New Subclasses concerning Some Analytic and Univalent Functions

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1. Introduction and Preliminaries

Let \( A \) be the class of functions \( f(z) \) which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( f(0) = 0 \) and \( f'(0) = 1 \).

Let \( S \) denote the subclass of \( A \) consisting of functions \( f(z) \in A \) which are univalent in \( U \). Also, let \( S^*(\beta) \) be the subclass of \( S \) consisting of functions \( f(z) \) which are starlike of order \( \beta \) \((0 \leq \beta < 1)\) in \( U \). Further, we say that \( f(z) \in \mathcal{K}(\beta) \) if \( f(z) \in S \) satisfies \( zf'(z) \in S^*(\beta) \). A function \( f(z) \in \mathcal{K}(\beta) \) is said to be convex of order \( \beta \) in \( U \) (cf. [1–3]).

With the above definitions for classes \( \mathcal{K}(\beta) \), \( S^*(\beta) \), \( S \), and \( A \), it is known that

\[ \mathcal{K}(\beta) \subset S^*(\beta) \subset S \subset A \quad (1) \]

and \( f(z) \in S^*(\beta) \) if and only if \( \int_0^z (f(t)/t) \, dt \in \mathcal{K}(\beta) \).

The function \( f(z) \) given by

\[ f(z) = \frac{z}{1 - z^2} = z + z^3 + z^5 + \cdots \quad (z \in U) \quad (2) \]

is in the class \( S^*(0) \equiv S^* \) and the function \( f(z) \) given by

\[ f(z) = \frac{z}{1 - z^\alpha} = z + z^2 + z^3 + \cdots \quad (z \in U) \quad (3) \]

is in the class \( \mathcal{K}(0) \equiv \mathcal{K} \).

If we consider the function \( f(z) \) given by

\[ f_\alpha(z) = \frac{z}{1 - z^\alpha} = z + \sum_{n=1}^{\infty} z^{1+n\alpha} \quad (z \in U) \quad (4) \]

for some real \( \alpha \) \((0 < \alpha \leq 2)\), we discuss some properties between functions \( f(z) \) in (2) and (3), where we consider the principal value for \( z^{\alpha} \).

With the function \( f(z) \) given by (4), we introduce a class \( \mathcal{A}_\alpha \) of analytic functions \( f(z) \) with series expansion in \( U \) such that

\[ f(z) = z + \sum_{n=1}^{\infty} a_n z^{1+n\alpha} \quad (z \in U) \quad (5) \]

for some real \( \alpha \) \((0 < \alpha \leq 2)\), where we take the principal value for \( z^{\alpha} \). If \( f(z) \in \mathcal{A}_\alpha \) satisfies

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in U) \quad (6) \]

for some real \( \beta \) \((0 \leq \beta < 1)\), then we say that \( f(z) \in \delta^*_\alpha (\beta) \).
Also, if \( f(z) \in \mathcal{A}_a \) satisfies
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \quad (z \in U)
\]  
(7)
for some real \( \beta \) (0 \leq \beta < 1), then we say that \( f(z) \in \mathcal{K}_a(\beta) \).

With the above definitions for the classes \( S^*_a(\beta) \) and \( K_a(\beta) \), we have that \( f(z) \in \mathcal{K}_a(\beta) \) if and only if \( zf'(z) \in \mathcal{D}_a^*(\beta) \) and that \( f(z) \in \mathcal{K}_a^*(\beta) \) if and only if \( \int_0^zf(t)/t \, dt \in \mathcal{D}_a(\beta) \).

2. Some Properties

In this section, we consider some properties of functions with series expansion given by (4).

**Theorem 1.** If \( f(z) \) is given by (4), then \( f(z) \in S^*_a((2-\alpha)/2) \) for \( 0 < \alpha \leq 2 \) and \( f(z) \in \mathcal{K}_a(\alpha) \) for \( 0 < \alpha < 1 \).

**Proof.** For \( f(z) \) given by (4), we see that
\[
\frac{zf'(z)}{f(z)} = 1
\]
for \( z = 0 \). This shows that \( f(z) \in S^*_a((2-\alpha)/2) \) for \( 0 < \alpha \leq 2 \). Further, we have that \( 1 + zf''(z)/f'(z) = 1 \) for \( z = 0 \) and
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \Re \left( 1 + \frac{(\alpha - 1) z^\alpha}{1 - z^\alpha} + \frac{\alpha (\alpha - 1) z^\alpha}{1 - (\alpha - 1) z^\alpha} \right)
\]
\[
= 3\alpha - 1 + 2(1 - \alpha) \Re \left( \frac{1}{1 - e^{i\theta}} \right)
\]
\[
= 2\alpha - \alpha \Re \left( \frac{1}{1 + (\alpha - 1) e^{i\theta}} \right)
\]
for \( z = e^{i\theta} \) (0 \( \leq \theta < 2\pi \)). Letting
\[
g(t) = \frac{1 + (\alpha - 1) t}{1 + (\alpha - 1)^2 + 2(\alpha - 1)t} \quad (t = \cos(\alpha\theta)),
\]
we have that
\[
g'(t) = \frac{\alpha (\alpha - 1) (\alpha - 2)}{(1 + (\alpha - 1)^2 + 2(\alpha - 1)t)^2} > 0
\]
(0 \( < \alpha < 1 \)).

