Research Article

A Characterization of Bi-Jordan Homomorphisms on Banach Algebras

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For Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \), we show that if \( \mathcal{U} = \mathcal{A} \times \mathcal{B} \) is unital and commutative, each bi-Jordan homomorphism from \( \mathcal{U} \) into a semisimple commutative Banach algebra \( \mathcal{D} \) is a bihomomorphism.

1. Introduction

Let \( \mathcal{A} \) and \( \mathcal{B} \) be complex Banach algebras and let \( \varphi : \mathcal{A} \to \mathcal{B} \) be a linear map. Then \( \varphi \) is called \( n \)-homomorphism if for all \( a_1, a_2, \ldots, a_n \in \mathcal{A} \),

\[
\varphi (a_1 a_2 \cdots a_n) = \varphi (a_1) \varphi (a_2) \cdots \varphi (a_n).
\]

(1)

The concept of \( n \)-homomorphism was studied for complex algebras by Hejazian et al. in [1]. A 2-homomorphism is then just a homomorphism, in the usual sense. We refer the reader to [2], for certain properties of \( 3 \)-homomorphisms.

In [3], Eshaghi Gordji introduced the concept of an \( n \)-Jordan homomorphism. A linear map \( \varphi \) between Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \) is called an \( n \)-Jordan homomorphism if

\[
\varphi (a^n) = \varphi (a)^n, \quad (a \in \mathcal{A}).
\]

(2)

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

It is obvious that each \( n \)-homomorphism is an \( n \)-Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, Zelazko in [4] proved that every Jordan homomorphism from Banach algebra \( \mathcal{A} \) into a semisimple commutative Banach algebra \( \mathcal{B} \) is a homomorphism. See also [5] for another approach to the same result. The reader is referred to [6], for characterization of \( 3 \)-Jordan homomorphism.

Also it is shown in [3] that every \( n \)-Jordan homomorphism between two commutative Banach algebras is an \( n \)-homomorphism for \( n \in \{2, 3, 4\} \) and this result is extended to the case \( n = 5 \) in [7]. Some investigation has been done about Jordan derivations and Jordan centralizers in [8, 9].

Throughout the paper, let \( \mathcal{U} = \mathcal{A} \times \mathcal{B} \). Then \( \mathcal{U} \) is a Banach algebra for the multiplication

\[
(a, b) (x, y) = (ax, by), \quad (a, b), (x, y) \in \mathcal{U},
\]

(3)

and with norm

\[
\|(a, b)\| = \|a\| + \|b\|.
\]

(4)

Clearly, \( \mathcal{U} \) is commutative if and only if both \( \mathcal{A} \) and \( \mathcal{B} \) are commutative, and it is unital if and only if both \( \mathcal{A} \) and \( \mathcal{B} \) are unital. Without any confusion we denote by \( e \), the unit element of both \( \mathcal{A} \) and \( \mathcal{B} \).

Let \( \mathcal{D} \) be a complex Banach algebra. A bilinear map \( \varphi : \mathcal{U} \to \mathcal{D} \) such that for any \( a \in \mathcal{A} \) the map \( b \mapsto \varphi (a, b) \) is linear map from \( \mathcal{B} \) to \( \mathcal{D} \), and for any \( b \in \mathcal{B} \) the map \( a \mapsto \varphi (a, b) \) is linear map from \( \mathcal{A} \) to \( \mathcal{D} \).

A bilinear map \( \varphi \) is called bihomomorphism if for all \( (a,b),(x,y) \in \mathcal{U} \),

\[
\varphi (ax, by) = \varphi (a, b) \varphi (x, y),
\]

(5)

and it is called bi-Jordan homomorphism (BJH, for short) if

\[
\varphi (a^2, b^2) = \varphi (a, b)^2, \quad (a, b) \in \mathcal{U}.
\]

(6)
It is obvious that each bihomomorphism is BJH, but in general the converse is not true. For example, let

\[ A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}, \]

and \( B = A \cup \{ I \} \), where \( I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Define \( \varphi : A \times B \to A \) by \( \varphi(x, y) = xy \). Then \( \varphi \) is bilinear and

\[ \varphi(x^2, y^2) = x^2y^2 = (xy)^2 = (x, y)^2, \]  

for all \((x, y) \in U = A \times B\). Thus, \( \varphi \) is a bi-Jordan homomorphism, but it is not bihomomorphism. For instance, let

\[ u = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \]

\[ x = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \]

\[ v = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix}, \]

\[ y = I. \]

Then \((u, v), (x, y) \in U\) and

\[ \varphi(ux, vy) = \begin{bmatrix} asc & act \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} asc & asd \\ 0 & 0 \end{bmatrix} \]

\[ = \varphi(u, v) \varphi(x, y). \]

In this paper, we show that the converse statement holds under certain conditions.

