let f of form (3) be in the class $S_q(G, 1, -1)$. Then

$$|a_2| \leq \frac{1}{4(n+1)} \quad (43)$$

and for some $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{1}{2(n+1)(n+2)} \max \left\{ 1, \left| \left(\frac{\eta(n+2)}{8(n+1)} \right) \right| \right\} \quad (44)$$

and the result is sharp.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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References

- [1] M. S. Robertson, "Quasi-subordination and coefficient conjectures," *Bulletin (New Series) of the American Mathematical Society*, vol. 76, pp. 1–9, 1970.
- [2] F. Y. Ren, S. Owa, and S. Fukui, "Some inequalities on quasi-subordinate functions," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 2, pp. 317–324, 1991.
- [3] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, pp. 95–102, 1967.
- [4] K. Sakaguchi, "On a certain univalent mapping," *Journal of the Mathematical Society of Japan*, vol. 11, pp. 72–75, 1959.
- [5] Z.-G. Wang, C.-Y. Gao, and S.-M. Yuan, "On certain subclasses of close-to-convex and quasi-convex functions with respect to k -symmetric points," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 97–106, 2006.
- [6] B. A. Frasin, "Coefficient inequalities for certain classes of Sakaguchi type functions," *International Journal of Nonlinear Science*, vol. 10, no. 2, pp. 206–211, 2010.
- [7] S. P. Goyal, P. Vijaywargiya, and P. Goswami, "Sufficient conditions for Sakaguchi type functions of order β ," *European Journal of Pure and Applied Mathematics*, vol. 4, no. 3, pp. 230–236, 2009.
- [8] S. Owa, T. Sekine, and R. Yamakawa, "On Sakaguchi type functions," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 356–361, 2007.
- [9] M. Fekete and G. Szegő, "Eine Bemerkung Über UNGerade Schlichte Funktionen," *Journal of the London Mathematical Society*, pp. 85–89, 1933.
- [10] R. Bucur, L. Andrei, and D. Breaz, "Coefficient bounds and fekete-szegő problem for a class of analytic functions defined by using a new differential operator," *Applied Mathematical Sciences*, vol. 9, no. 25–28, pp. 1355–1368, 2015.
- [11] S. Bulut, "Fekete-Szegő problem for subclasses of analytic functions defined by Komatu integral operator," *Arabian Journal of Mathematics*, vol. 2, no. 2, pp. 177–183, 2013.
- [12] S. P. Goyal and P. Goswami, "Certain coefficient inequalities for Sakaguchi type functions and applications to fractional derivative operator," *Acta Universitatis Apulensis*, no. 19, pp. 159–166, 2009.
- [13] S. P. Goyal and R. Kumar, "Fekete-Szegő problem for a class of complex order related to Salagean operator," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 4, pp. 240–246, 2011.
- [14] F. M. Sakar, S. Aytaş, and H. Ö. Güney, "On the Fekete-Szegő problem for generalized class $M_{\alpha, \gamma}(\beta)$ defined by differential operator," *Süleyman Demirel University Journal of Natural and Applied Sciences*, vol. 20, no. 3, pp. 456–459, 2016.
- [15] H. M. Srivastava, A. K. Mishra, and M. K. Das, "The Fekete-Szegő problem for a subclass of close-to-convex functions," *Complex Variables, Theory and Application*, vol. 44, no. 2, pp. 145–163, 2001.
- [16] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.
- [17] H. S. Al-Amiri, "On Ruscheweyh derivatives," *Annales Polonici Mathematici*, vol. 38, no. 1, pp. 87–94, 1980.
- [18] O. A. Fadipe-Joseph, A. T. Oladipo, and A. Uzoamaka Ezeafulukwe, "Modified Sigmoid function in univalent function theory," *International Journal of Mathematical Sciences & Engineering*, pp. 313–317, 2013.
- [19] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 8–12, 1969.
- [20] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, NY, USA, 1st edition, 1952.