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Research Article

A Characterization of Bi-Jordan Homomorphisms on Banach Algebras

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For Banach algebras $\mathscr{A}$ and $\mathscr{B}$, we show that if $\mathscr{U} = \mathscr{A} \times \mathscr{B}$ is unital and commutative, each bi-Jordan homomorphism from $\mathscr{U}$ into a semisimple commutative Banach algebra $\mathscr{D}$ is a bihomomorphism.

1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be complex Banach algebras and let $\varphi : \mathscr{A} \to \mathscr{B}$ be a linear map. Then φ is called n-homomorphism if for all $a_1, a_2, \ldots, a_n \in \mathscr{A}$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n). \tag{1}$$

The concept of n-homomorphism was studied for complex algebras by Hejazian et al. in [1]. A 2-homomorphism is then just a homomorphism, in the usual sense. We refer the reader to [2], for certain properties of 3-homomorphisms.

In [3], Eshaghi Gordji introduced the concept of an n-Jordan homomorphism. A linear map φ between Banach algebras $\mathscr{A}$ and $\mathscr{B}$ is called an n-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \quad (a \in \mathscr{A}). \tag{2}$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

It is obvious that each n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, Zelazko in [4] proved that every Jordan homomorphism from Banach algebra $\mathscr{A}$ into a semisimple commutative Banach algebra $\mathscr{B}$ is a homomorphism. See also [5] for another approach to the same result. The reader is referred to [6], for characterization of 3-Jordan homomorphism.

Also it is shown in [3] that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism for $n \in \{2, 3, 4\}$ and this result is extended to the case $n = 5$ in [7]. Some investigation has been done about Jordan derivations and Jordan centralizers in [8, 9].

Throughout the paper, let $\mathscr{U} = \mathscr{A} \times \mathscr{B}$. Then $\mathscr{U}$ is a Banach algebra for the multiplication

$$(a, b)(x, y) = (ax, by), \quad (a, b), (x, y) \in \mathscr{U}, \tag{3}$$

and with norm

$$\|(a, b)\| = \|a\| + \|b\|. \tag{4}$$

Clearly, $\mathscr{U}$ is commutative if and only if both $\mathscr{A}$ and $\mathscr{B}$ are commutative, and it is unital if and only if both $\mathscr{A}$ and $\mathscr{B}$ are unital. Without any confusion we denote by e, the unit element of both $\mathscr{A}$ and $\mathscr{B}$.

Let $\mathscr{D}$ be a complex Banach algebra. A bilinear map is a function $\varphi : \mathscr{U} \to \mathscr{D}$ such that for any $a \in \mathscr{A}$ the map $b \mapsto \varphi(a, b)$ is linear map from $\mathscr{B}$ to $\mathscr{D}$, and for any $b \in \mathscr{B}$ the map $a \mapsto \varphi(a, b)$ is linear map from $\mathscr{A}$ to $\mathscr{D}$.

A bilinear map φ is called bihomomorphism if for all $(a, b), (x, y) \in \mathscr{U}$,

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y), \tag{5}$$

and it is called bi-Jordan homomorphism (BJH, for short) if

$$\varphi(a^2, b^2) = \varphi(a, b)^2, \quad (a, b) \in \mathscr{U}. \tag{6}$$

It is obvious that each bihomomorphism is BJH, but in general the converse is not true. For example, let

$$\mathscr{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}, \tag{7}$$

and $\mathscr{B} = \mathscr{A} \cup \{I\}$, where $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Define $\varphi : \mathscr{A} \times \mathscr{B} \to \mathscr{A}$ by $\varphi(x, y) = xy$. Then φ is bilinear and

$$\varphi\left(x^2, y^2\right) = x^2 y^2 = (xy)^2 = \varphi(x, y)^2, \tag{8}$$

for all $(x, y) \in \mathscr{U} = \mathscr{A} \times \mathscr{B}$. Thus, φ is a bi-Jordan homomorphism, but it is not bihomomorphism. For instance, let

$$\begin{aligned} u &= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \\ x &= \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \\ v &= \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix}, \\ y &= I. \end{aligned} \tag{9}$$

Then $(u, v), (x, y) \in \mathscr{U}$ and

$$\begin{aligned} \varphi(ux, vy) &= \begin{bmatrix} acs & act \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} asc & asd \\ 0 & 0 \end{bmatrix} \\ &= \varphi(u, v)\varphi(x, y). \end{aligned} \tag{10}$$

In this paper, we show that the converse statement holds under certain conditions.

