Research Article

On Generalized Growth of Analytic Functions Solutions of Linear Homogeneous Partial Differential Equation of Second Order

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The generalized growth parameters of analytic functions solutions of linear homogeneous partial differential equations of second order have been studied. Moreover, coefficients characterizations of generalized order and generalized type of the solutions represented in convergent series of Laguerre polynomials have been obtained.

1. Introduction

Hu and Yang [1, 2] studied the behavior of meromorphic solutions of the following homogeneous linear partial differential equations of the second order:

$$\begin{aligned} t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + (2t + 2) \frac{\partial u}{\partial t} - 2z \frac{\partial u}{\partial z} &= 0, \\ t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + t \frac{\partial u}{\partial t} - z \frac{\partial u}{\partial z} + t^2 u &= 0. \end{aligned} \quad (1)$$

They showed that these solutions are closely related to Bessel functions and Bessel polynomials for $(t, z) \in \mathbb{C}^2$. Several authors such as Berenstein and Li [3], Hu and Yang [4], Hu and Li [5], Li [6], and Li and Saleeby [7] have investigated the global solutions of some first-order partial differential equations. McCoy [8] and Kumar [9] studied the approximation of pseudoanalytic functions on the disk and obtained some coefficient and Bernstein-type growth theorems. Kapoor and Nautiyal [10] and Kumar and Basu [11] characterized the order and type of solutions (not necessarily entire solutions) of certain linear partial differential equations in terms of rates of decay of approximation errors in various norms. These solutions are related to Jacobi polynomials. Wang et al. [12] obtained the growth parameters order and

type of entire function solutions of the partial differential equation

$$t \frac{\partial^2 u}{\partial t^2} + (\delta + 1 - t) \frac{\partial u}{\partial t} + z \frac{\partial u}{\partial z} = 0 \quad (2)$$

for a real $\delta > 0$. These solutions are related to Laguerre polynomials. In this paper, our aim is to characterize the generalized growth parameters order and type of solutions of (2) which are analytic on bidisk of finite radius R . The generalized growth parameters have been studied by several authors such as Šeremeta [13], Shah [14], Srivastava and Kumar [15], and Vakarchuk and Zhir [16, 17]. Wang et al. [12] proved that the PDE (2) has an entire solution $u = f(t, z)$ on $\mathbb{C}^2$, if and only if $u = f(t, z)$ has a series expansion $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$. Here

$$L_n(\delta, t) = \sum_{k=0}^n \binom{n+\delta}{n-k} \frac{(-t)^k}{k!} \quad (3)$$

are the Laguerre polynomials. Bernstein theorem identifies a real analytic function on the closed unit disk as the restriction of an analytic function defined on an open disk of radius $R > 1$ by computing R from the sequence of minimal errors generated from optimal polynomials approximates.

The disk D_R of maximum radius on which analytic function $f(t, z)$ exists is denoted by $f(t, z) \in A(D_R \times D_R)$. If f is an entire function, it has no singularities in the finite positive $\mathbb{C}^2$ plane and write $f \in A(\mathbb{C}^2)$. Let α and β be two positive, strictly increasing, and differentiable functions from $(0, \infty) \rightarrow (0, \infty)$, which satisfies the following conditions for every $\gamma > 0$:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\alpha(\gamma x)}{\alpha(x)} &= 1, \\ \lim_{x \rightarrow \infty} \frac{\beta(1 + x\omega(x))}{\beta(x)} &= 1, \\ \lim_{x \rightarrow \infty} \omega(x) &= 0, \\ \lim_{x \rightarrow \infty} \frac{d(\beta^{-1}(\gamma\alpha(x)))}{d(\log x)} &\leq b, \\ \alpha\left(\frac{x}{\beta^{-1}(\gamma\alpha(x))}\right) &= (1 + o(1))\alpha(x), \quad x \rightarrow \infty; \end{aligned} \quad (4)$$

here $d(u)$ denotes the differential of u . Now we define the generalized order and generalized type of an entire $f(t, z) \in A(\mathbb{C}^2)$ by

$$\begin{aligned} \rho(\alpha, \beta) &= \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, r, f))}{\beta(\log r)}, \\ T(\alpha, \beta) &= \limsup_{r \rightarrow \infty} \frac{\alpha(M(r, r, f))}{[\beta(r)]^{\rho(\alpha, \beta)}}, \end{aligned} \quad (5)$$

where

$$M(r, r, f) = \sup_{z, t \in \overline{D}_R} |f(t, z)|, \quad r < R. \quad (6)$$