2. Main Results

We commence with some useful lemmas.

**Lemma 1.** Suppose that \( \varphi : U \to C \) is a BJH. Then

1. \( \varphi(xy + yx, b^2) = 2\varphi(x, b)\varphi(y, b); \)
2. \( \varphi(a^2, xy + yx) = 2\varphi(a, x)\varphi(a, y). \)

**Proof.** The proof is straightforward. \( \square \)

**Lemma 2.** Suppose that \( \varphi : U \to C \) is a BJH. If \( U \) is unital and commutative, then

1. \( \varphi(ax, e) = \varphi(a, e)\varphi(x, e); \)
2. \( \varphi(e, by) = \varphi(e, b)\varphi(e, y). \)

**Proof.** The proof follows from Lemma 1. \( \square \)

**Lemma 3.** Let \( U \) be unital, and let \( \varphi : U \to C \) be a nonzero BJH. Then \( \varphi(e, e) \neq 0 \).

**Proof.** Let \( \varphi \) be a BJH. Then for all \((a, b) \in U\), we get

\[ \varphi\left(a^2, b^2\right) = \varphi(a, b)^2. \]  

Replacing \( a \) by \( x + e \) and \( b \) by \( y + e \) in (11) gives

\[ \varphi\left(x^2 + 2x + e, y^2 + 2y + e\right) = \varphi(x + e, y + e)^2. \]  

Assume that \( \varphi(e, e) = 0 \); then by Lemma 1,

\[ \varphi(x, e) = \varphi(y, e) = 0, \quad (x, y) \in U. \]

It follows from (12) and (13) that

\[ \varphi\left(2x, 2y\right) + \varphi\left(2x, y^2\right) + \varphi\left(x^2, 2y\right) = 0, \]

for all \((x, y) \in U\). By Lemma 1,

\[ \varphi\left(2x, y^2\right) = 2\varphi(x, y)\varphi(e, y) = 0, \]

\[ \varphi\left(x^2, 2y\right) = 2\varphi(x, e)\varphi(x, y) = 0. \]

Thus, by (14) and (15), we get \( \varphi(x, y) = 0 \), for all \((x, y) \in U\), which is contradiction. \( \square \)

Now we state and prove the main theorem. The main idea of the proof can be found in [6].

**Theorem 4.** Let \( U \) be unital and commutative, and let \( \varphi \) be a BJH from \( U \) into a semisimple commutative Banach algebra \( D \). Then \( \varphi \) is a bihomomorphism.

**Proof.** We first assume that \( D = C \) and let \( \varphi : U \to C \) be a BJH. Then, for all \((a, b) \in U\), we get

\[ \varphi\left(a^2, b^2\right) = \varphi(a, b)^2. \]  

Replacing \( a \) by \( x + e \) and \( b \) by \( y + e \) in (16), we have

\[ \varphi\left(x^2 + 2x + e, y^2 + 2y + e\right) = \varphi(x + e, y + e)^2. \]

By Lemma 3, \( \varphi(e, e) \neq 0 \), so (16) gives

\[ \varphi(e, e) = 1. \]

It follows from Lemma 1 that

\[ \varphi\left(2x, y^2\right) = 2\varphi(x, y)\varphi(e, y), \]

\[ \varphi\left(x^2, 2y\right) = 2\varphi(x, e)\varphi(x, y). \]

Thus, by (17), (18), and (19) we get

\[ \varphi(x, y) = \varphi(x, e)\varphi(e, y), \]

for all \((x, y) \in U\). Replacing \( x \) by \( ax \) and \( y \) by \( by \) in (20) gives

\[ \varphi(ax, by) = \varphi(ax, e)\varphi(e, by). \]

By (20) and Lemma 2 we deduce

\[ \varphi(a, b)\varphi(x, y) = \left[ \varphi(a, e)\varphi(x, e) \right] \left[ \varphi(e, b)\varphi(e, y) \right] \]

\[ = \varphi(ax, e)\varphi(e, by). \]
By (21) and (22),
\[ \varphi(ax, by) = \varphi(a, b) \varphi(x, y), \]
for all \((a, b), (x, y) \in \mathcal{U}\), and so \(\varphi\) is bihomomorphism.