2. Main Results

We commence with some useful lemmas.

Lemma 1. *Suppose that $\varphi : \mathscr{U} \to \mathbb{C}$ is a BJH. Then*

(1) $\varphi(xy + yx, b^2) = 2\varphi(x, b)\varphi(y, b)$;

(2) $\varphi(a^2, xy + yx) = 2\varphi(a, x)\varphi(a, y)$.

Proof. The proof is straightforward. □

Lemma 2. *Suppose that $\varphi : \mathscr{U} \to \mathbb{C}$ is a BJH. If $\mathscr{U}$ is unital and commutative, then*

(1) $\varphi(ax, e) = \varphi(a, e)\varphi(x, e)$;

(2) $\varphi(e, by) = \varphi(e, b)\varphi(e, y)$.

Proof. The proof follows from Lemma 1. □

Lemma 3. *Let $\mathscr{U}$ be unital, and let $\varphi : \mathscr{U} \to \mathbb{C}$ be a nonzero BJH. Then $\varphi(e, e) \neq 0$.*

Proof. Let φ be a BJH. Then for all $(a, b) \in \mathscr{U}$, we get

$$\varphi\left(a^2, b^2\right) = \varphi(a, b)^2. \tag{11}$$

Replacing a by $x + e$ and b by $y + e$ in (11) gives

$$\varphi\left(x^2 + 2x + e, y^2 + 2y + e\right) = \varphi(x + e, y + e)^2. \tag{12}$$

Assume that $\varphi(e, e) = 0$; then by Lemma 1,

$$\varphi(x, e) = \varphi(e, y) = 0, \quad (x, y) \in \mathscr{U}. \tag{13}$$

It follows from (12) and (13) that

$$\varphi(2x, 2y) + \varphi\left(2x, y^2\right) + \varphi\left(x^2, 2y\right) = 0, \tag{14}$$

for all $(x, y) \in \mathscr{U}$. By Lemma 1,

$$\begin{aligned} \varphi\left(2x, y^2\right) &= 2\varphi(x, y)\varphi(e, y) = 0, \\ \varphi\left(x^2, 2y\right) &= 2\varphi(x, e)\varphi(x, y) = 0. \end{aligned} \tag{15}$$

Thus, by (14) and (15), we get $\varphi(x, y) = 0$, for all $(x, y) \in \mathscr{U}$, which is contradiction. □

Now we state and prove the main theorem. The main idea of the proof can be found in [6].

Theorem 4. *Let $\mathscr{U}$ be unital and commutative, and let φ be a BJH from $\mathscr{U}$ into a semisimple commutative Banach algebra $\mathscr{D}$. Then φ is a bihomomorphism.*

Proof. We first assume that $\mathscr{D} = \mathbb{C}$ and let $\varphi : \mathscr{U} \to \mathbb{C}$ be a BJH. Then, for all $(a, b) \in \mathscr{U}$, we get

$$\varphi\left(a^2, b^2\right) = \varphi(a, b)^2. \tag{16}$$

Replacing a by $x + e$ and b by $y + e$ in (16), we have

$$\varphi\left(x^2 + 2x + e, y^2 + 2y + e\right) = \varphi(x + e, y + e)^2. \tag{17}$$

By Lemma 3, $\varphi(e, e) \neq 0$, so (16) gives

$$\varphi(e, e) = 1. \tag{18}$$

It follows from Lemma 1 that

$$\begin{aligned} \varphi\left(2x, y^2\right) &= 2\varphi(x, y)\varphi(e, y), \\ \varphi\left(x^2, 2y\right) &= 2\varphi(x, e)\varphi(x, y). \end{aligned} \tag{19}$$

Thus, by (17), (18), and (19) we get

$$\varphi(x, y) = \varphi(x, e)\varphi(e, y), \tag{20}$$

for all $(x, y) \in \mathscr{U}$. Replacing x by ax and y by by in (20) gives

$$\varphi(ax, by) = \varphi(ax, e)\varphi(e, by). \tag{21}$$

By (20) and Lemma 2 we deduce

$$\begin{aligned} \varphi(a, b)\varphi(x, y) &= [\varphi(a, e)\varphi(x, e)][\varphi(e, b)\varphi(e, y)] \\ &= \varphi(ax, e)\varphi(e, by). \end{aligned} \tag{22}$$

By (21) and (22),

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y), \tag{23}$$

for all $(a, b), (x, y) \in \mathcal{U}$, and so φ is bihomomorphism.