In view of the concept introduced by MacLane [18] to the measures of order and type for an analytic function on a disk $|z| < R$, we normalize above definitions relative to the boundary under the transformation $r \rightarrow R/(R - r)$. Thus an analytic function $f(t, z) \in A(D_R \times D_R)$ with radial limits is said to be of generalized regular growth $(\rho_o(\alpha, \beta), T_o(\alpha, \beta))$ if it satisfies

$$\begin{aligned} \rho_o(\alpha, \beta) &= \limsup_{r \rightarrow R} \frac{\alpha(\log M(r, r, f))}{\beta(R/(R - r))}, \\ T_o(\alpha, \beta) &= \limsup_{r \rightarrow R} \frac{\alpha(M(r, r, f))}{[\beta(R/(R - r))]^{\rho_o(\alpha, \beta)}}, \quad 1 < r < R, \end{aligned} \quad (7)$$

where $\rho_o(\alpha, \beta)$ is referred to as the (α, β) -order of $f(t, z)$ provided that $0 < \rho_o(\alpha, \beta) < \infty$ and $T_o(\alpha, \beta)$ is referred to as the (α, β) -type.

Example A. $f(t, z) = \exp\{(1/(1 - t))(1/(1 - z))\} \in A(D_1 \times D_1)$ has (α, β) -order 2 and (α, β) -type 1 for $\alpha(x) = \log x$ and $\beta(x) = x$.

In a neighborhood of origin, the function $f(t, z)$ has the local expansion

$$f(t, z) = \sum_{n=0}^{\infty} \frac{w_n(t)}{n!} z^n, \quad (8)$$

where $w_n(t) = (\partial^n f / \partial z^n)(t, 0)$ is an entire solution of the ordinary differential equation

$$t \frac{d^2 w}{dt^2} + (\delta + 1 - t) \frac{dw}{dt} + nw = 0, \quad (9)$$

where $w_n(t)z^n = n!a_n L_n(\delta, t)z^n$. Following the method of Frobenius [19], a second independent solution of (9) can be obtained as

$$X_n(\delta, t) = qL_n(\delta, t) \log t + \sum_{i=0}^{\infty} p_i t^i, \quad (10)$$

where $q \neq 0$ and p_i are constants. So there exist a_n and b_n satisfying

$$w_n(t) = n!a_n L_n(\delta, t) + b_n X_n(\delta, t). \quad (11)$$

Because of the singularity of $X_n(\delta, t)$ at $t = 0$, it leads to $b_n = 0$. Thus

$$f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n. \quad (12)$$

Since $f(0, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, 0) z^n$, we have the estimate $|w_n(0)z^n| = |n!a_n L_n(\delta, 0)z^n| \sim n!|a_n|(n^\delta/\Gamma(\delta + 1))r^n$ for large n . Set

$$\limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\beta[n/\log((n^\delta/\Gamma(\delta + 1))|a_n|R^n)]} = \mu(\alpha, \beta). \quad (13)$$

2. Auxiliary and Main Results

First we prove the following lemma.