Thus, we see that
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)
\]
(12)
for \( 0 < \alpha < 1 \). This completes the proof of the theorem. \( \square \)

**Corollary 2.** A function
\[
f(z) = \frac{z}{1 - \sqrt{z}} \quad (z \in U)
\]
(13)
belongs to the class \( S^*_{1/2}(3/4) \) and \( \mathcal{K}_{1/2}(1/2) \).

Next, we discuss some properties of functions \( f(z) \) for \( A_a(\alpha) \).

**Theorem 3.** If \( f(z) \) given by (5) satisfies
\[
\sum_{n=1}^{\infty} (\alpha + 1 - \beta) |a_n| \leq 1 - \beta
\]
(14)
for some \( \beta \) (0 \( \leq \beta < 1 \)), then \( f(z) \in \mathcal{K}_a(\alpha) \).

The equality holds true for \( f(z) \) given by
\[
f(z) = z + \sum_{n=1}^{\infty} \frac{(1 - \beta) e^{i\pi n}}{n(n+1)(\alpha + 1 - \beta)} z^{1+na}.
\]
(15)

**Proof.** Let the function \( f(z) \) be given by (5); then, we have that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \sum_{n=1}^{\infty} \frac{na_n}{1 + \sum_{n=1}^{\infty} a_n} z^{na} \right| \leq \sum_{n=1}^{\infty} n a_n |a_n| |z|^{\alpha n}
\]
\[
< \sum_{n=1}^{\infty} n a_n |a_n| |z|^{\alpha n} \leq 1 - \beta
\]
if \( f(z) \) satisfies (14). This shows that \( f(z) \in \mathcal{D}_a^*(\beta) \). Further, if we consider a function \( f(z) \) given by (15), then we see that
\[
\sum_{n=1}^{\infty} (\alpha + 1 - \beta) |a_n| = \sum_{n=1}^{\infty} \frac{1 - \beta}{n(n+1)}
\]
(17)
\[
= (1 - \beta) \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]
\[
= 1 - \beta.
\]
\( \square \)

**Theorem 4.** If \( f(z) \) given by (5) satisfies
\[
\sum_{n=1}^{\infty} (\alpha + 1) (\alpha + 1 - \beta) |a_n| \leq 1 - \beta
\]
(18)
for some \( \beta \) (0 \( \leq \beta < 1 \)), then \( f(z) \in \mathcal{K}_a(\beta) \).
The equality in (18) holds true for \( f(z) \) given by
\[
f(z) = z + \sum_{n=1}^{\infty} \left( \frac{1 - \beta}{n(\alpha + 1)} \right) \frac{e^{i\pi}}{z^{1+\alpha}}.
\]

Further, we obtain the following.

**Theorem 5.** Let \( f(z) \) be given by (5) with \( \arg a_n = \pi - n\alpha \theta \) \( (0 < \theta < 2\pi) \). Then, \( f(z) \in \Delta^\alpha_\alpha(\beta) \) if and only if
\[
\sum_{n=1}^{\infty} (\alpha n + 1 - \beta) |a_n| \leq 1 - \beta
\]
for some \( \beta \) \( (0 \leq \beta < 1) \). The equality holds true for
\[
f(z) = z + \sum_{n=1}^{\infty} \left( \frac{1 - \beta}{n(n + 1)} \right) \frac{e^{i\pi}}{z^{1+\alpha}}.
\]

**3. Radius Problems**

In this section, we consider
\[
g(z) = \frac{z}{1 - z^\alpha} \quad (z \in U)
\]
for some real \( \alpha > 2 \). Then, we say that \( g(z) \notin \Delta^\alpha_\alpha(\beta) \) and \( g(z) \notin \mathcal{K}_\alpha(\beta) \) for any real \( \beta \) \( (0 \leq \beta < 1) \).

Now, we derive the following.