Now suppose that \(\mathcal{D}\) is semisimple and commutative. Let \(\mathcal{M}(\mathcal{D})\) be the maximal ideal space of \(\mathcal{D}\). We associate with each \(f \in \mathcal{M}(\mathcal{D})\) a function \(\varphi_f : \mathcal{U} \rightarrow \mathbb{C}\) defined by
\[ \varphi_f(a, b) = f(\varphi(a, b)), \quad (a, b) \in \mathcal{U}. \]

(24)

Pick \(f \in \mathcal{M}(\mathcal{D})\) arbitrary. It is easy to see that \(\varphi_f\) is a BJH, so by the above argument it is a bihomomorphism. Thus by the definition of \(\varphi_f\) we have
\[ f(\varphi(ax, by)) = f(\varphi(a, b) f(x, y)) = f(\varphi(a, b) \varphi(x, y)). \]

(25)

Since \(f \in \mathcal{M}(\mathcal{D})\) was arbitrary and \(\mathcal{D}\) is assumed to be semisimple, we obtain
\[ \varphi(ax, by) = \varphi(a, b) \varphi(x, y), \]
for all \((a, b), (x, y) \in \mathcal{U}\).

The second dual space \(\mathcal{A}''\) of a Banach algebra \(\mathcal{A}\) admits two Banach algebra multiplications known as the first and second Arens products, each extending the product on \(\mathcal{A}\). These products which we denote by \(\square\) and \(\diamond\), respectively, can be defined as follows:
\[ F \square G = w^* - \lim_j a_j b_j, \]
\[ F \diamond G = w^* - \lim_i a_i b_j, \]

where \((a_i)\) and \((b_j)\) are nets in \(\mathcal{A}\) that converge, in \(w^*\)-topologies, to \(F\) and \(G\), respectively. The Banach algebra \(\mathcal{A}\) is said to be Arens regular if \(F \square G = F \diamond G\) on the whole of \(\mathcal{A}''\). Some significant results related to the Arens regularity of certain bilinear maps and Banach algebra obtained in [10]. For more information on the Arens products, we refer the reader to [11, 12], for example.

It is shown in [11] that every \(C^*\)-algebra \(\mathcal{A}\) is Arens regular and semisimple. Also the second dual of each \(C^*\)-algebra is also a \(C^*\)-algebra.

Theorem 5. If \(k\) is the natural embedding of a Banach space \(X\) into \(X''\), then \(k(X)\) is \(w^*\)-dense in \(X''\).

Proof. See [13]. □

Theorem 6. Let \(\varphi : \mathcal{U} \rightarrow \mathcal{D}\) be a continuous BJH. Then the second adjoint \(\varphi'' : \mathcal{U}'' \rightarrow \mathcal{D}''\) of \(\varphi\) is also a BJH.

Proof. By Theorem 5, there are bounded nets \((a_i)\) and \((b_j)\) in \(\mathcal{A}\) and \(\mathcal{D}\) that converge, in \(w^*\)-topologies, to \(F\) and \(G\), respectively. Then
\[ \varphi''(F^2, G^2) = \varphi'' \left( w^* - \lim_i a_i^2, w^* - \lim_j b_j^2 \right) \]
\[ = w^* - \lim_i \varphi(a_i^2, b_j^2) \]
\[ = w^* - \lim j \varphi(a_i, b_j)^2 \]
\[ = w^* - \lim j \varphi''(a_i, b_j)^2 \]
\[ = \varphi'' \left( w^* - \lim a_i, w^* - \lim b_j \right)^2 \]
\[ = \varphi''(F, G)^2. \]

Thus, \(\varphi''\) is BJH, as claimed. □

Since the second dual of every \(C^*\)-algebra is unital [14], we deduce the following result from Theorems 4 and 6.