Now suppose that $\mathcal{D}$ is semisimple and commutative. Let $\mathfrak{M}(\mathcal{D})$ be the maximal ideal space of $\mathcal{D}$. We associate with each $f \in \mathfrak{M}(\mathcal{D})$ a function $\varphi_f : \mathcal{U} \to \mathbb{C}$ defined by

$$\varphi_f(a, b) := f(\varphi(a, b)), \quad (a, b) \in \mathcal{U}. \tag{24}$$

Pick $f \in \mathfrak{M}(\mathcal{D})$ arbitrary. It is easy to see that φ_f is a BJH, so by the above argument it is a bihomomorphism. Thus by the definition of φ_f we have

$$\begin{aligned} f(\varphi(ax, by)) &= f(\varphi(a, b)) f(\varphi(x, y)) \\ &= f(\varphi(a, b)\varphi(x, y)). \end{aligned} \tag{25}$$

Since $f \in \mathfrak{M}(\mathcal{D})$ was arbitrary and $\mathcal{D}$ is assumed to be semisimple, we obtain

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y), \tag{26}$$

for all $(a, b), (x, y) \in \mathcal{U}$. □

The second dual space $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$ admits two Banach algebra multiplications known as the first and second Arens products, each extending the product on $\mathcal{A}$. These products which we denote by $\square$ and $\diamondsuit$, respectively, can be defined as follows:

$$\begin{aligned} F\square G &= w^* - \lim_i \lim_j a_i b_j, \\ F\diamondsuit G &= w^* - \lim_j \lim_i a_i b_j, \end{aligned} \tag{27}$$

where (a_i) and (b_j) are nets in $\mathcal{A}$ that converge, in w^*-topologies, to F and G, respectively. The Banach algebra $\mathcal{A}$ is said to be Arens regular if $F\square G = F\diamondsuit G$ on the whole of $\mathcal{A}''$. Some significant results related to the Arens regularity of certain bilinear maps and Banach algebra obtained in [10]. For more information on the Arens products, we refer the reader to [11, 12], for example.

It is shown in [11] that every C^*-algebra $\mathcal{A}$ is Arens regular and semisimple. Also the second dual of each C^*-algebra is also a C^*-algebra.

Theorem 5. *If k is the natural embedding of a Banach space X into X'', then $k(X)$ is w^*-dense in X''.*

Proof. See [13]. □

Theorem 6. *Let $\varphi : \mathcal{U} \to \mathcal{D}$ be a continuous BJH. Then the second adjoint $\varphi'' : \mathcal{U}'' \to \mathcal{D}''$ of φ is also a BJH.*

Proof. By Theorem 5, there are bounded nets (a_i) and (b_j) in $\mathcal{A}$ and $\mathcal{B}$ that converge, in w^*-topologies, to F and G, respectively. Then

$$\begin{aligned} \varphi''(F^2, G^2) &= \varphi''\left(w^* - \lim_i a_i^2, w^* - \lim_j b_j^2\right) \\ &= \varphi\left(w^* - \lim_i a_i^2, w^* - \lim_j b_j^2\right) \\ &= w^* - \lim_i \lim_j \varphi(a_i^2, b_j^2) \\ &= w^* - \lim_i \lim_j \varphi(a_i, b_j)^2 \\ &= w^* - \lim_i \lim_j \varphi''(a_i, b_j)^2 \\ &= \varphi''\left(w^* - \lim_i a_i, w^* - \lim_j b_j\right)^2 \\ &= \varphi''(F, G)^2. \end{aligned} \tag{28}$$

Thus, φ'' is BJH, as claimed. □

Since the second dual of every C^*-algebra is unital [14], we deduce the following result from Theorems 4 and 6.

Corollary 7. *Let $\varphi : \mathcal{U} \to \mathcal{D}$ be a continuous BJH between commutative C^*-algebras. Then $\varphi'' : \mathcal{U}'' \to \mathcal{D}''$ is a bihomomorphism.*

Theorem 8. *Suppose that φ is a BJH from the unital Banach algebra $\mathcal{U}$ into $\mathbb{C}$. Then φ is a n-BJH; that is,*

$$\varphi(a^n, b^n) = \varphi(a, b)^n, \tag{29}$$

for all $n \geq 3$.