Lemma 1. Let $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ and $f \in A(D_R \times D_R)$. For every $1 < r < R$, we set

$$\begin{aligned} \overline{M}(r, r, f) &= \sup_{n \in \mathbb{N}} \{\|a_n L_n(\delta, t)\| \cdot r^n, \quad r > 0\}, \\ \overline{\rho}_o(\alpha, \beta) &= \limsup_{r \rightarrow R} \frac{\alpha(\log \overline{M}(r, r, f))}{\beta(R/(R - r))}; \end{aligned} \quad (14)$$

then

$$\begin{aligned} \overline{\rho}_o(\alpha, \beta) &\leq \mu(\alpha, \beta), \\ \rho_o(\alpha, \beta) &\leq \overline{\rho}_o(\alpha, \beta). \end{aligned} \quad (15)$$

Proof. From (13), for r sufficiently close to R , we have

$$\log\left(\frac{n^\delta}{\Gamma(\delta + 1)}|a_n|R^n\right) \leq \frac{n}{\beta^{-1}(\alpha(n)/\overline{\mu})}, \quad \overline{\mu} = \mu + \varepsilon \quad (16)$$

or

$$\begin{aligned} \log \left(\frac{n^\delta}{\Gamma(\delta+1)} |a_n| R^n \right) \\ \leq n \log \left(\frac{r}{R} \right) + \frac{n}{\beta^{-1}(\alpha(n)/\bar{\mu})}. \end{aligned} \quad (17)$$

Using the result

$$\begin{aligned} \lim_{x \rightarrow \infty} \left| \frac{d(\beta^{-1}(\alpha(x)))}{d(\log x)} \right| \leq b \log(1 + \xi) = (1 + o(\xi)), \\ \xi \rightarrow 0, \end{aligned} \quad (18)$$

with the method of Calculus that, for every $r > 1$ and $\mu > 0$, the maximum of the function

$$x \log \left(\frac{r}{R} \right) + \frac{x}{\beta^{-1}(\alpha(x)/\bar{\mu})} \quad (19)$$

is reached at

$$\begin{aligned} x \\ = \alpha^{-1} \left\{ \bar{\mu} \beta \left[\frac{1 - d \log(\beta^{-1}(\alpha(x)/\bar{\mu})) / d(\log x)}{\log(R/r)} \right] \right\}, \end{aligned} \quad (20)$$

we get

$$x = (1 + o(1)) \alpha^{-1} \left(\bar{\mu} \beta \left(\frac{R}{(R-r)} \right) \right), \quad r \rightarrow \infty. \quad (21)$$

Now we have

$$\begin{aligned} \log \left(\frac{n^\delta}{\Gamma(\delta+1)} |a_n| r^n \right) \leq C_o \alpha^{-1} \left(\bar{\mu} \beta \left(\frac{R}{(R-r)} \right) \right), \\ \log M(r, r, f) \leq C_o \alpha^{-1} \left(\bar{\mu} \beta \left(\frac{R}{(R-r)} \right) \right). \end{aligned} \quad (22)$$

Using the property of α , we get

$$\frac{\alpha(\log \bar{M}(r, r, f))}{\beta(R/(R-r))} \leq \bar{\mu}. \quad (23)$$

Applying the limit supremum as $r \rightarrow R$, we obtain

$$\bar{\rho}_o(\alpha, \beta) \leq \mu(\alpha, \beta). \quad (24)$$

Now consider

$$f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, 0) z^n; \quad (25)$$

putting $r = \sqrt{r \cdot R} \sqrt{r/R}$ in the above, we get

$$\begin{aligned} M(r, r, f) \leq \sum_{n=0}^{\infty} |a_n| |L_n(\delta, t)| (\sqrt{r \cdot R})^n \left(\sqrt{\frac{r}{R}} \right)^n, \\ \left(\frac{r}{R} \right) < 1, \end{aligned} \quad (26)$$

or

$$\begin{aligned} M(r, r, f) \\ \leq \sum_{n=0}^{\infty} \sup \left(|a_n| |L_n(\delta, t)| (\sqrt{r \cdot R})^n \left(\sqrt{\frac{r}{R}} \right)^n \right), \end{aligned} \quad (27)$$

or

$$\begin{aligned} M(r, r, f) \leq \bar{M}(r', r', f) \sum_{n=0}^{\infty} \left(\sqrt{\frac{r}{R}} \right)^n \\ \leq \bar{M}(r', r', f) \left(\frac{1}{1 - \sqrt{r/R}} \right), \\ r' = \sqrt{r \cdot R}, \end{aligned} \quad (28)$$