Corollary 7. Let \(\varphi : \mathcal{U} \rightarrow \mathcal{D}\) be a continuous BJH between commutative \(C^*\)-algebras. Then \(\varphi'' : \mathcal{U}'' \rightarrow \mathcal{D}''\) is a bihomomorphism.

Theorem 8. Suppose that \(\varphi\) is a BJH from the unital Banach algebra \(\mathcal{U}\) into \(\mathbb{C}\). Then \(\varphi\) is a \(n\)-BJH; that is,
\[ \varphi(a^n, b^n) = \varphi(a, b)^n, \]
for all \(n \geq 3\).

Proof. Let \(\varphi : \mathcal{U} \rightarrow \mathbb{C}\) be a BJH. Then, for all \((a, b) \in \mathcal{U}\),
\[ \varphi(a^n, b^n) = \varphi(a, b)^n. \]

Replacing \(a\) by \(a + x\) and \(b\) by \(b + y\) in (30), we have
\[ \varphi(ax + xa, by + yb) = 2\varphi(a, b) \varphi(x, y) + 2\varphi(a, y) \varphi(x, b). \]

Replacing \(a\) by \(a^2\) and \(b\) by \(b^2\) in (31) gives
\[ \varphi(a^2x + xa^2, b^2y + yb^2) = 2\varphi(a^2, b^2) \varphi(x, y) + 2\varphi(a^2, y) \varphi(x, b^2). \]

Replacing \(x\) by \(a\) and \(y\) by \(b\) in (32), we obtain
\[ \varphi(2a^3, 2b^3) = 2\varphi(a^2, b^2) \varphi(a, b) + 2\varphi(a^2, b) \varphi(a, b^2). \]
Replacing \( x \) and \( y \) by \( e \) in (31), we get
\[
\varphi(2a, 2b) = 2\varphi(a, b) \varphi(e, e) + 2\varphi(a, e) \varphi(e, b).
\] (34)

Since \( \varphi(e, e) = 1 \), so (34) gives
\[
\varphi(a, b) = \varphi(a, e) \varphi(e, b),
\] (35)
for all \((a, b) \in \mathcal{U}\). By Lemma 1,
\[
\varphi(a^2, b) = \varphi(a, b) \varphi(a, e),
\] (36)
\[
\varphi(a, b^2) = \varphi(a, b) \varphi(e, b).
\]

It follows from (35) and (36) that
\[
\varphi(a^2, b) \varphi(a, b^2) = \varphi(a^2) \varphi(a, e) \varphi(e, b)
\] (37)
\[= \varphi(a, b)^3.
\]

By (33) and (37),
\[
\varphi(a^3, b^3) = \varphi(a, b)^3,
\] (38)
for all \((a, b) \in \mathcal{U}\). Thus, the result is established for \( n = 3 \). An easy induction argument now finishes the proof. \(\blacksquare\)

As a consequence of Theorem 8 we have the next result.

**Corollary 9.** Suppose that \( \varphi \) is a BJH from the unital Banach algebra \( \mathcal{U} \) into a semisimple commutative Banach algebra \( \mathcal{D} \). Then \( \varphi \) is an \( n \)-BJH.

Baker in [15] proved that an almost multiplicative function is either bounded or multiplicative. Now we prove an analogous result of Baker’s theorem forbihomomorphism.

**Theorem 10.** Let \( \delta > 0 \) and \( \varphi : \mathcal{U} \to \mathbb{C} \) satisfy
\[
|\varphi(ax, by) - \varphi(a, b) \varphi(x, y)| \leq \delta,
\] (39)
for all \((a, b), (x, y) \in \mathcal{U}\). Then either \( \varphi \) is bihomomorphism or
\[
|\varphi(x, y)| \leq \frac{1}{2} \left( 1 + \sqrt{1 + 4\delta} \right) = \varepsilon,
\] (40)
for all \((x, y) \in \mathcal{U}\).