Proof. Let $\varphi : \mathcal{U} \to \mathbb{C}$ be a BJH. Then, for all $(a, b) \in \mathcal{U}$,

$$\varphi(a^2, b^2) = \varphi(a, b)^2. \tag{30}$$

Replacing a by $a + x$ and b by $b + y$ in (30), we have

$$\begin{aligned} \varphi(ax + xa, by + yb) &= 2\varphi(a, b)\varphi(x, y) \\ &\quad + 2\varphi(a, y)\varphi(x, b). \end{aligned} \tag{31}$$

Replacing a by a^2 and b by b^2 in (31) gives

$$\begin{aligned} &\varphi(a^2x + xa^2, b^2y + yb^2) \\ &\quad = 2\varphi(a^2, b^2)\varphi(x, y) + 2\varphi(a^2, y)\varphi(x, b^2). \end{aligned} \tag{32}$$

Replacing x by a and y by b in (32), we obtain

$$\begin{aligned} \varphi(2a^3, 2b^3) &= 2\varphi(a^2, b^2)\varphi(a, b) \\ &\quad + 2\varphi(a^2, b)\varphi(a, b^2). \end{aligned} \tag{33}$$

Replacing x and y by e in (31), we get

$$\varphi(2a,2b) = 2\varphi(a,b)\varphi(e,e) + 2\varphi(a,e)\varphi(e,b). \tag{34}$$

Since $\varphi(e,e) = 1$, so (34) gives

$$\varphi(a,b) = \varphi(a,e)\varphi(e,b), \tag{35}$$

for all $(a,b) \in \mathcal{U}$. By Lemma 1,

$$\begin{aligned} \varphi\left(a^2,b\right) &= \varphi(a,b)\varphi(a,e), \\ \varphi\left(a,b^2\right) &= \varphi(a,b)\varphi(e,b). \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned} \varphi\left(a^2,b\right)\varphi\left(a,b^2\right) &= \varphi(a,b)^2\varphi(a,e)\varphi(e,b) \\ &= \varphi(a,b)^3. \end{aligned} \tag{37}$$

By (33) and (37),

$$\varphi\left(a^3,b^3\right) = \varphi(a,b)^3, \tag{38}$$

for all $(a,b) \in \mathcal{U}$. Thus, the result is established for $n = 3$. An easy induction argument now finishes the proof. □

As a consequence of Theorem 8 we have the next result.

Corollary 9. *Suppose that φ is a BJH from the unital Banach algebra $\mathcal{U}$ into a semisimple commutative Banach algebra $\mathcal{D}$. Then φ is an n-BJH.*

Baker in [15] proved that an almost multiplicative function is either bounded or multiplicative. Now we prove an analogous result of Baker's theorem for bihomomorphism.

Theorem 10. *Let $\delta > 0$ and $\varphi : \mathcal{U} \to \mathbb{C}$ satisfy*

$$\left|\varphi(ax,by) - \varphi(a,b)\varphi(x,y)\right| \le \delta, \tag{39}$$

for all $(a,b),(x,y) \in \mathcal{U}$. Then either φ is bihomomorphism or

$$\left|\varphi(x,y)\right| \le \frac{1}{2}\left(1 + \sqrt{1+4\delta}\right) := \varepsilon, \tag{40}$$

for all $(x,y) \in \mathcal{U}$.

Proof. Let $\varepsilon^2 - \varepsilon = \delta$ and $\varepsilon > 1$. Suppose that there exist $(s,t) \in \mathcal{U}$ such that $|\varphi(s,t)| > \varepsilon$, so that $|\varphi(s,t)| = \varepsilon + p$, where $p > 0$. Then we have

$$\begin{aligned} \left|\varphi\left(s^2,t^2\right)\right| &\ge \left|\varphi(s,t)^2 - \left(\varphi(s,t)^2 - \varphi\left(s^2,t^2\right)\right)\right| \\ &\ge \left|\varphi(s,t)\right|^2 - \left|\varphi(s,t)^2 - \varphi\left(s^2,t^2\right)\right|; \end{aligned} \tag{41}$$

hence

$$\begin{aligned} \left|\varphi\left(s^2,t^2\right)\right| &\ge \left|\varphi(s,t)\right|^2 - \delta = (\varepsilon+p)^2 - \delta \\ &= 2\varepsilon p + p^2 + \varepsilon > \varepsilon + 2p. \end{aligned} \tag{42}$$