$$\begin{aligned} \log M(r, r, f) \leq \log \bar{M}(r', r', f) \\ - \log \left(1 - \sqrt{\frac{r}{R}} \right), \end{aligned}$$

or

$$\begin{aligned} \frac{\alpha(\log M(r, r, f))}{\beta(R/(R-r))} \\ \leq \frac{\alpha(\log \bar{M}(r', r', f) - \log(1 - \sqrt{r/R}))}{\beta(R/(R - \sqrt{r \cdot R}))} \\ \cdot \frac{\beta(R/(R - \sqrt{r \cdot R}))}{\beta(R/(R-r))}. \end{aligned} \quad (29)$$

Proceeding to limits, we obtain

$$\rho_o(\alpha, \beta) \leq \bar{\rho}_o(\alpha, \beta). \quad (30)$$

Combining (24) and (30), we get

$$\rho_o(\alpha, \beta) \leq \mu(\alpha, \beta). \quad (31)$$

□

Theorem 2. Let $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\beta[n/\log((n^\delta/\Gamma(\delta+1)) |a_n| R^n)]} = \mu(\alpha, \beta) \\ < \infty; \end{aligned} \quad (32)$$

then f is the restriction of analytic function in $(D_R \times D_R)$ ($R > 1$) and its (α, β) -order $\rho(\alpha, \beta) = \mu(\alpha, \beta)$.

Proof. Since $f(0, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, 0) z^n$ is an analytic function in D_R , we have

$$\limsup_{n \rightarrow \infty} (|a_n| L_n(\delta, 0))^{1/n} = \frac{1}{R}, \quad R > 1. \quad (33)$$

Using $L_n(\delta, 0) = n^\delta/\Gamma(\delta+1)$, we get

$$\limsup_{n \rightarrow \infty} (|a_n|)^{1/n} = \frac{1}{R}, \quad (34)$$

which is necessary and sufficient condition for $f \in A(D_R \times D_R)$. So, for every $1 < r < R$, the series $\sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ is convergent in $D_R \times D_R$ when $\sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ is analytic in $D_R \times D_R$.

Now we have to prove that $\mu(\alpha, \beta)$ is the (α, β) -order of $f(t, z)$.

In order to complete the proof by Lemma 1, it is only to show that $\rho(\alpha, \beta) \geq \mu(\alpha, \beta)$. In view of the definition of $\rho(\alpha, \beta)$, we have, for every $\delta > 0$, that there exists $1 < r_\varepsilon < R$ such that, for every $r_\varepsilon < r < R$,

$$\log M(r, r, f) \leq \alpha^{-1} \left[(\rho(\alpha, \beta) + \varepsilon) \beta \left(\frac{R}{(R-r)} \right) \right]. \quad (35)$$

Using Cauchy's estimate of analytic functions, we have

$$\left| \frac{\partial^n f}{\partial z^n}(0, 0) \right| \leq n r^{-n} M(r, r, f) \quad (36)$$

with the coefficients formula of the Taylor expansion,

$$\frac{\partial^n f}{\partial z^n}(0, 0) = a_n L_n(\delta, 0) n!; \quad (37)$$

we get $M(r, r, f) \geq |a_n| |L_n(\delta, 0)| r^n$. Since $|L_n(\delta, 0)| \geq 1$, it gives

$$|a_n| r^n \leq M(r, r, f). \quad (38)$$

Now using (35) in (38), we get

$$\begin{aligned} & \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta+1)} R^n \right) \\ & \leq -n \log \left(\frac{r}{R} \right) + \delta \log n - \log(\Gamma(\delta+1)) \\ & \quad + \alpha^{-1} \left[(\rho(\alpha, \beta) + \varepsilon) \beta \left(\frac{R}{(R-r)} \right) \right]. \end{aligned} \quad (39)$$