**Proof.** Let \( \varepsilon^2 - \varepsilon = \delta \) and \( \varepsilon > 1 \). Suppose that there exist \((s, t) \in \mathcal{U}\) such that \( |\varphi(s, t)| > \varepsilon \), so that \( |\varphi(s, t)| = \varepsilon + p \), where \( p > 0 \). Then we have
\[
|\varphi(s^2, t^2)| \geq |\varphi(s, t)^2 - (\varphi(s, t)^2 - \varphi(s^2, t^2))|.
\] (41)
hence
\[
|\varphi(s^2, t^2)| \geq |\varphi(s, t)^2 - \varphi(s^2, t^2)|.
\]

Now make the induction assumption:
\[
|\varphi(2^n s, 2^n t)| > \varepsilon + (n + 1) p.
\] (43)

Let
\[
M_n = \varphi(s^{2^n}, t^{2^n}).
\] (44)

Then
\[
|M_{n+1}| = |M_n^2 - (M_n - M_{n+1})| \geq |M_n^2| - \delta
\]
\[= |M_n^2| + \varepsilon - \varepsilon^2,
\]
whence
\[
|M_{n+1}| \geq (\varepsilon + (n + 1) p)^2 + \varepsilon - \varepsilon^2
\]
\[= (2n + 2) \varepsilon p + \varepsilon + (n + 1) \varepsilon^2
\]
\[> \varepsilon + (n + 2) \varepsilon
\]
and (43) is established for all \( n \in \mathbb{N} \).

Given that \((a, b), (x, y), (u, v) \in \mathcal{U}\), we have
\[
|\varphi(axu, byv) - \varphi(a, b) \varphi(xu, yv)| \leq \delta,
\]
\[
|\varphi(axu, byv) - \varphi(ax, by) \varphi(u, v)| \leq \delta.
\] (47)

Thus,
\[
|\varphi(ax, by) \varphi(u, v) - \varphi(a, b) \varphi(xu, yv)| \leq 2\delta.
\] (48)

Hence
\[
|\varphi(ax, by) \varphi(u, v) - \varphi(a, b) \varphi(x, y) \varphi(u, v)|
\]
\[\leq |\varphi(ax, by) \varphi(u, v) - \varphi(a, b) \varphi(xu, yv)|
\]
\[+ |\varphi(a, b) \varphi(xu, yv) - \varphi(a, b) \varphi(x, y) \varphi(u, v)|
\]
\[\leq 2\delta + \delta |\varphi(a, b)|.
\] (49)

Therefore
\[
|\varphi(u, v)| |\varphi(ax, by) - \varphi(a, b) \varphi(x, y)|
\]
\[\leq 2\delta + \delta |\varphi(a, b)|.
\] (50)

Now put \((u, v) = (s^{2^n}, t^{2^n})\) to obtain
\[
|\varphi(ax, by) - \varphi(a, b) \varphi(x, y)| \leq \frac{2\delta + \delta |\varphi(a, b)|}{|M_n|}
\]
\[\leq \frac{2\delta + \delta |\varphi(a, b)|}{\varepsilon + (n + 1) p},
\] (51)
for all \( n \in \mathbb{N} \) and \((a, b) \in \mathcal{U}\). Letting \( n \to \infty \), we get \( \varphi(ax, by) = \varphi(a, b) \varphi(x, y) \) and so \( \varphi \) is a bihomomorphism. \(\blacksquare\)
As Baker pointed out in his article, the above proof works for functions \( \varphi : \mathcal{U} \to \mathcal{D} \), where \( \mathcal{D} \) is a Banach algebra in which the norm is multiplicative; that is, \( \|xy\| = \|x\|\|y\| \) for all \( x, y \in \mathcal{D} \). For norm algebra \( \mathcal{D} \) for which the norm is not multiplicative, the situation is false. For example, let \( \delta > 0 \) and choose \( \varepsilon > 0 \) so that \( |\varepsilon - \varepsilon^2| = \delta \), and define \( \varphi : \mathbb{R} \times \mathbb{R} \to M_3(\mathbb{R}) \) by

\[
\varphi(x, y) = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \quad x, y \in \mathbb{R}. \tag{52}
\]

Then with the usual matrix norm

\[
\|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)\| = \delta, \tag{53}
\]

for all \((a, b), (x, y) \in \mathbb{R} \times \mathbb{R}\). Clearly, \( \varphi \) is unbounded, but \( \varphi \) is not bihomomorphism.

**Competing Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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