**Theorem 7.** If \( g(z) \) is given by (29) with \( \alpha > 2 \), then
\[
\text{Re} \left( \frac{z g'(z)}{g(z)} \right) > \frac{1 - (\alpha - 1) r^\alpha}{1 + r^\alpha} \quad (0 < |z| = r < 1).
\]

**Proof.** For \( g(z) \) given by (29), we have that
\[
\frac{z g'(z)}{g(z)} = \frac{1 + (\alpha - 1) r^\alpha e^{i\alpha \theta}}{1 - r^\alpha e^{i\alpha \theta}} = \frac{e^{-i\alpha \theta} + (\alpha - 1) r^\alpha}{e^{-i\alpha \theta} - r^\alpha}
\]
for \( z = x + iy \). This gives us
\[
\text{Re} \left( \frac{z g'(z)}{g(z)} \right) = \frac{1 + (\alpha - 2) r^\alpha \cos \alpha \theta - (\alpha - 1) r^{2\alpha}}{1 + r^{2\alpha} - 2 r^{\alpha} \cos \alpha \theta}.
\]

Letting
\[
h(t) = \frac{1 + (\alpha - 2) r^\alpha t - (\alpha - 1) r^{2\alpha}}{1 + r^{2\alpha} - 2 r^\alpha t} \quad (t = \cos \alpha \theta),
\]
we see that \( h'(t) > 0 \). This gives us
\[
\text{Re} \left( \frac{z g'(z)}{g(z)} \right) > \frac{1 - (\alpha - 1) r^\alpha}{1 + r^\alpha}.
\]

**Corollary 8.** If \( g(z) \) is given by (29) with \( \alpha > 2 \), then
\[
\text{Re} \left( \frac{z g'(z)}{g(z)} \right) > \beta \quad (0 \leq \beta < 1)
\]
for \( 0 < |z| \leq \sqrt{(1 - \beta)/(\beta + \alpha - 1)} < 1 \).

**Proof.** If we consider
\[
\text{Re} \left( \frac{z g'(z)}{g(z)} \right) > \frac{1 - (\alpha - 1) r^\alpha}{1 + r^\alpha} \geq \beta,
\]
then
\[
0 < r \leq \sqrt{\frac{1 - \beta}{\beta + \alpha - 1}} < 1.
\]

**Remark 9.** If \( \beta = 0 \) in (35), then
\[
0 < |z| \leq \sqrt{\frac{1}{\alpha - 1}} < 1,
\]
and if \( \beta = 1/2 \), then
\[
0 < |z| \leq \sqrt{\frac{1}{2\alpha - 1}} < 1.
\]
4. Partial Sums

Finally, we consider the partial sums of \( f(z) \) given by (5). In view of (5), we write

\[ f_n(z) = z + a_n z^1 \quad (n = 1, 2, 3, \ldots) \]  

(40)

for some real \( 0 < \alpha \leq 2 \). Recently, Darus and Ibrahim [4] and Hayami et al. [5] have shown some interesting results for some partial sums of analytic functions.

Now, we derive the following.

**Theorem 10.** Let \( f_n(z) \) be given by (40) with \( |a_n| \leq 1 \). Then,

\[ \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right) > \frac{1 - (n \alpha + 1)|a_n|}{1 - |a_n|^r} \quad (z \in U) \]  

(41)

\[ \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right) \geq \frac{1 - (n \alpha + 1) r^{\alpha}}{1 - r^{\alpha}} \quad (|z| = r < 1). \]  

(42)

**Proof.** It follows that

\[ \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right) = \text{Re} \left( 1 + \frac{n \alpha a_n z^{\alpha}}{1 + a_n z^{\alpha}} \right) = 1 \]

(43)

where \( a_n = |a_n| e^{i \varphi} \) and \( z = r e^{i \theta} \). This gives us

\[ \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right) = 1 + \frac{n \alpha |a_n| r^{\alpha} \cos(n \alpha \theta + \varphi) + i |a_n| r^{\alpha} \sin(n \alpha \theta + \varphi)}{1 + 2 |a_n| r^{\alpha} \cos(n \alpha \theta + \varphi) + |a_n|^2 r^{\alpha}}. \]  

(44)

Defining \( h(t) \) by

\[ h(t) = \frac{|a_n| r^{\alpha} + t}{1 + 2 |a_n| r^{\alpha} + |a_n|^2 r^{\alpha}} \]

(45)

\( (t = \cos(n \alpha \theta + \varphi)) \),

we have that \( h'(t) > 0 \) with \( |a_n| \leq 1 \).

Thus, we obtain

\[ \text{Re} \left( \frac{z f'_n(z)}{f_n(z)} \right) > 1 - \frac{n \alpha |a_n| r^{\alpha}}{1 - |a_n|^r} \quad (0 \leq r < 1). \]  

(46)

Making \( r \to 1 \) in (46), we see (41). Also letting \( |a_n| = 1 \) in (46), we see (42).

**Corollary 11.** Let \( f_n(z) \) be given by (40) with \( |a_n| \leq \frac{1 - \beta}{(n \alpha + 1 - \beta)} \) \((0 \leq \beta < 1)\). Then, \( f_n(z) \in \mathcal{S}_\alpha^*(\beta) \).

**Proof.** Since \( |a_n| < 1 \), \( f_n(z) \) satisfies (41). Therefore, for \( |a_n| \leq \frac{1 - \beta}{(n \alpha + 1 - \beta)} \), (41) gives us \( f_n(z) \in \mathcal{S}_\alpha^*(\beta) \).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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