The minimum value of right-hand side is estimated at

$$\begin{aligned} r &= R \left\{ 1 \right. \\ & \quad \left. - \frac{1}{\beta^{-1} \left((1/(\rho + \varepsilon)) \alpha(n/\beta^{-1}(\alpha(n)/(\rho + \varepsilon))) \right)} \right\}, \quad (40) \\ \rho &= \rho(\alpha, \beta). \end{aligned}$$

Using the properties of functions α and β ,

$$\alpha \left(\frac{x}{\beta^{-1}(c\alpha(x))} \right) \approx (1 + o(1)) \alpha(x); \quad (41)$$

for $c > 0$, $x \rightarrow \infty$, and the properties of logarithm, we get

$$\begin{aligned} & \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta+1)} R^n \right) \\ & \leq C_1 \left(\frac{n}{\beta^{-1}(\alpha(n)/(\rho + \varepsilon))} \right), \end{aligned} \quad (42)$$

where C_1 is a constant. Hence,

$$\beta \left(\frac{C_1 n}{\log(|a_n| (n^\delta/\Gamma(\delta+1)) R^n)} \right) \geq \frac{\alpha(n)}{(\rho + \varepsilon)}. \quad (43)$$

Proceeding to limit supremum as $n \rightarrow \infty$, we obtain

$$\mu(\alpha, \beta) \leq \rho(\alpha, \beta). \quad (44)$$

Hence the proof is completed. $\square$

Example B. $f(t, z) = \sum_{m+n=0}^{\infty} (1+m)^l (1+n)^l \exp\{m^\mu n^\mu\} t^m z^n \in A(D_1 \times D_1)$ has (α, β) -order $(\mu/(1-\mu))$ for $\alpha(x) = \log x$ and $\beta(x) = x$, where l is the positive integer.

Let $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ be analytic function of (α, β) -order $\rho = \rho(\alpha, \beta)$ and write

$$\begin{aligned} T(\alpha, \beta) &= \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\{\beta[n/\log((n^\delta/\Gamma(\delta+1)) |a_n| R^n)]\}^{\rho(\alpha, \beta)}}. \end{aligned} \quad (45)$$

Lemma 3. Let $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$. For every $1 < r < R$,

$$\bar{\sigma}_1(\alpha, \beta) = \limsup_{r \rightarrow R} \frac{\alpha(\log \bar{M}(r, r, f))}{[\beta(R/(R-r))]^{\rho(\alpha, \beta)}}; \quad (46)$$

then

$$\sigma(\alpha, \beta) \leq \bar{\sigma}_1(\alpha, \beta). \quad (47)$$

Proof. Following the same reasoning as in the proof of Lemma 1, we obtain

$$\begin{aligned} & \frac{\alpha(\log M(r, r, f))}{[\beta(R/(R-r))]^{\rho(\alpha, \beta)}} \\ & \leq \frac{\alpha(\log \bar{M}(r', r', f) - \log(1 - \sqrt{r/R}))}{[\beta(R/(R - \sqrt{r \cdot R}))]^{\rho(\alpha, \beta)}} \\ & \quad \cdot \frac{[\beta(R/(R - \sqrt{r \cdot R}))]^{\rho(\alpha, \beta)}}{[\beta(R/(R-r))]^{\rho(\alpha, \beta)}}. \end{aligned} \quad (48)$$

Applying the limit supremum as $r \rightarrow R$, we get

$$\sigma(\alpha, \beta) \leq \bar{\sigma}_1(\alpha, \beta). \quad (49)$$

$\square$

Theorem 4. Let $f(t, z) = \sum_{n=0}^{\infty} a_n L_n(\delta, t) z^n$ be of finite generalized (α, β) -order $\rho(\alpha, \beta)$ and

$$\begin{aligned} T(\alpha, \beta) &= \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\{\beta[n/\log((n^\delta/\Gamma(\delta+1)) |a_n| R^n)]\}^{\rho(\alpha, \beta)}} \\ &< \infty. \end{aligned} \quad (50)$$

Then f is the restriction of an analytic function in $(D_R \times D_R)$ ($R > 1$) and its (α, β) -type $\sigma(\alpha, \beta) = T(\alpha, \beta)$.