Now make the induction assumption:

$$\left|\varphi\left(s^{2^n},t^{2^n}\right)\right| > \varepsilon + (n+1)p. \tag{43}$$

Let

$$M_n = \varphi\left(s^{2^n},t^{2^n}\right). \tag{44}$$

Then

$$\begin{aligned} \left|M_{n+1}\right| &= \left|M_n^2 - \left(M_n^2 - M_{n+1}\right)\right| \ge \left|M_n^2\right| - \delta \\ &= \left|M_n\right|^2 + \varepsilon - \varepsilon^2, \end{aligned} \tag{45}$$

whence

$$\begin{aligned} \left|M_{n+1}\right| &\ge (\varepsilon + (n+1)p)^2 + \varepsilon - \varepsilon^2 \\ &= (2n+2)\varepsilon p + \varepsilon + (n+1)^2 p^2 \\ &> \varepsilon + (n+2)p, \end{aligned} \tag{46}$$

and (43) is established for all $n \in \mathbb{N}$.

Given that $(a,b),(x,y),(u,v) \in \mathcal{U}$, we have

$$\begin{aligned} \left|\varphi(axu,byv) - \varphi(a,b)\varphi(xu,yv)\right| &\le \delta, \\ \left|\varphi(axu,byv) - \varphi(ax,by)\varphi(u,v)\right| &\le \delta. \end{aligned} \tag{47}$$

Thus,

$$\left|\varphi(ax,by)\varphi(u,v) - \varphi(a,b)\varphi(xu,yv)\right| \le 2\delta. \tag{48}$$

Hence

$$\begin{aligned} &\left|\varphi(ax,by)\varphi(u,v) - \varphi(a,b)\varphi(x,y)\varphi(u,v)\right| \\ &\quad\le \left|\varphi(ax,by)\varphi(u,v) - \varphi(a,b)\varphi(xu,yv)\right| \\ &\qquad + \left|\varphi(a,b)\varphi(xu,yv) - \varphi(a,b)\varphi(x,y)\varphi(u,v)\right| \\ &\quad\le 2\delta + \delta\left|\varphi(a,b)\right|. \end{aligned} \tag{49}$$

Therefore

$$\begin{aligned} &\left|\varphi(u,v)\right|\left|\varphi(ax,by) - \varphi(a,b)\varphi(x,y)\right| \\ &\quad\le 2\delta + \delta\left|\varphi(a,b)\right|. \end{aligned} \tag{50}$$

Now put $(u,v) = (s^{2^n},t^{2^n})$ to obtain

$$\begin{aligned} \left|\varphi(ax,by) - \varphi(a,b)\varphi(x,y)\right| &\le \frac{2\delta + \delta\left|\varphi(a,b)\right|}{\left|M_n\right|} \\ &\le \frac{2\delta + \delta\left|\varphi(a,b)\right|}{\varepsilon + (n+1)p}, \end{aligned} \tag{51}$$

for all $n \in \mathbb{N}$ and $(a,b) \in \mathcal{U}$. Letting $n \to \infty$, we get $\varphi(ax,by) = \varphi(a,b)\varphi(x,y)$ and so φ is a bihomomorphism. □

As Baker pointed out in his article, the above proof works for functions $\varphi : \mathscr{U} \to \mathscr{D}$, where $\mathscr{D}$ is a Banach algebra in which the norm is multiplicative; that is, $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathscr{D}$. For norm algebra $\mathscr{D}$ for which the norm is not multiplicative, the situation is false. For example, let $\delta > 0$ and choose $\varepsilon > 0$ so that $|\varepsilon - \varepsilon^2| = \delta$, and define $\varphi : \mathbb{R} \times \mathbb{R} \to M_3(\mathbb{R})$ by

$$\varphi(x, y) = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \quad x, y \in \mathbb{R}. \tag{52}$$

Then with the usual matrix norm

$$\|\varphi(ax, by) - \varphi(a, b)\varphi(x, y)\| = \delta, \tag{53}$$

for all $(a, b), (x, y) \in \mathbb{R} \times \mathbb{R}$. Clearly, φ is unbounded, but φ is not bihomomorphism.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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