Proof. We have proven in Theorem 2 that f is the restriction of an analytic function in $(D_R \times D_R)$. Now, in order to complete the proof, first we shall prove that $\sigma(\alpha, \beta) \leq T(\alpha, \beta)$. In view of the definition of T , for every $\varepsilon > 0$, there exists $n \geq n_\varepsilon$:

$$\begin{aligned} \alpha(n) &\leq (T + \varepsilon) \left\{ \beta \left[\frac{n}{\log((n^\delta / \Gamma(\delta + 1)) |a_n| R^n)} \right] \right\}^{\rho(\alpha, \beta)}, \\ T &\equiv T(\alpha, \beta) < \infty, \end{aligned} \quad (51)$$

$$\begin{aligned} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} R^n \right) &\leq \left(\frac{n}{\beta^{-1} [(\alpha(n) / (\bar{T}))]^{1/\rho}} \right), \\ \bar{T} &= T + \varepsilon, \end{aligned}$$

since

$$\begin{aligned} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} r^n \right) &\leq n \log \left(\frac{r}{R} \right) + \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} R^n \right). \end{aligned} \quad (52)$$

From inequality (51), we get

$$\begin{aligned} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} r^n \right) &\leq n \log \left(\frac{r}{R} \right) + \left(\frac{n}{\beta^{-1} [(\alpha(n) / (\bar{T}))]^{1/\rho}} \right). \end{aligned} \quad (53)$$

For every $1 < r < R$ and r sufficiently close to R , we put

$$\varphi(x, r) = x \log \left(\frac{r}{R} \right) + \left(\frac{x}{\beta^{-1} [(\alpha(x) / (\bar{T}))]^{1/\rho}} \right). \quad (54)$$

The maximum value of $\varphi(x, r)$ is reached at

$$\begin{aligned} x = x_r &= \alpha^{-1} \left\{ \left[\bar{T} \beta \left[\frac{1 - d \log \left(\beta^{-1} (\alpha(x) / \bar{T})^{1/\rho} \right) / d(\log x)}{\log(R/r)} \right] \right]^\rho \right\}. \end{aligned} \quad (55)$$

Using the relation

$$\log \left(\frac{r}{R} \right) = \log \left(\frac{r - R}{R} + 1 \right) \sim \frac{r - R}{R} \quad (56)$$

(as $(r - R)/R \rightarrow 0$) and

$$\left| \frac{d \left(\log \left(\beta^{-1} (\alpha(x) / \bar{T})^{1/\rho} \right) \right)}{d(\log x)} \right| \leq k_o, \quad (57)$$

where k_o is a positive constant, it gives

$$x_r = (1 + o(1)) \alpha^{-1} \left(\bar{T} \left(\beta \left(\frac{R}{R - r} \right) \right)^\rho \right). \quad (58)$$

Now, from relation (53), we have

$$\log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} r^n \right) \leq \sup \varphi(x, r) \quad (59)$$

or

$$\begin{aligned} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} r^n \right) &\leq \frac{(1 + o(1)) \alpha^{-1} \left(\bar{T} \left(\beta \left(\frac{R}{R - r} \right) \right)^\rho \right)}{R / (R - r)}. \end{aligned} \quad (60)$$

Since $R/(R - r) > 1$, it gives

$$\log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} r^n \right) \leq k_1 \alpha^{-1} \left(\bar{T} \left(\beta \left(\frac{R}{R - r} \right) \right)^\rho \right). \quad (61)$$

Then

$$\log M(r, r, f) \leq k_1 \alpha^{-1} \left(\bar{T} \left(\beta \left(\frac{R}{R - r} \right) \right)^\rho \right), \quad (62)$$

or

$$\frac{\alpha(\log M(r, r, f))}{(\beta(R/(R - r)))^\rho} \leq \bar{T}, \quad (63)$$

or

$$\sigma(\alpha, \beta) \leq T. \quad (64)$$

The inequality obviously holds for $T = \infty$. Now we shall prove that $\sigma(\alpha, \beta) \geq T(\alpha, \beta)$. Let $\sigma(\alpha, \beta) < \infty$. In view of definition of $\sigma(\alpha, \beta)$, we have, for every $\varepsilon > 0$, that there exist $1 < r_\varepsilon < R$, such that, for every $r > r_\varepsilon$ ($R > r > r_\varepsilon > 1$),

$$\begin{aligned} \log M(r, r, f) &\leq \alpha^{-1} \left(\bar{\sigma} \left(\beta \left(\frac{R}{R - r} \right) \right)^\rho \right), \\ \bar{\sigma} &= \sigma + \varepsilon. \end{aligned} \quad (65)$$

Now, using (51), we get

$$\begin{aligned} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta + 1)} R^n \right) &\leq -n \log \left(\frac{r}{R} \right) + \delta \log n - \log(\Gamma(\delta + 1)) \\ &\quad + \alpha^{-1} \left[\bar{\sigma} \left(\beta \left(\frac{R}{(R - r)} \right) \right)^\rho \right], \end{aligned} \quad (66)$$

or

$$\begin{aligned} \frac{\log \left(|a_n| \left(n^\delta / \Gamma(\delta + 1) \right) R^n \right)}{n} &\leq -\log \left(\frac{r}{R} \right) \\ &\quad + \frac{1}{n} \left[\delta \log n - \log(\Gamma(\delta + 1)) \right] \\ &\quad + \alpha^{-1} \left[\bar{\sigma} \left(\beta \left(\frac{R}{(R - r)} \right) \right)^\rho \right], \end{aligned} \quad (67)$$

or

$$\frac{1}{n} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta+1)} R^n \right) \leq (1 + o(1)) \log \left(\frac{R}{r} \right), \quad (68)$$

as $n \rightarrow \infty$. Now, using $\log(R/r) \sim (R-r)/r$ as $r \rightarrow R$, we have

$$\begin{aligned} \frac{1}{n} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta+1)} R^n \right) \\ \leq (1 + o(1)) \log \left(\left(\frac{R}{r} \right) - 1 \right). \end{aligned} \quad (69)$$

Set

$$\frac{r}{R} = \frac{\beta^{-1} [(\alpha(n)/(\bar{\sigma}))]^{1/\rho}}{1 + \beta^{-1} [(\alpha(n)/(\bar{\sigma}))]^{1/\rho}}; \quad (70)$$

in above inequality, we get

$$\frac{1}{n} \log \left(|a_n| \frac{n^\delta}{\Gamma(\delta+1)} R^n \right) \leq \frac{1}{\beta^{-1} [(\alpha(n)/(\bar{\sigma}))]^{1/\rho}} \quad (71)$$

or

$$\frac{\alpha(n)}{\beta(n/\log(|a_n| (n^\delta/\Gamma(\delta+1)) R^n))} \leq \bar{\sigma} = \sigma + \varepsilon. \quad (72)$$

Applying the limit supremum as $n \rightarrow \infty$, we obtain

$$\sigma(\alpha, \beta) \geq T(\alpha, \beta). \quad (73)$$

The result is obviously true for $\sigma(\alpha, \beta) = \infty$. This completes the proof. $\square$

Example C. $f(t, z) = \sum_{m+n=0}^{\infty} \exp(\delta^*(m+n)^{\gamma^*}) t^m z^n \in A(D_1 \times D_1)$, where $0 < \gamma^* < 1$ and $0 < \delta^* < \infty$, and has (α, β) -order $\gamma^*/(1-\gamma^*)$ and (α, β) -type $(1-\gamma^*)(\gamma^*)^{\gamma^*/(1-\gamma^*)}(\delta^*)^{1/(1-\gamma^*)}$ for $\alpha(x) = \log x$ and $\beta(x) = x$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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